Chapter 1

BOSONS AFTER SYMMETRY BREAKING IN QUANTUM FIELD THEORY

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Abstract

We present a unified description of the spontaneous symmetry breaking and its associated bosons in fermion field theory. There is no Goldstone boson in the fermion field theory models of Nambu-Jona-Lasinio, Thirring and QCD_2 after the chiral symmetry is spontaneously broken in the new vacuum. The defect of the Goldstone theorem is clarified, and the "massless boson" predicted by the theorem is virtual and corresponds to just a $free$ massless fermion and antifermion pair. Further, we discuss the exact spectrum of the Thirring model by the Bethe ansatz solutions, and the analytical expressions of all the physical observables enable us to understand the essence of the spontaneous symmetry breaking in depth. Also, we examine the boson spectrum in QCD_2, and show that bosons always have a finite mass for $SU(N_c)$ colors. The problem of the light cone prescription in QCD_2 is discussed, and it is shown that the trivial light cone vacuum is responsible for the wrong prediction of the boson mass.

1 Introduction

Most of the field theory models should possess some kind of symmetries. Apart from the basic symmetry like the Lorentz invariance, there are symmetries which play a fundamental role in determining the structure of the vacuum state. The chiral symmetry is one of the most popular ones in quantum field theory of fermions.

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The physics of the chiral symmetry and its spontaneous symmetry breaking in fermion field theory models has been discussed quite extensively since the invent of the Goldstone theorem [1, 2]. In particular, the current current interaction model of Nambu and Jona-Lasinio (NJL) has been studied by many people since it is believed that the NJL model can present a good example of exhibiting the Goldstone boson after the chiral symmetry is spontaneously broken [3].

However, recent careful studies clarify that there appears no massless boson in the NJL model [4, 5]. Further, it is shown that the Goldstone theorem cannot be applied to the fermion field theory models due to a serious defect in the course of proving the theorem [6]. That is, the existence of a massless boson that should be proved as the result of the Goldstone equation has to be assumed as the initial input of the equation, and this is of course no proof at all.

In the NJL model, the Lagrangian density possesses the chiral symmetry, but the vacuum state prefers the chiral symmetry broken state since it has the lower energy than the one with preserving its symmetry. In this procedure, the chiral current is conserved and therefore the symmetry breaking is considered to be spontaneous due to the definition of the *spontaneous* symmetry breaking in proving the Goldstone theorem. However, if one calculates the boson mass properly, then there appears no massless boson in the NJL model. In this case, a question may arise as to why people obtained the massless boson in the NJL model. Not only Nambu and Jona-Lasinio but also quite a few physicists found a massless boson in their boson mass calculation [7]. Surprisingly, the reason why they found a massless boson is simple. They calculated the boson mass by summing up one loop Feynman diagrams, but their calculation is based on the perturbative vacuum state. However, after the spontaneous symmetry breaking, one finds the new vacuum which has the lower energy than the perturbative vacuum state. Therefore, the physical vacuum state is of course the new vacuum that breaks the chiral symmetry, and thus if one wishes to calculate any physical observables in field theory, then one must employ the formulation which is based on the physical vacuum state. In fact, the quantum field theory is constructed on the physical vacuum state, and therefore it is essential to make the formulation which is based on the physical vacuum state. Those calculations which start from the perturbative vacuum state should find a unphysical boson. Indeed, this unphysical massless boson is just what Nambu and all other people obtained in their calculations.

In this chapter, we review the spontaneous symmetry breaking and its appearance of bosons associated with the symmetry breaking phenomena. The symmetry breaking of the vacuum occurs in the boson field theory as well as in the fermion field theory models. In the boson field theory, the symmetry breaking is clear in four dimensions, but the symmetry should not be broken in two dimensions since there should not exist a physical massless boson due to the infra-red divergence of its propagator. In the fermion field theory, the symmetry is broken in the vacuum in two and four dimensions. This is quite simple because there is no Goldstone boson and therefore the two dimensional field theory is not special any more.

The picture of the chiral symmetry breaking in the fermion field theory can be drawn

in the following way. One starts from the perturbative vacuum which is the same as the free vacuum state of fermions. This preserves the chiral symmetry. When one takes the interactions into account, then the distributions of the negative energy particles in the vacuum state change and the vacuum energy becomes lower than that of the free vacuum state. The momentum distributions of the negative energy particles become different for right and left mover fermions so that the chiral symmetry is broken in the new vacuum. In addition, the distribution of the negative energy particles in the new vacuum is rearranged such that the finite gap is seen in the excitation spectrum. This is essentially all that happens to the symmetry breaking phenomena in the fermion field theory models like NJL, Thirring model, QED_2 and QCD_2 with massless fermions. On this vacuum state, one may find bosons or there may be no boson, and this depends on the interactions between fermions and antifermions. For the current current interaction models like the NJL or the massless Thirring model, one cannot make any bosons since the interaction is a $\delta-$function type potential. On the other hand, the gauge field theory models of QED_2 and QCD_2 find massive bosons since the interaction is a confining potential in two dimensions. At this point, it is probably fair to mention that, if one finds the true vacuum state in quantum field theory model, then it means that one could solve this field theory model exactly. However, the NJL model is not exactly solvable, and we believe the solution of the vacuum state constructed by Bogoliubov transformation method is not exact. However, the new vacuum state has the lower energy than the perturbative vacuum state, and the qualitatively right picture of the symmetry breaking phenomena should be obtained from this approximate vacuum state.

For the symmetry broken vacuum states, the chiral condensate value is finite in all of the massless fermion field theory models. This is a representation of the vacuum structure, and it is quite natural that the symmetry broken vacuum state has a complicated vacuum structure with a finite condensate value. However, QED_2 and QCD_2 with massive fermions have no chiral symmetry, and therefore one cannot discuss about the symmetry broken vacuum. Nevertheless, the vacuum state has a finite condensate value, and it keeps the complicated vacuum structure. In these field theory models of QED_2 and QCD_2, the fermion mass term seems to play a role of just like a perturbative interaction term. The boson mass increases linearly as the function of the fermion mass m_0. The behavior of the chiral condensate is somewhat similar to the boson mass, and it decreases as the function of the fermion mass m_0 and the condensate value goes to zero at the very large fermion mass where the system becomes nonrelativistic. In this respect, it is still not very clear how the chiral condensate value can be related to the symmetry breaking phenomena.

This chapter is organized in the following way. In the second section, we review the Goldstone theorem and discuss its problem related to the fermion field theory models. This theorem is originally meant for the boson field theory models, but it was believed that the theorem should be valid for the fermion field theory models as well. However, one can easily notice the basic defect of the theorem, that is, the existence of the massless boson that should be proved as a final goal has to be assumed in the initial equation. Here, we present a proof why the Goldstone theorem is justified for the boson field case while it does not hold for the fermion field case.

In section 3, we treat the spontaneous symmetry breaking in boson field theory models. This is well established in four dimensions, and we present it only for the clarification of the essence of the physics of the spontaneous symmetry breaking phenomena and the appearance of the massless boson associated with the symmetry breaking. However, the symmetry breaking and the spectrum of the boson field theory in two dimensions is not so clear as in four dimensions, and we discuss problems behind the two dimensional boson fields.

In section 4, we discuss the non-appearance of a massless boson after the spontaneous symmetry breaking in fermion field theory models. Since the spontaneous symmetry breaking has a long history, the non-appearance of the Goldstone boson is indeed a surprising result. But it was due to the lack of the deep understanding of the vacuum structure, and, in a sense, it should have been very difficult to clearly realize the importance of the change in the vacuum structure which arises from the symmetry breaking. Here, we present the symmetry breaking and its boson associated with the symmetry breaking in the NJL and the Thirring models. We first give intuitive discussions why the chiral symmetry is broken in the vacuum of the NJL and the Thirring models, and clarify why there should not appear any massless boson. Further, we carry out more elaborate calculations of the symmetry breaking in these fermion field theory models based on the Bogoliubov transformation method. In this calculation, one sees that there should be a massive boson depending on the strength of the coupling constant, but the boson is not a consequence of the symmetry breaking, but it is due to the result of the approximate scheme of the Bogoliubov transformation. It is most probable that there should be no boson in the NJL and the Thirring models.

In section 5, we present the Bethe ansatz solutions of the massless Thirring model. This is quite important to understand the structure of the new vacuum and its change of the negative energy particle distribution in the vacuum state. One clearly sees that the new vacuum state that breaks the chiral symmetry has the lower energy than the symmetric vacuum state. In this case, the momentum distribution of the negative energy state changes drastically. This means that the symmetry is spontaneously broken even though the Thirring model is a two dimensional field theory model. However, there is neither a massless boson nor a massive boson. There is only a finite gap for the excitation spectrum. In this respect, one can learn a lot about the symmetry breaking and its boson associated with the spontaneous symmetry breaking. The non-existence of a massless boson is very reasonable since there should not exist any massless boson in two dimensions. Here, we also show that the bosonization procedure which is commonly used for the massless Thirring model has a serious defect in that one cannot find the corresponding degree of freedom for the zero mode. Therefore, the massless Thirring model is not bosonized properly, and therefore it is indeed consistent with the finite gap in the spectrum.

In section 6, we discuss the Schwinger model, and one knows that it is well understood. There is no new thing added to this section. However, we believe that it should be important to understand the origin of the chiral symmetry breaking. In the Schwinger model, the chiral symmetry is broken, but it is not spontaneous since the chiral current is not conserved any more due to the anomaly term after the regularization of the vacuum state. The anomaly

term arises from the conflict between the gauge invariance and the chiral current conservation. However, it is interesting to note that the chiral condensate is a smooth function of the fermion mass, and it is not very clear yet whether the chiral condensate is a consequence of the chiral symmetry breaking or not.

In section 7, we present the recent results of the numerical calculations of SU(N_c) QCD in two dimensions in terms of the Bogoliubov transformation method. The calculations are carried out up to very large values of the N_c color degree, and it is shown that the SU(50) calculation is already quite similar to the result with the $N_c \to \infty$. However, it turns out that the light cone calculation cannot reproduce neither the right boson spectrum nor the right condensate values. This must be due to the fact that the light cone vacuum is trivial even though the real vacuum has a complicated structure with a finite condensate value. In fact, 't Hooft calculation is not an exception and gives wrong results of the spectrum since he employed the light cone vacuum, even though the large N_c expansion itself is a right and good scheme for SU(N_c) QCD$_2$. Therefore, if one wishes to find the correct spectrum of the field theory model, then one has to start from the right vacuum state as a minimum condition. Here, the calculated results of the spectrum and the condensate values by the Bogoliubov transformation method indicate that the symmetry is spontaneously broken in the vacuum state, and there is no massless boson. This is again consistent with the fact that there should be no physical massless boson in two dimensions.

In section 8, we summarize what we have clarified in the spontaneous symmetry breaking and its boson associated with the symmetry breaking phenomena. Some comments on the Heisenberg XXZ model and the lattice field theory are included.

2 Goldstone Theorem and its Applicability

The Goldstone theorem has played a central role for understanding the symmetry breaking and its massless boson after the spontaneous symmetry breaking. When the Lagrangian density has some continuous symmetry which can be represented by the unitary operator $U(\alpha)$, there is a conserved current associated with the symmetry

$$\partial_\mu j^\mu = 0. \tag{2.1}$$

In this case, there is a conserved charge Q which is defined as

$$Q = \int j^0(x)d^3x. \tag{2.2}$$

The Hamiltonian of this system H is invariant under the unitary transformation $U(\alpha)$,

$$U(\alpha)HU(\alpha)^{-1} = H. \tag{2.3a}$$

Writing the $U(\alpha)$ explicitly as $U(\alpha) = e^{i\alpha Q}$, we obtain

$$QH = HQ. \tag{2.3b}$$

Now, the vacuum state can break the symmetry, and we define the symmetric vacuum $|0\rangle$ and symmetry broken vacuum $|\Omega\rangle$, respectively, which satisfy the following equations,

$$U(\alpha)|0\rangle = |0\rangle \tag{2.4a}$$

$$U(\alpha)|\Omega\rangle \neq |\Omega\rangle. \tag{2.4b}$$

These equations can be written in terms of the charge operator Q as

$$Q|0\rangle = 0 \tag{2.5a}$$

$$Q|\Omega\rangle \neq 0. \tag{2.5b}$$

In the Goldstone theorem, it is assumed that the current conservation arising from the symmetry should hold after the symmetry is broken in the vacuum state. Therefore, the commutation relation between the charge and some boson field operator $\phi(x)$ is time independent, and we write it as

$$[Q(t), \phi(x)] = \hat{C} \tag{2.6}$$

where $\hat{C}$ is some operator that is described by the field operators. This is of course an identity equation, but the choice of the field $\phi(x)$ itself is not at all trivial. It should also be important to note that eq.(2.6) is derived independently from the Hamiltonian of field theory models.

Now, we take the expectation value of eq. (2.6) with the vacuum state, and there the information of the field theory model should be put in eq.(2.6) through the vacuum state. First, we employ the symmetric vacuum $|0\rangle$,

$$\langle 0 \,|\, [Q(t), \phi(x)] \,|\, 0\rangle = \langle 0 \,|\, \hat{C} \,|\, 0\rangle. \tag{2.7a}$$

In this case, the left hand side vanishes. Therefore, the right hand side must also vanish, and eq. (2.6) gives just the identity equation as expected.

Next, we take the expectation value of eq.(2.6) with the symmetry broken vacuum $|\Omega\rangle$,

$$\langle \Omega \,|\, [Q(t), \phi(x)] \,|\, \Omega\rangle = \langle \Omega \,|\, \hat{C} \,|\, \Omega\rangle \neq 0. \tag{2.7b}$$

If the right hand side is non-zero, then the vacuum of the system has the symmetry broken state since the left hand side survives only when the operator Q satisfies eq.(2.5b).

The Goldstone theorem starts from the vacuum expectation value of the commutation relation eq.(2.7b) with the symmetry broken vacuum $|\Omega\rangle$, and the boson field ϕ eventually corresponds to a massless boson.

Further, we assume that the field ϕ and the current density $j^0(x)$ satisfies the following translational property

$$\phi(x) = e^{ipx}\phi(0)e^{-ipx} \tag{2.8a}$$

$$j^0(x) = e^{ipx}j^0(0)e^{-ipx}. \tag{2.8b}$$

Now, we insert a complete set of intermediate bosonic states $|n\rangle$ in eq.(2.7b). Since it should be excited by the charge operator Q, this state $|n\rangle$ should have the same momentum as the vacuum state, and this means that the momentum of the bosonic state $|n\rangle$ is zero.

Thus, we obtain from eq.(2.7b),

$$\sum_n (2\pi)^3 \delta(\boldsymbol{p}_n) \left[\langle\Omega|j^0(0)|n\rangle\langle n|\phi(0)|\Omega\rangle e^{-iE_n t} - \langle\Omega|\phi(0)|n\rangle\langle n|j^0(0)|\Omega\rangle e^{iE_n t} \right] \neq 0. \tag{2.9}$$

The right hand side is non-zero and is also time-independent. However, in the left hand side, the positive and negative energy terms cannot cancel with each other as long as the energy E_n is non-zero. Therefore, the time dependence of eq.(2.9) in the left hand side can be taken away only when the following condition is satisfied,

$$E_n = 0 \quad \text{for} \quad \boldsymbol{p}_n = 0. \tag{2.10}$$

From this constraint, one learns that if this bosonic state is an isolated system, then this should correspond to a massless boson. Thus, there should appear a massless boson after the spontaneous symmetry breaking. However, there is a serious difference between the boson field and fermion field theory models, and we show that the proof of the Goldstone theorem cannot be applied to the fermion field theory models since the existence of the boson field has to be assumed while the boson field itself is, however, the one that must be proved as a result.

The difficult part in eq.(2.9) is to find the boson field operator ϕ, and if one can find it properly, then one can obtain some physical information from the identity equation. It should be noted that, normally, one cannot get any important information from the identity equation since it is not directly related to the dynamics of the field theory model.

At this point, we should comment on a possible degeneracy of the symmetry broken vacuum state associated with the charge operator Q in the total Hamiltonian system since there is a belief that the symmetry broken vacuum may have infinite degenerate states. First, we define the symmetry broken vacuum energy which is the eigenstate of the Hamiltonian H,

$$H|\Omega\rangle = E_\Omega|\Omega\rangle. \tag{2.11}$$

Now, if we define a new state $|\varphi_n\rangle$ by

$$|\varphi_n\rangle \equiv \mathcal{N}Q^n|\Omega\rangle \tag{2.12}$$

with $\mathcal{N}$ a normalization constant, then the state $|\varphi_n\rangle$ has the same vacuum energy E_Ω because of eq.(2.3b),

$$H|\varphi_n\rangle = \mathcal{N}HQ^n|\Omega\rangle = \mathcal{N}Q^n H|\Omega\rangle = E_\Omega|\varphi_n\rangle. \tag{2.13}$$

This equation seems to indicate that the vacuum state has $n-$degeneracy. However, Q has the same eigenstate as H due to eq.(2.3b), and therefore we write with its eigenvalue q

$$Q|\Omega\rangle = q|\Omega\rangle. \tag{2.14}$$

Thus, we obtain

$$|\varphi_n\rangle = \mathcal{N}q^n|\Omega\rangle = |\Omega\rangle \tag{2.15}$$

since we can choose the normalization constant $\mathcal{N}$ as $\mathcal{N} = q^{-n}$. Therefore, the state $|\varphi_n\rangle$ is nothing but the vacuum state $|\Omega\rangle$ itself, and there is no degeneracy because the charge operator Q cannot change the vacuum structure. This degeneracy is spurious, but the degeneracy of the potential vacuum in the double well potential problem in the boson field theory model is real, and this will be discussed in section 3.

2.1   Boson Field Theory Model

The Goldstone theorem is proved by employing eqs.(2.7) and (2.8). In the boson field theory models, the boson field ϕ exists from the beginning. This is of course a trivial thing in the boson field theory models. This boson field ϕ in eq.(2.7) corresponds eventually to the Goldstone boson. Here, the most important point is that boson is characterized by its mass, and therefore the determination of the boson mass is the only concern for the boson field theory models. Also, eq.(2.8) has no problem since the boson should exist as an elementary boson state and therefore it satisfies the translational invariance.

From eqs.(2.9), one obtains the constraint on the bosonic state as given by eq.(2.10). This constraint does not necessarily mean that this bosonic state should have the dispersion relation of a massless boson. But if this system is an isolated one, then this is just the dispersion relation of a massless boson, and therefore there should be a massless boson, and this is just the Goldstone boson.

2.2   Fermion Field Theory Model

Now, we discuss the fermion field theory models [6]. It is important to note that, in the fermion field theory models, bosons must be constructed by the fermions and antifermions as their bound states. There is no elementary boson field in this field theory model itself.

In this case, we should ask ourselves what is the boson field ϕ in eq.(2.7) in the fermion field theory models. This boson field ϕ should eventually correspond to a massless boson if at all exists in the fermion field theory models. But who shows that there are any bound states of the fermions and antifermions in this field theory models ? This should involve dynamics and it should be very hard to solve any of the fermion field theory models until one finds bound states.

Now, in the proof of the Goldstone theorem, the existence of the boson field ϕ is assumed, and this is just the one that should be proved as a final goal. Thus, it is clear that eq.(2.7) cannot be applied to the fermion field theory models. Eq.(2.7) can give one information which is the dispersion relation, but one cannot take out the information on the existence of the bound state between fermions and antifermions. Further, if one wishes to evaluate the commutation relation between the conserved charge and the boson field ϕ in eq.(2.7), then one has to be able to describe the boson field ϕ in terms of the fermion field operators. This is quite clear since the charge $Q_5(t) = \int j_5^0(x)d^3x$ is written in terms of

the fermion field operators, and therefore one should have the expression of the boson field ϕ by the fermion fields.

$$\phi = F[\bar{\psi}, \psi]. \tag{2.16}$$

The functional dependence of $F[\bar{\psi}, \psi]$ should be determined by solving the dynamics, and it should be extremely difficult to find the functional dependence of $F[\bar{\psi}, \psi]$. In fact, it is practically impossible to find the functional dependence of $F[\bar{\psi}, \psi]$ unless the field theory model is exactly solvable. In two dimensional field theory models, there is one example which is solved exactly, and that is the Schwinger model as we treat it in section 6. In this case, one can describe the boson field in terms of the fermion field operators.

Here, we show a common mistake which is often found in the textbook to describe how the Goldstone theorem holds in fermion field theory model. One says that one may take the following ϕ in the case of the chiral symmetry breaking

$$\phi(x) = \bar{\psi}(x)\gamma_5\psi(x). \tag{2.17}$$

In this case, one can easily prove that this ϕ satisfies eq.(2.6). In fact, one obtains for eq. (2.6)

$$\left[Q_5(t), \bar{\psi}(x)\gamma_5\psi(x)\right] = 2\bar{\psi}(x)\psi(x) \tag{2.18}$$

where Q_5 denotes the chiral charge which is a conserved quantity for the chiral symmetry preserving system. It should be quite important to note that eq.(2.18) is derived independently from the shape of the interaction Lagrangian density. The only condition is that the interaction term should be invariant under the chiral symmetry. Therefore, eq.(2.18) does not carry any information about the dynamics of the fermion and anti-fermion system, and therefore there is no chance that one obtains any information about the bound state of fermion and ant-fermion from eq.(2.18).

If we take the expectation value of eq.(2.18) with the symmetry broken vacuum state, then we obtain

$$\sum_n (2\pi)^3\delta(\boldsymbol{p}_n)\left[\langle\Omega|j_5^0|n\rangle\langle n|\bar{\psi}\gamma_5\psi|\Omega\rangle e^{-iE_nt} - \langle\Omega|\bar{\psi}\gamma_5\psi|n\rangle\langle n|j_5^0|\Omega\rangle e^{iE_nt}\right] \neq 0 \tag{2.19}$$

where $|n\rangle$ denotes the complete set of the fermion number zero states of the field theory model one considers. Therefore, bosonic states as well as the massless free fermion and antifermion states should be included in the intermediate states. Eq.(2.19) is just the same equation as the boson case, and therefore, it gives an impression that the Goldstone theorem is meaningful for the fermion field theory models as well.

However, one easily notices that this ϕ has nothing to do with any bound state of fermions and antifermions since, as we mention above, eq. (2.19) does not contain any information on the interaction term. Namely, we obtain from eq.(2.19),

$$E_n = 0 \quad \text{for} \quad \boldsymbol{p}_n = 0 \tag{2.20}$$

where

$$E_n = E_f + E_{\bar{f}} \tag{2.21a}$$

$$\boldsymbol{p}_n = \boldsymbol{p}_f + \boldsymbol{p}_{\bar{f}} \tag{2.21b}$$

where $\boldsymbol{p}_f$ ($\boldsymbol{p}_{\bar{f}}$) and E_f ($E_{\bar{f}}$) denote the momentum and energy of the fermion (anti-fermion), respectively. For the free massless fermion and anti-fermion pair, eq.(2.20) is indeed satisfied. This energy dispersion looks like a massless boson, but of course it has nothing to do with the massless boson.

Obviously, the existence of the boson field ϕ can be confirmed only after the whole dynamics of this field theory model is completely solved. As mentioned above, eq. (2.19) does not contain any information on the interaction term of the Lagrangian density, and therefore it is natural that eq. (2.19) cannot prove the existence of the bosonic states in the corresponding field theory model. Further, even if one could solve the dynamics properly, it would not mean that the ϕ can be expressed in terms of fermion field operators.

3 Goldstone Boson in Boson Field Theory

The physics of the spontaneous symmetry breaking started from the boson field theory models, and Goldstone discovered that there should appear a massless boson when the symmetry is spontaneously broken in the vacuum state. The basic point in this theorem is that the vacuum state always prefers the lowest energy state of the total Hamiltonian and therefore when the minimum energy state of the interaction field energy is located at the point which breaks the symmetry of the Lagrangian density, then one should find the vacuum state which breaks this symmetry. The interesting discovery of Goldstone is that there should appear a massless boson when one adds the kinetic energy terms of the boson field to the interaction field energy term. This is quite similar to the situation where the degenerate states in quantum mechanics are split into several states due to the perturbative interaction. Here, the role of the perturbative interaction is played by the boson's kinetic energy terms, and this is quite important to realize since the degeneracy is resolved by the kinetic energy term of the boson field, and thus this indeed leads to a massless boson in the Goldstone theorem. In this respect, one says that the degrees of freedom of the degeneracy of the symmetry become the Goldstone boson since the interaction that breaks the degeneracy is the boson's kinetic energy term.

3.1 Symmetry Breaking in Four Dimensional Boson Fields

Now the discussion of the spontaneous symmetry breaking in boson field theory in four dimensions can be found in any field theory text books, and therefore we only sketch the simple picture why the massless boson appears in the spontaneous symmetry breaking.

The Hamiltonian density for complex boson fields can be written as

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{p}\phi^{\dagger})(\boldsymbol{p}\phi) + U\left(|\phi|\right). \tag{3.1}$$

This has a $U(1)$ symmetry, and when one takes the potential as

$$U\left(|\phi|\right) = U_0\left(|\phi|^2 - \lambda^2\right)^2 \tag{3.2}$$

then, the minimum of the potential $U(|\phi|)$ can be found at $|\phi| = \lambda$. But one must notice that this is a minimum of the potential, but not the minimum of the total energy.

The minimum of the total energy must be found together with the kinetic energy term. When one rewrites the complex field as

$$\phi = (\lambda + \rho)e^{i\frac{\xi}{\lambda}} \tag{3.3}$$

then, one can rewrite eq.(3.1) as

$$\mathcal{H} = \frac{1}{2}\big[(\boldsymbol{p}\xi)(\boldsymbol{p}\xi) + (\boldsymbol{p}\rho)(\boldsymbol{p}\rho)\big] + U(|\lambda + \rho|) + ... \tag{3.4}$$

Here, one finds the massless boson ξ which is associated with the degeneracy of the vacuum energy. The important point is that this infinite degeneracy of the potential vacuum is converted into the massless boson degrees of freedom when the degeneracy of the potential vacuum is resolved by the kinetic energy term.

Also, it should be noted that the massless boson appears at the time when the new vacuum is determined. Namely, the massless boson and the new vacuum creation after the spontaneous symmetry breaking should occur at the same time because the appearance of the Goldstone boson is the consequence of the symmetry breaking of the vacuum state. Eq.(3.4) shows that the excitation spectrum of the boson system with respect to the field ρ has nothing to do with the symmetry breaking.

3.2 Symmetry Breaking in Two Dimensional Boson Fields

The spontaneous symmetry breaking should not occur in two dimensional field theory models due to Coleman's theorem [8]. However, as discussed in section 2, there appears no massless boson after the symmetry breaking in fermion field theory models, and therefore the symmetry can be spontaneously broken in two dimensional field theory models of fermions. Indeed, the massless Thirring model and two dimensional QCD breaks the chiral symmetry in the vacuum state, and there is no massless boson in these models.

Now, the boson field theory models in two dimensions cannot break the symmetry spontaneously since there should appear a massless boson associated with the symmetry breaking. In this case, however, it is interesting to ask ourselves what then happens to the spectrum with the double well potential case as an example as we discussed in the preceding subsection.

It is clear that there should not be any continuum spectrum arising from a massless boson since there is no physical massless boson in two dimensions. Does this imply that the appearance of the massless boson $Hamiltonian$ is forbidden after the spontaneous symmetry breaking ?

This is not a very easy question to answer since clearly the vacuum should prefer the lower energy state to the symmetric vacuum. What one can say with confidence is that the massless boson cannot become a physical particle in two dimensions. Since the boson field ξ should be coupled to the other boson field ρ in higher order interaction Hamiltonian, one

may not conclude that the vacuum should be found at the symmetry preserving or broken state, before one obtains the spectrum of this boson field theory model by solving it exactly.

As discussed in detail in section 2, the Goldstone theorem states that there should be a bosonic state which has the energy dispersion with $E = 0$ for $\boldsymbol{p} = 0$ when the symmetry is spontaneously broken. It has been believed that this state should be a massless boson. But this is, of course, a too strong statement. It only says that the dispersion of the state should be $E = 0$ for $\boldsymbol{p} = 0$, and could well be more complicated than the massless one, that is, $E = |\boldsymbol{p}|$ if the system is not an isolated one.

In other words, the degrees of freedom of ξ field may not become independent of the field ρ, and in this case, there is no reason to claim that there should appear a massless boson. It can be said that the state which satisfies the condition of $E = 0$ for $\boldsymbol{p} = 0$ does not necessarily correspond to a massless boson unless it is an independent field. If this state couples to other fields, then this complex field can survive free from the infra-red singularity of the massless boson propagator. In this sense, the condition of $E = 0$ for $\boldsymbol{p} = 0$ is not sufficient to forbid the existence of the symmetry broken vacuum.

Therefore, it is still an open question what kind of spectrum should emerge from the boson field theory model with the double well potential in two dimensions. We believe that the spectrum in this boson field theory model should help us understand the symmetry breaking in two dimensions in depth.

4 No Goldstone Boson in Fermion Field Theory

It has been believed that, when the symmetry of the Lagrangian density is spontaneously broken in the vacuum state, there should appear a massless boson in the fermion field theory models in the same way as the boson field theory. This belief started from the original work by Nambu and Jona-Lasinio (NJL) who studied the current current interaction model of the fermion field theory. By now, it is called the NJL model and it has the chiral symmetry. In their study, they showed that the vacuum prefers the chiral symmetry broken state and there should appear a massless boson associated with the chiral symmetry breaking. However, they calculated the boson mass by summing up one loop Feynman diagrams based on the perturbative vacuum state. But it is obvious that the field theory calculation should be based on the physical vacuum state, and the physical vacuum in this NJL model is of course the new vacuum that breaks the chiral symmetry. Therefore, any physical observables should be evaluated starting from the symmetry broken vacuum. Otherwise one obtains a boson mass which is unphysical, and this unphysical massless boson is exactly what Nambu and Jona-Lasinio obtained. The reality is that there is no massless boson, and in addition there should be no boson in the NJL model even though the latter claim is not proved yet.

In the NJL model, the massless fermion acquires an induced mass when we employ the Bogoliubov transformation method. Thus, the NJL model becomes a massive fermion field theory after the spontaneous symmetry breaking. This effective fermion mass was supposed to be the nucleon mass in the original paper of Nambu and Jona-Lasinio. However, we believe that this mechanism of the effective fermion mass is spurious, and it is only due to

the approximation of the Bogoliubov transformation method. The spontaneous symmetry breaking phenomena can arise from the change of the vacuum state of the field theory model, and this should not change any property of the elementary fermion field itself. This is just in contrast to the boson field theory where the property of the boson field is basically determined by the boson mass, and bosons can be easily created or destroyed. But one cannot create any fermions since they are fundamental particles and there is no way to induce the mass scale for the massless fermion from the renormalization procedure. This point can be seen quite nicely in the Bethe ansatz solution in the massless Thirring model, and we will discuss it later.

4.1   Intuitive Discussion

Here, we present an intuitive discussion of the chiral symmetry breaking in the NJL models and show that there should not appear any massless boson at all [4]. The treatment here is far from rigorous, but we believe that the essential physics of the spontaneous symmetry breaking phenomena and bosons associated with the symmetry breaking in fermion field theory models should be clarified since there is still a misunderstanding in this problem. The treatment is somewhat similar to the Bogoliubov transformation method which is originally employed by Nambu and Jona-Lasinio when they calculated the vacuum energy after the symmetry breaking in their model. Here, the determination of the vacuum energy is done by an educated guess although the result is quite similar to the one which is obtained by the Bogoliubov transformation method.

The Hamiltonian density of the NJL model is written as

$$\mathcal{H} = \psi^\dagger \boldsymbol{p} \cdot \boldsymbol{\alpha} \psi - \frac{1}{2} G \left[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right]. \tag{4.1}$$

Here, we take the chiral representation, and denote the ψ as

$$\psi(\boldsymbol{n}, s) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \psi_1 \chi^{(s)} \\ \psi_2 \chi^{(s)} \end{array} \right) \tag{4.2}$$

where $\chi^{(s)}$ denotes the spin part. In this case, the Hamiltonian density for the NJL model can be written after the summation of s is taken

$$\mathcal{H} = \psi_1^\dagger (\boldsymbol{p} \cdot \hat{\boldsymbol{n}}) \psi_1 - \psi_2^\dagger (\boldsymbol{p} \cdot \hat{\boldsymbol{n}}) \psi_2 + 2G(\psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2). \tag{4.3}$$

In the same way as the boson case, we can define the potential $U(\psi_1, \psi_2)$ as

$$U(\psi_1, \psi_2) = 2G|\psi_1|^2 |\psi_2|^2. \tag{4.4}$$

It is clear from this equation that the potential of the fermion field theory models does not have any nontrivial minimum, apart from the trivial one $\psi_1^\dagger \psi_1 = 0$, $\psi_2^\dagger \psi_2 = 0$. This is in contrast to the boson case where there is a nontrivial minimum in the potential. Therefore,

there is no degeneracy of the true vacuum state since the minimum of the potential here is only a trivial one.

Where can one find the new vacuum that breaks the chiral symmetry ? The answer is simple. One has to consider the kinetic energy term. In the fermion system, the kinetic energy is negative for the vacuum state.

In what follows, we present a simple and intuitive argument of obtaining a new vacuum state including the kinetic energy term. This treatment is schematic, but one can learn the essence of the physics of the chiral symmetry breaking in fermion field theory. The treatment by employing the Bogoliubov transformation method will be given in the next section. Now, we can take an average value of the kinetic energy $(-\Lambda_0)$ for the negative energy state, and thus we write eq.(4.3) as

$$\mathcal{H} \approx -\Lambda_0 \left(|\psi_1|^2 + |\psi_2|^2 \right) + 2G|\psi_1|^2|\psi_2|^2. \tag{4.5}$$

This can be rewritten as

$$\mathcal{H} = 2G \left(|\psi_1|^2 - \frac{\Lambda_0}{2G} \right) \left(|\psi_2|^2 - \frac{\Lambda_0}{2G} \right) - \frac{\Lambda_0^2}{2G}. \tag{4.6}$$

Therefore, it is easy to find the $|\psi_1|^2$ and $|\psi_2|^2$ for the new vacuum state, that is,

$$|\psi_1|^2 = \frac{\Lambda_0}{2G}, \qquad |\psi_2|^2 = \frac{\Lambda_0}{2G}. \tag{4.7}$$

This result is somewhat similar to the mean field approximation and indeed the mean field approximation gives rise to the chiral symmetry breaking.

In this case, the vacuum energy E_{vac} and the condensate C become

$$E_{vac} = -\frac{\Lambda_0^2 V}{2G} \tag{4.8a}$$

$$C = \frac{\Lambda_0}{G} \tag{4.8b}$$

where V denotes the volume of the system. Now, we write the ψ_1 and ψ_2 as

$$\psi_i = \psi_0 + \tilde{\psi}_i, \quad (i = 1, 2) \tag{4.9}$$

where ψ_0 denotes the fermion field which is assumed to satisfy the following relations

$$\psi_0^* \psi_0 = \frac{\Lambda_0}{2G} \tag{4.10a}$$

$$\psi_0 \psi_0 = 0. \tag{4.10b}$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = -\frac{\Lambda_0^2}{2G} + \Lambda_0 \left(\tilde{\psi}_1^\dagger \tilde{\psi}_2 + h.c. \right) + \tilde{\psi}_1^\dagger (\boldsymbol{p} \cdot \hat{\boldsymbol{n}}) \tilde{\psi}_1 - \tilde{\psi}_2^\dagger (\boldsymbol{p} \cdot \hat{\boldsymbol{n}}) \tilde{\psi}_2$$
$$+ 2G|\tilde{\psi}_1|^2 |\tilde{\psi}_2|^2 + O(\tilde{\psi}_1, \tilde{\psi}_2^\dagger). \tag{4.11}$$

Now, it is clear that the second term is the mass term. Therefore, one notices that after the chiral symmetry breaking, the fermion acquires the finite mass, and the induced mass M becomes $M = \Lambda_0$. Therefore, at this point, the symmetry breaking problem is completed. The rest of the field theory becomes just the massive fermion field theory. For example, the Thirring model becomes the massive Thirring model where one knows well that there exists one massive boson, and the mass spectrum is obtained as the function of the coupling constant [9, 33, 10].

This means that one cannot find a massless boson in the Hamiltonian of the fermion system. It is also quite important to note that the new Hamiltonian is still described by the same number of the fermion degrees of freedom as the original one. This is in contrast to the boson case where one of the complex field freedom becomes the massless boson ξ.

Therefore, if one wants to find any boson in the NJL model, then one has to solve the dynamics since the kinematics cannot produce any Goldstone boson in fermion field theory. However, it is difficult to find a massless boson as a bound state of fermion and antifermion system, regardless the strength of the coupling constant in the system of the finite fermion mass. In any fermion field theory models, the bound state energy should depend on the strength of the interaction, and if there exists a massless boson in the massive fermion field theory model, this must be the strong coupling limit of the interaction strength. However, Nambu claimed the existence of a massless boson, regardless the strength of the coupling constant, and one can easily see that this claim is physically out of question.

Indeed there is no massless boson in the NJL as well as in the massless Thirring models if one solves the dynamics properly as will be seen below in the next section.

Here, it is interesting to note that the vacuum energy and the condensate with this value of the Λ_0 [eq.(4.8a) and eq.(4.8b)] become

$$E_{vac} = -\frac{M^2 V}{2G} \qquad (4.12a)$$

$$C = \frac{M}{G} \qquad (4.12b)$$

which are quite close to the Bogoliubov transformation calculations. Also, the chiral charge Q_5 of the vacuum can be evaluated and is found to be

$$Q_5 = 0 \qquad (4.13)$$

which is also consistent with the one calculated by the Bogoliubov transformation method.

We summarize the intuitive discussions here for the fermion field theory. Even though there is no nontrivial minimum in the potential, one finds a new vacuum if one considers the kinetic energy terms of the negative energy particles in the vacuum state. The chiral symmetry is broken in the new vacuum state of the NJL and the Thirring models. Namely, the momentum distributions of the negative energy particles in the vacuum states are rearranged such that the new vacuum energy becomes lower than the perturbative vacuum state. In this process, the left and right moving fermions change the momentum distributions in the vacuum state such that the chirality is broken since the broken state has simply a lower

energy than the symmetry preserving vacuum state. After the symmetry breaking, the massless fermion acquires the effective mass though it is an approximate scheme. But there is no massless boson since the degree of freedom for the massless boson does not exist. The mass of the boson predicted in the field theory of the finite fermion mass has therefore nothing to do with the symmetry breaking business since it is just the same as asking for the excitation spectrum of the field ρ in eq.(3.4).

4.2   Bogoliubov Transformation

Now, we carry out the calculation which is based on the Bogoliubov transformation, and show that the chiral symmetry is indeed broken. However, we also show that there appears no massless boson in this regularized NJL model.

Here, we first quantize the fermion field in a box L^3

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}} \sum_{\boldsymbol{n},s} \left[a_{\boldsymbol{n},s} u(\boldsymbol{n},s) e^{i\frac{2\pi}{L}\boldsymbol{n}\cdot\boldsymbol{r}} + b_{\boldsymbol{n},s}^\dagger v(\boldsymbol{n},s) e^{-i\frac{2\pi}{L}\boldsymbol{n}\cdot\boldsymbol{r}} \right] \tag{4.14}$$

where s denotes the spin index, and $s = \pm 1$. Also, the spinors are defined as

$$u(\boldsymbol{n},s) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\chi^{(s)} \\ \chi^{(s)} \end{array} \right),$$

$$v(\boldsymbol{n},s) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \chi^{(s)} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\chi^{(s)} \end{array} \right).$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$c_{\boldsymbol{n},s} = e^{-\mathcal{A}} a_{\boldsymbol{n},s} e^{\mathcal{A}} = \cos\theta_{\boldsymbol{n}} a_{\boldsymbol{n},s} + s\sin\theta_{\boldsymbol{n}} b_{-\boldsymbol{n},s}^\dagger \tag{4.15a}$$

$$d_{-\boldsymbol{n},s}^\dagger = e^{-\mathcal{A}} b_{-\boldsymbol{n},s}^\dagger e^{\mathcal{A}} = \cos\theta_{\boldsymbol{n}} b_{-\boldsymbol{n},s}^\dagger - s\sin\theta_{\boldsymbol{n}} a_{\boldsymbol{n},s} \tag{4.15b}$$

where the generator of the Bogoliubov transformation is given by

$$\mathcal{A} = \sum_{\boldsymbol{n},s} s\theta_{\boldsymbol{n}} \left(a_{\boldsymbol{n},s}^\dagger b_{-\boldsymbol{n},s}^\dagger - b_{-\boldsymbol{n},s} a_{\boldsymbol{n},s} \right). \tag{4.16}$$

$\theta_{\boldsymbol{n}}$ denotes the Bogoliubov angle which can be determined by the condition that the vacuum energy is minimized. In this case, the new vacuum state is obtained as

$$| \Omega \rangle = e^{-\mathcal{A}} |0\rangle. \tag{4.17}$$

In what follows, we treat the NJL Hamiltonian with the Bogoliubov transformed vacuum state. In order to clearly see some important difference between the massive fermion and massless fermion cases, we treat the two cases separately.

4.3 Massive Fermion Case

The Lagrangian density for the NJL model with the massive fermion can be written as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m_0\bar{\psi}\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right]. \tag{4.18}$$

Now, we can obtain the new Hamiltonian under the Bogoliubov transformation,

$$H = \sum_{\boldsymbol{n},s}\left\{|\boldsymbol{p_n}|\cos 2\theta_{\boldsymbol{n}} + \left(m_0 + \frac{2G}{L^3}\mathcal{B}\right)\sin 2\theta_{\boldsymbol{n}}\right\}\left(c^\dagger_{\boldsymbol{n},s}c_{\boldsymbol{n},s} + d^\dagger_{-\boldsymbol{n},s}d_{-\boldsymbol{n},s}\right)$$

$$+ \sum_{\boldsymbol{n},s}\left\{-|\boldsymbol{p_n}|s\sin 2\theta_{\boldsymbol{n}} + \left(m_0 + \frac{2G}{L^3}\mathcal{B}\right)s\cos 2\theta_{\boldsymbol{n}}\right\}\left(c^\dagger_{\boldsymbol{n},s}d^\dagger_{-\boldsymbol{n},s} + d_{-\boldsymbol{n},s}c_{\boldsymbol{n},s}\right) + H'_{int}$$

$$\tag{4.19}$$

where H'_{int} is the interaction term. Since the H'_{int} is quite complicated, and besides its explicit expression is not needed in this context, we will not write it here. $\mathcal{B}$ is defined as

$$\mathcal{B} = \sum_{\mathbf{n},s}\sin 2\theta_{\boldsymbol{n}}.$$

Now, we can define the renormalized fermion mass m

$$m = m_0 + \frac{2G}{L^3}\mathcal{B}. \tag{4.20}$$

The Bogoliubov angle $\theta_{\boldsymbol{n}}$ can be determined by imposing the condition that the cd term in eq.(4.19) must vanish. Therefore, we obtain

$$\cot 2\theta_{\boldsymbol{n}} = \frac{|\boldsymbol{p_n}|}{m}. \tag{4.21}$$

This Bogoliubov angle $\theta_{\boldsymbol{n}}$ does not change when the mass varies from m_0 to m. In this case, the vacuum is just the same as the trivial vacuum of the massive case, except that the fermion mass is replaced by the renormalized mass m. The rest of the theory becomes identical to the massive NJL model with the same interaction Hamiltonian H'_{int}. Therefore, there is no symmetry breaking, and this vacuum has no condensate.

4.4 Massless Fermion Case

Here, we present the same procedure for the massless fermion case in order to understand why the fermion has to become massive.

We start from the Lagrangian density with no mass term in eq.(4.18). Under the Bogoliubov transformation, we obtain the new Hamiltonian

$$H = \sum_{\boldsymbol{n},s}\left\{|\boldsymbol{p_n}|\cos 2\theta_{\boldsymbol{n}} + \frac{2G}{L^3}\mathcal{B}\sin 2\theta_{\boldsymbol{n}}\right\}\left(c^\dagger_{\boldsymbol{n},s}c_{\boldsymbol{n},s} + d^\dagger_{-\boldsymbol{n},s}d_{-\boldsymbol{n},s}\right)$$

$$+ \sum_{\mathbf{n},s}\left\{-|\boldsymbol{p_n}|s\sin 2\theta_{\boldsymbol{n}} + \frac{2G}{L^3}\mathcal{B}s\cos 2\theta_{\boldsymbol{n}}\right\}\left(c^\dagger_{\boldsymbol{n},s}d^\dagger_{-\boldsymbol{n},s} + d_{-\boldsymbol{n},s}c_{\boldsymbol{n},s}\right) + H'_{int} \tag{4.22}$$

where H'_{int} is just the same as the one given in eq.(4.19). From this Hamiltonian, we get to know that the mass term is generated in the same way as the massive case. But we cannot make any renormalization since there is no mass term. Further, the new term is the only mass scale in this Hamiltonian since the coupling constant cannot serve as the mass scale. In fact, it is even worse than the dimensionless coupling constant case, since the coupling constant in the NJL model is proportional to the inverse square of the mass dimension. Thus, we define the new fermion mass M_N by

$$M_N = \frac{2G}{L^3}\mathcal{B}. \tag{4.23}$$

The Bogoliubov angle $\theta_{\boldsymbol{n}}$ can be determined from the following equation

$$\cot 2\theta_{\boldsymbol{n}} = \frac{|\boldsymbol{p_n}|}{M_N}. \tag{4.24}$$

In this case, the vacuum changes drastically since the original vacuum is trivial.

Further, the constraints of eqs.(4.23) and (4.24) give rise to the equation that determines the relation between the induced fermion mass M_N and the cutoff momentum Λ

$$M_N = \frac{4G}{(3\pi)^3} \int^{\Lambda} d^3p \frac{M_N}{\sqrt{M_N^2 + p^2}}. \tag{4.25}$$

This equation has a nontrivial solution for M_N, and the vacuum energy becomes lower than the trivial vacuum ($M_N = 0$). Therefore, M_N can be expressed in terms of Λ as

$$M_N = \gamma \Lambda$$

where γ is a simple numerical constant.

It should be noted that the treatment up to now is exactly the same as the one given by Nambu and Jona-Lasinio [3]. Further, we stress that the induced fermion mass M_N can never be set to zero, and it is always finite.

4.5 Boson Mass in NJL Model

The boson state $|B\rangle$ can be expressed as

$$|B\rangle = \sum_{\boldsymbol{n},s} f_{\boldsymbol{n}} c^{\dagger}_{\boldsymbol{n},s} d^{\dagger}_{-\boldsymbol{n},s} |\Omega\rangle, \tag{4.26}$$

where $f_{\boldsymbol{n}}$ is a wave function in momentum space, and $|\Omega\rangle$ denotes the Bogoliubov vacuum state. The equation for the boson mass $\mathcal{M}$ for the NJL model is written in terms of the Fock space expansion at the large L limit

$$\mathcal{M}f(p) = 2E_p f(p) - \frac{2G}{(3\pi)^3} \int^{\Lambda} d^3q f(q) \left(1 + \frac{M^2}{E_p E_q} + \frac{\boldsymbol{p} \cdot \boldsymbol{q}}{E_p E_q}\right) \tag{4.27}$$

where M should be taken to be $M = m$ for the massive case, and $M = M_N$ for the massless case. It is important to note that the fermion mass M after the Bogoliubov transformation, therefore, cannot become zero.

Here, again, we note that the RPA calculation gives the similar boson spectrum to the Fock space expansion. But we do not know whether the RPA calculation is better than the Fock space expansion or not, since the derivation of the RPA equation in field theory is not based on the fundamental principle. In principle, the RPA calculation may take into account the effect of the deformation of the vacuum in the presence of the particle and antiparticle. However, this is extremely difficult to do it properly, and indeed the RPA eigenvalue equation is not Hermite, and thus it is not clear whether the effect is taken into account in a better way or worse. The examination and the validity of the RPA equation will be given else where.

The solution of eq.(4.25) can be easily obtained, and the boson mass spectrum for the NJL model is shown in Fig. 1. Note that the boson mass is measured in units of the cutoff momentum Λ. As can be seen from the figure, there is a massive boson for some regions of the values of the coupling constant. Here, as we will see later, the NJL and the Thirring models are quite similar to each other. This is mainly because the current-current interaction is essentially a delta function potential in coordinate space. Indeed, as is well known, the delta function potential in one dimension can always bind the fermion and anti-fermion while the delta function potential in three dimension cannot normally bind them. Due to the finite cut off momentum, the delta function potential in three dimensions can make a weak bound state, depending on the strength of the coupling constant. This result of the delta function potential in quantum mechanics is almost the same as what is just shown in Fig. 1.
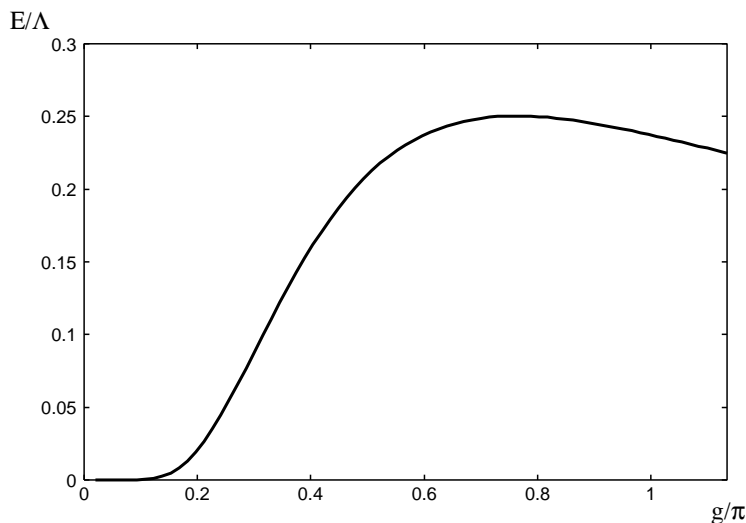

Figure 1: The boson mass for the NJL model is plotted as the function of $G\Lambda^2$. It is measured by the cutoff Λ.

Further, we should note that Kleinert and Van den Bossche [12] also found that the bosons after the symmetry breaking are all massive, which is just consistent with our claim. Their method and approach are quite different from the present calculation, but both of the calculations agree with each other that there is no massless boson in the NJL model.

Here, we should add that there is no serious difficulty of proving the non-existence of the massless boson from the calculated spectrum. However, if it were to prove the existence of the massless boson, it would have been extremely difficult to do it. For the massless boson, there should be a continuum spectrum, and this continuum spectrum of the massless boson should be differentiated from the continuum spectrum arising from the many body nature of the system. This differentiation must have been an extremely difficult task without having some analytic expression of the spectrum. In fact, even if one finds a continuum spectrum which has, for example, the dispersion of $E = c_0 p^2$ as often discussed in solid state physics, one sees that the spectrum has nothing to do with the Goldstone boson.

4.6   Boson Mass in Thirring Model

The massless Thirring model can be treated just in the same way as the NJL model in terms of the Bogoliubov transformation method. We do not repeat the detailed calculations, but instead we present the summary of the calculated results.

First, we can determine the Bogoliubov angles and from the consistency condition we can determine the induced mass. The induced mass M can be expressed in terms of the cutoff Λ,

$$M = \frac{\Lambda}{\sinh\left(\frac{\pi}{g}\right)}. \tag{4.28}$$

Further, the vacuum energy E_{vac} as measured from the trivial vacuum is given

$$E_{vac} = -\frac{L}{2\pi}\frac{\Lambda^2}{\sinh\left(\frac{\pi}{g}\right)}e^{-\frac{\pi}{g}}. \tag{4.29}$$

From this value of the vacuum energy, we get to know that the new vacuum energy is indeed lower than the trivial one. Therefore, the chiral symmetry is broken in the new vacuum state and, effectively, the fermion becomes massive.

In the same manner as [5], we carry out the calculations of the spectrum of the bosons in the Fock space expansion. The boson state $|B\rangle$ can be expressed as

$$|B\rangle = \sum_n f_n c_n^\dagger d_{-n}^\dagger |\Omega\rangle, \tag{4.30}$$

where f_n is a wave function in momentum space, and $|\Omega\rangle$ denotes the Bogoliubov vacuum state. The energy eigenvalue of the Hamiltonian for the large L limit can be written as

$$\mathcal{M}f(p) = 2E_p f(p) - \frac{g}{2\pi}\int dq f(q)\left(1 + \frac{M^2}{E_p E_q} + \frac{pq}{E_p E_q}\right) \tag{4.31}$$

where $\mathcal{M}$ denotes the boson mass. E_p is given as

$$E_p = \sqrt{M^2 + p^2}. \tag{4.32}$$

Eq.(4.31) can be solved exactly as shown in [5]. First, we define A and B by

$$A = \int_{-\Lambda}^{\Lambda} dp\, f(p) \tag{4.33a}$$

$$B = \int_{-\Lambda}^{\Lambda} dp\, \frac{f(p)}{E_p}. \tag{4.33b}$$

Using A and B, we can solve Eq. (4.31) for $f(p)$ and obtain

$$f(p) = \frac{g/2\pi}{2E_p - \mathcal{M}} \left(A + \frac{M^2}{E_p} B \right). \tag{4.34}$$

Putting this $f(p)$ back into Eqs. (4.31), we obtain the matrix equations

$$A = \frac{g}{2\pi} \int_0^{\Lambda} \frac{2dp}{2E_p - \mathcal{M}} \left(A + \frac{M^2}{E_p} B \right) \tag{4.35a}$$

$$B = \frac{g}{2\pi} \int_0^{\Lambda} \frac{2dp}{(2E_p - \mathcal{M})E_p} \left(A + \frac{M^2}{E_p} B \right). \tag{4.35b}$$

Since the model is already regularized, we can easily calculate the boson spectrum which is given in Fig. 2 as the function of the coupling constant g/π. As can be seen from Fig. 2, there is no massless boson in this spectrum even though the boson mass for the very small coupling constant g is exponentially small.

5 Bethe Ansatz Solutions in Thirring Model

The structure of the vacuum and the spectrum of the current current interaction models of the NJL and the Thirring model have been discussed in terms of the Bogoliubov transformation method in section 4. It is clear that there should not be any fundamental differences between the NJL and the Thirring models as far as the vacuum structure is concerned once one accepts the reliability of the Bogoliubov transformation method. For the perturbative point of view, however, there is a serious difference between them since the former is an unrenormalizable field theory model in four dimensions while the latter is a renormalizable field theory model in two dimensions. Therefore, there is no guarantee that the NJL model can give any reliable predictions when one evaluates physical observables in the perturbative calculations. In this respect, it is always very important to have some exact solutions of the field theory model. In two dimensions, the Thirring model can be indeed solved exactly by the Bethe ansatz method. From the exact solutions, one can learn a lot about the vacuum
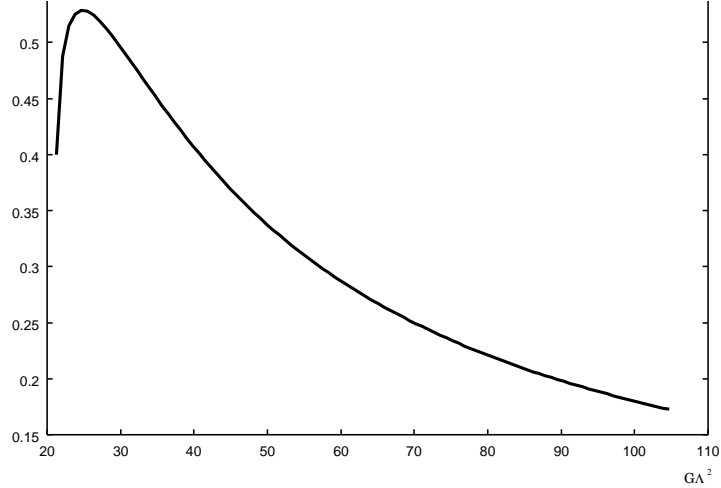
Figure 2: The boson mass for the massless Thirring model is plotted as the function of g/π. It is measured by the cutoff Λ.

structure and the spectrum of the excitation in the Thirring model. In particular, we can discuss the symmetry breaking of the vacuum state from this solution.

In two dimensions, however, the symmetry in the field theory is considered to be *not* broken spontaneously in the vacuum state. In fact, Coleman [8] presented the proof that the two dimensional field theory models cannot spontaneously break the symmetry even though the vacuum state may prefer the symmetry broken state. However, his proof of the nonexistence of the spontaneous symmetry breaking in two dimensions is essentially based on the Goldstone theorem. The Goldstone theorem [1, 2] states that the spontaneous symmetry breaking should accompany a massless boson when the vacuum prefers the broken symmetric state. However, the massless boson cannot exist in two dimensions since it cannot propagate due to the infra-red singularity of the propagator. Since this non-existence of the massless boson should hold rigorously, it naturally means that the spontaneous symmetry breaking should not occur in two dimensions as long as the Goldstone theorem is right. Coleman's theorem looks reasonable, and indeed until recently it has been believed to hold true for fermion field theory models as well.

However, as we see in the preceding sections, the Goldstone theorem does not hold in the fermion field theory models. Therefore, there is no massless boson in the fermion field theory after the spontaneous symmetry breaking. This suggests that Coleman's theorem has lost its basis in the proof of the theorem. Indeed, as we saw in section 4, the chiral symmetry of the massless Thirring model is spontaneously broken by the Bogoliubov vacuum state [4, 5]. There, the energy of the new vacuum is lower than that of the free vacuum state, and it breaks the chiral symmetry, but there appears no massless boson.

For this claim, however, people may insist that the Bogoliubov transformation does

not have to be exact, and therefore there might be some excuse for the symmetry breaking phenomena that occurred in the Thirring model.

In this section, we present a new discovery of the symmetry broken vacuum of the Bethe ansatz solution in the Thirring model, and show that the energy of the new vacuum state is indeed lower than that of the symmetric vacuum state even though the symmetric vacuum was considered to be the lowest state in the Thirring model. The new vacuum state breaks the chiral symmetry, and becomes a massive fermion field theory model.

Further, we evaluate the energy spectrum of the one particle-one hole states, and show that the excitation spectrum has indeed a finite gap. This gap energy turns out to be consistent with the effective fermion mass deduced from the momentum distribution of the negative energy particles in the new vacuum state. This confirms the consistency of the calculation of the Bethe ansatz solutions in the Thirring model.

After carrying out the numerical calculations, we get to know that the energies of the vacuum as well as the lowest one particle-one hole state can be expressed analytically. This is quite nice since we know clearly which of the vacuum state is the lowest. Also, in the thermodynamic limit, the lowest one particle-one hole state can be reduced to the effective fermion mass M_N which is described in terms of the cutoff Λ.

It turns out that there is no massive boson in the Bethe ansatz solutions, contrary to the prediction of the Bogoliubov transformation method [5, 16]. However, qualitative properties of the symmetry breaking phenomena between the Bethe ansatz calculations and the Bogoliubov method agree with each other.

Even though the Bethe ansatz calculations confirm that there is no massless boson in the massless Thirring model, some people may claim that the massless Thirring model can be bosonized and is reduced to a massless boson Hamiltonian. Here, we show that the well-known procedure of bosonization of the massless Thirring model is incomplete because the zero mode of the boson field cannot be defined and quantized. In other words, the zero mode of the field $\Phi(0)$ identically vanishes in the massless Thirring model. This is in contrast to the Schwinger model in which one finds the zero mode of the field $\Phi(0)$ by the gauge field A^1. Also, it is interesting to note that the massive Thirring model has the zero mode through the mass term, and this clearly indicates that the massless limit of the massive Thirring model is indeed a singular point with respect to the dynamics of the field theory.

Therefore, the massless Thirring model cannot be reduced to a free massless boson even though it has a similar mathematical structure to the massless boson. The spectrum of the massless Thirring model has a finite gap, and this is consistent with the fact that there should not be any physical massless boson in two dimensions. Even though the defect of the bosonization of the massless Thirring model is only one point of the boson field, that is, zero mode, it is interesting and surprising that $nature$ knows it in advance.

5.1   Thirring Model and Bethe Ansatz Solutions

The massless Thirring model is a 1+1 dimensional field theory with current current interactions [3]. Its Hamiltonian can be written as

$$H = \int dx \left\{ -i \left(\psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2 \right) + 2g\psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right\}. \tag{5.1}$$

The Hamiltonian eq.(5.1) can be diagonalized by the Bethe ansatz wave function for N particles [9, 13, 14]

$$| k_1, \cdots, k_N \rangle = \int dx_1 \cdots dx_{N_1} dy_1 \cdots dy_{N_2} \prod_{i=1}^{N_1} \exp(ik_i x_i) \prod_{j=1}^{N_2} \exp\left(ik_{N_1+j} y_j\right)$$

$$\times \prod_{i,j} [1 + \lambda\theta(x_i - y_j)] \prod_{i=1}^{N_1} \psi_1^\dagger(x_i) \prod_{j=1}^{N_2} \psi_2^\dagger(y_j) |0\rangle, \tag{5.2}$$

with $N_1 + N_2 = N$. $\theta(x)$ denotes the step function. k_i represents the momentum of the $i-$th particle. λ is determined to be [3]

$$\lambda = -\frac{g}{2} S_{ij} \tag{5.3}$$

where S_{ij} is defined as

$$S_{ij} = \frac{k_i E_j - k_j E_i}{k_i k_j - E_i E_j - \epsilon^2} \tag{5.4}$$

where ϵ denotes the infra-red regulator which is eventually set to zero. We note that all of the momenta and any of the physical observables do not depend on the regulator ϵ when we solve the PBC equations as we discuss below.

In this case, the eigenvalue equation becomes

$$H | k_1, \cdots, k_N \rangle = \sum_{i=1}^{N} E_i | k_1, \cdots, k_N \rangle. \tag{5.5}$$

From the periodic boundary condition (PBC), one obtains the following PBC equations,

$$k_i = \frac{2\pi n_i}{L} + \frac{2}{L} \sum_{j \neq i}^{N} \tan^{-1} \left(\frac{g}{2} S_{ij} \right) \tag{5.6}$$

where n_i's are integer, and runs as $n_i = 0, \pm 1, \pm 2, \cdots, N_0$ where

$$N_0 = \frac{1}{2}(N - 1).$$

5.2   Vacuum State

First, we want to make a vacuum. We write the PBC equations for the vacuum which is filled with negative energy particles [9, 15]

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L} \sum_{\substack{i \neq j \\ k_i \neq k_j}}^{N} \tan^{-1}\left(\frac{g}{2} \frac{k_i|k_j| - k_j|k_i|}{k_i k_j - |k_i||k_j| - \epsilon^2}\right). \tag{5.7}$$

Although, the expression of the phase shift function is somewhat different from that of ref. [18, 19], it produces the same values of the momentum solution of the vacuum state. However, we believe that the expression of eq.(5.4) with the infra-red regulator must be better since it is transparent and clear.

Here, we first fix the maximum momentum of the negative energy particles, and denote it by the cut off momentum Λ. Next, we take the specific value of N, and this leads to the determination of L

$$L = \frac{2\pi N_0}{\Lambda}. \tag{5.8}$$

If we solve eq.(5.7), then we can determine the vacuum state, and the vacuum energy E_v can be written as

$$E_v = - \sum_{i=1}^{N} |k_i|. \tag{5.9}$$

It should be noted that physical observables are obtained by taking the thermodynamic limit where we let $L \to \infty$ and $N \to \infty$, keeping Λ finite. If there is other scale like the mass, then one should take the Λ which is sufficiently large compared to the other scale. However, there is no other scale in the massless Thirring model or four dimensional QCD with massless fermions, and therefore all the physical observables are measured by the Λ. Here, we can take all the necessary steps, if required, since all the physical quantities are given analytically. In fact, as we see below, the excitation energy and the effective fermion mass are expressed in terms of the Λ in the thermodynamic limit.

5.3   Symmetric Vacuum State

The solution of eq.(5.7) has been known and is written as [18, 19]

$$k_1 = 0 \tag{5.10a}$$

for $n_1 = 0$,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L} \tan^{-1}\left(\frac{g}{2}\right) \tag{5.10b}$$

for $n_i = 1, 2, \cdots, N_0$,

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L} \tan^{-1}\left(\frac{g}{2}\right) \tag{5.10c}$$

for $n_i = -1, -2, \cdots, -N_0$. This gives a symmetric vacuum state, and was considered to be the lowest state.

The vacuum energy E_v^{sym} can be written as

$$E_v^{\mathrm{sym}} = -\Lambda \left\{ N_0 + 1 + \frac{2N_0}{\pi} \tan^{-1} \left(\frac{g}{2} \right) \right\}. \tag{5.11}$$

5.4   True Vacuum State

It is surprising that eq.(5.7) has a completely different solution from the above analytical solutions. By the numerical calculation of eq.(5.7), we first find the new vacuum state. After that, we get to know that the solutions can be analytically written like the symmetric case,

$$k_1 = \frac{2N_0}{L} \tan^{-1} \left(\frac{g}{2} \right) \tag{5.12a}$$

for $n_1 = 0$,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L} \tan^{-1} \left(\frac{g}{2} \right) \tag{5.12b}$$

for $n_i = 1, 2, \cdots, N_0$,

$$k_i = \frac{2\pi n_i}{L} - \frac{2(N_0 + 1)}{L} \tan^{-1} \left(\frac{g}{2} \right) \tag{5.12c}$$

for $n_i = -1, -2, \cdots, -N_0$. The new vacuum has no $k_i = 0$ solution, and breaks the left-right symmetry. Instead, all of the momenta of the negative energy particles become finite.

The energy E_v^{true} of the true vacuum state can be written as

$$E_v^{\mathrm{true}} = -\Lambda \left\{ N_0 + 1 + \frac{2(N_0 + 1)}{\pi} \tan^{-1} \left(\frac{g}{2} \right) \right\}. \tag{5.13}$$

From the distributions of the negative energy particles, one sees that this solution breaks the chiral symmetry. This situation can be easily seen from the analytical solutions since the absolute value of the momentum of the negative energy particles is higher than $\frac{\Lambda}{\pi} \tan^{-1} \left(\frac{g}{2} \right)$. Therefore, we can define the effective fermion mass M_N by

$$M_N = \frac{\Lambda}{\pi} \tan^{-1} \left(\frac{g}{2} \right). \tag{5.14}$$

In Table 1, we show the calculated results of the new vacuum as well as the symmetric vacuum energies as the function of the particle number N. Here, we present the case with the coupling constant of $g = \pi$.

Table 1: We show the calculated results of the vacuum energy of Bethe ansatz solutions at $g = \pi$ with the particle number $N = 401$ and $N = 1601$. $\mathcal{E}_{\mathrm{sym}}$ and $\mathcal{E}_{\mathrm{true}}$ denote the symmetric vacuum and the true vacuum energies, respectively. We also show the effective fermion mass $\mathcal{M}_N$ deduced from the vacuum momentum distributions. All the energies are measured in units of Λ, namely, $\mathcal{E} \equiv E/\Lambda$ and $\mathcal{M}_N \equiv M_N/\Lambda$.

N	$\mathcal{E}_{\mathrm{sym}}$	$\mathcal{E}_{\mathrm{true}}$	$\mathcal{M}_N$
401	-328.819	-329.458	0.320
1601	-1312.274	-1312.913	0.320

5.5 $1p - 1h$ **State**

Next, we evaluate one particle-one hole $(1p - 1h)$ states. There, we take out one negative energy particle (i_0-th particle) and put it into a positive energy state. In this case, the PBC equations become

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L} \tan^{-1}\left(\frac{g}{2} \frac{k_i|k_{i_0}| + k_{i_0}|k_i|}{k_i k_{i_0} + |k_i||k_{i_0}| + \epsilon^2} \right)$$

$$- \frac{2}{L} \sum_{\substack{j \neq i, i_0 \\ k_j \neq k_i, k_{i_0}}}^{N} \tan^{-1}\left(\frac{g}{2} \frac{k_i|k_j| - k_j|k_i|}{k_i k_j - |k_i||k_j| - \epsilon^2} \right) \tag{5.15a}$$

for $i \neq i_0$.

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2}{L} \sum_{\substack{j \neq i_0 \\ k_j \neq -k_{i_0}}}^{N} \tan^{-1}\left(\frac{g}{2} \frac{k_{i_0}|k_j| + k_j|k_{i_0}|}{k_{i_0} k_j + |k_{i_0}||k_j| + \epsilon^2} \right) \tag{5.15b}$$

for $i = i_0$. In this case, the energy of the one particle-one hole states $E^{1p1h}_{(i_0)}$ is given as,

$$E^{1p1h}_{(i_0)} = |k_{i_0}| - \sum_{\substack{i=1 \\ i \neq i_0}}^{N} |k_i|. \tag{5.16}$$

It turns out that the solutions of eqs.(5.15) can be found at the specific value of n_{i_0} and then from this n_{i_0} value on, we find continuous spectrum of the $1p - 1h$ states.

Here, we show the analytical solution of eqs.(5.15) for the lowest $1p - 1h$ state.

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2N_0}{L} \tan^{-1}\left(\frac{g}{2} \right) \tag{5.17a}$$

for n_{i_0},

$$k_i = \frac{2\pi n_i}{L} + \frac{2(N_0 + 1)}{L} \tan^{-1}\left(\frac{g}{2}\right) \qquad (5.17b)$$

for $n_i = 0, 1, 2, \cdots, N_0$

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L} \tan^{-1}\left(\frac{g}{2}\right) \qquad (5.17c)$$

for $n_i = -1, -2, \cdots, -N_0$. n_{i_0} is given by

$$n_{i_0} = \left[\frac{N_0}{\pi} \tan^{-1}\left(\frac{g}{2}\right)\right], \qquad (5.18)$$

where $[X]$ denotes the smallest integer value which is larger than X. In this case, we can express the lowest $1p - 1h$ state energy analytically

$$E_0^{1p-1h} = -\Lambda \left\{(N_0 + 1) - \frac{2n_{i_0}}{N_0} + \frac{2(N_0 + 1)}{\pi} \tan^{-1}\left(\frac{g}{\pi}\right)\right\}. \qquad (5.19)$$

Therefore, the lowest excitation energy ΔE_0^{1p-1h} with respect to the true vacuum state becomes

$$\Delta E_0^{1p-1h} \equiv E_0^{1p-1h} - E_v^{\text{true}} = \frac{2\Lambda}{N_0} n_{i_0}. \qquad (5.20)$$

If we take the thermodynamic limit, that is, $N \to \infty$ and $L \to \infty$, then eq.(5.18) can be reduced to

$$\Delta E_0^{1p-1h} = \frac{2\Lambda}{\pi} \tan^{-1}\left(\frac{g}{2}\right) = 2M_N. \qquad (5.21)$$

In Table 2, we show the lowest five states of the $1p-1h$ energy by the numerical calculation. From this, we can determine the gap energy.

Table 2: We show several lowest states of the calculated results of the 1p-1h states energy $\mathcal{E}$ of eqs.(4.12) at $g = \pi$ with $N = 1601$. The gap energy $\Delta\mathcal{E} \equiv \mathcal{E}^{(1p1h)} - \mathcal{E}_v$ is also shown. All the energies are measured in units of Λ.

	$\mathcal{E}$	$\Delta\mathcal{E}$
vacuum	-1312.913	
$1p - 1h$ (1)	-1312.273	0.640
$1p - 1h$ (2)	-1312.272	0.641
$1p - 1h$ (3)	-1312.271	0.642
$1p - 1h$ (4)	-1312.269	0.644
$1p - 1h$ (5)	-1312.268	0.645

From this gap energy, we can obtain the effective fermion mass which is one half of the lowest gap energy. This can be easily given as

$$M_N = 0.320 \, \Lambda. \qquad (5.22)$$

This is consistent with the effective fermion mass deduced from the negative energy distribution of the vacuum. This confirms the consistency of the present calculations.

5.6   Boson State

In this calculation, we do not find any boson state, contrary to the prediction of the Bogoliubov transformation method. Since the present calculation is exact, we believe that the Bogoliubov calculation overestimates the attraction between the particle hole states. The main difference between the Bethe solutions and the Bogoliubov vacuum arises from the dispersion relation of the negative energy particles. From the Bethe ansatz solutions, it is clear that one cannot make a simple free particle dispersion with the fermion mass term while the Bogoliubov method assumes the free fermion dispersion relation for the negative energy particles. This should generate slightly stronger attraction for the Bogoliubov vacuum state than for the Bethe ansatz solution.

However, as far as the symmetry breaking mechanism is concerned, the Bogoliubov transformation gives a sufficiently reliable description of the dynamics in the spontaneous symmetry breaking phenomena.

5.7   Bosonization of Massless Thirring Model

Here, we briefly review the bosonization procedure in two dimensional field theory models. In particular, we discuss the massless and massive Thirring models and show that the massless Thirring model cannot be bosonized properly due to the lack of the zero mode of the boson field.

It has been believed that the massless Thirring model can be bosonized [20] in the same way as above, and its Hamiltonian is written

$$H = \frac{1}{2} \sum_{p \neq 0} \left\{ \left(1 - \frac{g}{2\pi}\right) \Pi^\dagger(p)\Pi(p) + \left(1 + \frac{g}{2\pi}\right) p^2 \Phi^\dagger(p)\Phi(p) \right\}. \qquad (5,23)$$

This looks plausible, but one knows at the same time that the $p = 0$ part is not included. In fact, there is a serious problem in the definition of the boson field $\Phi(0)$ and $\Pi(0)$ at the zero momentum $p = 0$. From eqs.(4.1), it is clear that one cannot define the zero mode of the boson field. In the Schwinger model, one finds the $\Phi(0)$ due to the anomaly equation. However, the Thirring model has no anomaly, and therefore the $\Phi(0)$ identically vanishes. That is,

$$\Phi(0) = 0. \qquad (5.24)$$

There is no way to find the corresponding zero mode of the boson field in the massless Thirring model since the axial vector current is always conserved.

Therefore, the Hamiltonian of the massless Thirring model eq.(5.5) does not correspond to the massless boson. It is interesting to notice that the problem is closely related to the zero mode which exhibits the infra-red property of the Hamiltonian. This is just consistent with the non-existence of the massless boson due to the infra-red singularity of the propagator

in two dimensions [43]. Further, as discussed in the previous section, the Bethe ansatz solutions confirm the finite gap of the massless Thirring spectrum, and this rules out a possibility of any excuse of the massless boson in the massless Thirring model.

5.8 Physics of Zero Mode

What is the physics behind the Hamiltonian without the zero mode ? Here, we discuss the effect of the zero mode and the eigenvalues of the Hamiltonian in a simplified way. The Hamiltonian eq.(5.23) can be rewritten as

$$H = H_B - \frac{1}{2}\left(1 - \frac{g}{2\pi}\right)\Pi^\dagger(0)\Pi(0) \tag{5.25}$$

where the $\Pi(0)$ field is introduced by hand, and the existence of the $\Pi(0)$ and $\Phi(0)$ fields is assumed. Here, H_B denotes the free boson Hamiltonian and is written as

$$H_B = \frac{1}{2}\sum_p \left\{\left(1 - \frac{g}{2\pi}\right)\Pi^\dagger(p)\Pi(p) + \left(1 + \frac{g}{2\pi}\right)p^2\Phi^\dagger(p)\Phi(p)\right\}. \tag{5.26}$$

Now, we assume the following eigenstates for H_B and $\Pi^\dagger(0)\Pi(0)$ by

$$H_B|p\rangle = E_p|p\rangle \tag{5.27a}$$

$$\Pi^\dagger(0)\Pi(0)|\Lambda\rangle = \Lambda|\Lambda\rangle \tag{5.27b}$$

where $E_p = \frac{2\pi}{L}p$ with $p = 0, 1, 2, \cdots$, and Λ is related to the box length L by $\Lambda = \frac{c_0}{L}$ with c_0 constant.

Eq. (5.27a) is just the normal eigenvalue equation for the massless boson and its spectrum. On the other hand, eq.(5.27b) is somewhat artificial since the state $|\Lambda\rangle$ is introduced by hand. The zero mode state of the Hamiltonian H_B should couple with the state $|\Lambda\rangle$, and therefore new states can be made by the superposition of the two states

$$|v\rangle = c_1|\Lambda\rangle + c_2|0\rangle \tag{5.28}$$

where c_1 and c_2 are constants. Further, we assume for simplicity that the overlapping integral between the $|0\rangle$ and the $|\Lambda\rangle$ states is small and is given by ϵ

$$\langle 0|\Lambda\rangle = \epsilon. \tag{5.29}$$

In this case, the energy eigenvalues $\langle v|H|v\rangle$ of eq.(5.25) become at the order of $O(\epsilon)$

$$E_\Lambda = \langle\Lambda|H_B|\Lambda\rangle - \frac{1}{2}\left(1 - \frac{g}{2\pi}\right)\Lambda \tag{5.30a}$$

$$E_0 = -\frac{1}{2}\left(1 - \frac{g}{2\pi}\right)\langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle. \tag{5.30b}$$

If we assume that the magnitude of the $\langle\Lambda|H_B|\Lambda\rangle$ and $\langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle$ should be appreciably smaller than the Λ,

$$\langle\Lambda|H_B|\Lambda\rangle \ll \Lambda \tag{5.31a}$$

$$\langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle \ll \Lambda \tag{5, .31b}$$

then the spectrum of the Hamiltonian eq.(5.25) has a finite gap, and the continuum states of the massless excitations start right above the gap. This is just the same as the spectrum obtained from the Bethe ansatz solutions discussed in the previous section.

5.9 Bosonization of Massive Thirring Model

It is well known that the massive Thirring model is equivalent to the sine-Gordon field theory [21]. The proof of the equivalence is based on the observation that the arbitrary number of the correlation functions between the two models agree with each other if some constants and the fields of the two models are properly identified between them. This indicates that the massive Thirring model must be well bosonized.

This is now quite clear since the axial vector current conservation is violated by the mass term,

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma_5\psi \tag{5.32}$$

where j_μ^5 is defined as

$$j_\mu^5 = \bar{\psi}\gamma_5\gamma_\mu\psi. \tag{5.33}$$

It should be noted that the j_0^5 is equal to j_1 in two dimensions.

Therefore, one can always define the $\dot{Q}_5$ by

$$\dot{Q}_5 = 2im\int \bar{\psi}\gamma_5\psi dx. \tag{5.34}$$

Therefore, one obtains the field $\Phi(0)$ of the boson

$$\Phi(0) = \frac{2im\pi}{g\sqrt{L}}\int \bar{\psi}\gamma_5\psi dx. \tag{5.35}$$

where $\Phi(0)$ of the boson will be discussed in detail in the next section in connection to the bosonization of the Schwinger model.

5.10 Bosons of Massive Thirring Model

The massive Thirring model has some bound states which are composed out of fermions and antifermions. The number of the bound states has been debated since the semiclassical calculation of the sine-Gordon model predicted that there should be many bound states in the massive Thirring model [22]. On the other hand, the infinite momentum frame calculation and the Bogoliubov transformation method predict that there is only one bound state [33, 11]. This is quite reasonable since the interaction of the massive Thirring model is in

fact the $\delta-$function potential which can normally possess one bound state. In addition, the Bethe ansatz calculations of the massive Thirring model also confirm that there is only one bound state in the massive Thirring model [9, 15]. In particular, the Bethe ansatz equations are analytically solved in the strong coupling limit where the semiclassical method predicts many bound states [15]. The analytic solutions clearly show that there is only one bound state even in the strong coupling limit in this field theory model.

5.11   Summary of Thirring Model

In this section, we have presented a symmetry broken vacuum of the Bethe ansatz solutions in the Thirring model, and have shown that the true vacuum energy is indeed lower than the symmetric vacuum energy. This is quite surprising since the symmetry preserving state often gives the lowest energy state in quantum mechanics. However, in the field theory model, there is also the case in which the symmetry is spontaneously broken in the vacuum state, and this is indeed what is realized and observed in the Thirring model.

In this new vacuum state, the chiral symmetry is broken, and therefore the momentum distribution of the negative energy state becomes similar to a massive fermion theory. From the distribution of the vacuum momentum, we can deduce the effective fermion mass. However, we should note that the fermion should be massless in reality, and we cannot approximate the system by the massive fermion field theory for a boson mass evaluation. Even though some of the physical observables may be calculated by the approximate scheme with the massive fermion theory, one should keep the massless fermion scheme in general.

We have also calculated the one particle-one hole excitation spectrum, and found that the spectrum has a finite gap. From this gap energy, we can determine the effective fermion mass, and confirm that the effective fermion mass from the gap energy agrees with the one which is estimated from the vacuum momentum distribution.

Also, we have shown that the bosonization procedure of the massless Thirring model has a serious defect since there is no corresponding zero mode of the boson field and that the massless Thirring model therefore cannot be fully bosonized.

Since the massless Thirring model cannot be bosonized properly, there is no massless excitation spectrum in the model, and this is consistent with the Bethe ansatz solutions that the massless Thirring model has a finite gap and then the continuum spectrum starts right above the gap.

Also, we should stress that the bosonization of the massless Thirring model has a subtlety, and one must be very careful for treating it. If one makes a small approximation or a subtle mistake in calculating the spectrum of the Hamiltonian, then one would easily obtain unphysical massless excitations from the massless Thirring model. We believe that the same care must be taken for the $SU(N)$ Thirring model where some approximations like the $1/N$ expansion are made and the massless boson is predicted [23]. When we discuss the large N expansion, there are serious problems related to the $1/N$ approximation. The basic point is that they cannot take into account the subtlety of the dynamics. In particular, if one makes first the large N limit, then one loses some important interactions which contribute to the boson mass. As Gross and Neveu pointed out in their paper [24], the massless

boson does not exist if they were to calculate to the higher orders in $1/N$. The existence of massless boson will give rise to infrared infinities arising from virtual states. This means that the lowest order approximation in $1/N$ is meaningless, and to investigate the infrared stability of the theory one has to work to all orders in $1/N$. This infra-red problems become particularly important when treating the bound state like boson mass.

It is clear by now that the present results are in contradiction with Coleman's theorem [8]. Here, we have presented counter examples against Coleman's theorem, and the exact solutions in two dimensional field theory should correspond to "experimental facts". Therefore, one should figure out the mathematical reason why Coleman's theorem is violated in fermion field theory model. In addition to the massless Thirring model, QCD with massless fermions in two dimensions spontaneously breaks the chiral symmetry with the axial vector current conservation as we will see later. Therefore, the massless QCD$_2$ is also in contradiction with Coleman's theorem, but there is no massless boson [4], and in this sense, it does not violate the theorem that there should not exist any massless boson in two dimensions. In reality, there is no example of fermion field theory models in which the symmetry of the vacuum state is *not* broken due to Coleman's theorem. However, the basic and mathematical problem with Coleman's theorem is still unsolved here. But we believe that the basic problem of the symmetry breaking business in two dimensions must come from the Goldstone theorem itself for the fermion field theory as discussed in sections 2 and 4.

6 Schwinger Boson in Two Dimensional QED

The best known model of the bosonization is the Schwinger model [25] which is the two dimensional QED with massless fermions. In the Schwinger model, the coupling constant has a mass dimension, and, due to this super-renormalizability, one can treat the model quite easily in many respects. There is no infinity at the large momentum, and therefore one does not need any cutoff momentum. In the Schwinger model, therefore, all the physical observables are described in terms of the coupling constant g.

6.1 Schwinger Model

The Schwinger model is the two dimensional QED with massless fermions. This is exactly solved, and this exact solution in field theory means that the Schwinger model can be rewritten into a new free field theory model. Indeed, the Schwinger model can be bosonized and becomes a free massive boson field theory model. This is quite interesting, but we believe that the Schwinger model is very special in that the interaction between fermions and antifermions is attractive, but it is a confining potential. Therefore, there exists no free fermion state, and that should be a strong reason why the Schwinger model becomes identical to the free bosonic fields.

The Lagrangian density for the Schwinger model can be written as

$$\mathcal{L} = \bar{\psi} i \gamma_\mu D^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{6.1}$$

where

$$D_\mu = \partial_\mu + igA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The equation of motion for the gauge field is

$$\partial_\mu F^{\mu\nu} = gj^\nu \tag{6.2}$$

where the fermion current j_ν is given as

$$j_\nu =: \bar{\psi}\gamma_\nu\psi : \tag{6.3}$$

where $:\ :$ denotes a normal ordering. The Dirac equation becomes

$$\gamma^\mu(-i\partial_\mu + gA_\mu)\psi = 0. \tag{6.4}$$

Now, we quantize the fermion field in a box with the length L

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{i\frac{2\pi}{L}nx}. \tag{6.5}$$

Here, we take the Coulomb gauge fixing

$$\partial_1 A^1 = 0.$$

In this case, the Hamiltonian of the Schwinger model can be written as

$$H = \frac{L}{2}\dot{A}^{1^2} + \sum_n \left(\frac{2\pi}{L}n + gA^1\right) a_n^\dagger a_n + \sum_n \left(-\frac{2\pi}{L}n - gA^1\right) b_n^\dagger b_n + \frac{g^2 L}{8\pi^2} \sum_{p\neq 0} \frac{1}{p^2} \tilde{j}_0(p)\tilde{j}_0(-p) \tag{6.6}$$

where $\tilde{j}_0(p)$ denotes the momentum representation of the fermion current $j_0(x)$.

    The Schwinger model is solved by several methods. The bosonization is one of them and we will discuss it below. Also, the Schwinger model has been solved by the Bogoliubov transformation method. In principle, the Bogoliubov transformation method is an approximate scheme for the four fermion interaction models. However, the correct mass of the Schwinger boson is obtained by the Bogoliubov transformation method. Until now, it is not clarified why the Bogoliubov transformation method can give an exact mass for the Schwinger model. Further, the Bogoliubov transformation method reproduces the right condensate value of the Schwinger model which is obtained analytically. This suggest that the Bogoliubov vacuum state may well be a good vacuum state since the condensate value should exhibit some information of the vacuum structure.

6.2 Bosonization of Schwinger Model

In the Schwinger model, the Coulomb gauge is taken, and in this case, the space part of the vector potential A^1 depends on time and corresponds to the zero mode of the boson field

[26]. Since the fermion current j_μ is defined in eq.(6.3), the momentum representation $\tilde{j}_\mu$ of the current can be written in terms of $\rho_a(p)$ and $\rho_b(p)$

$$\tilde{j}_0(p) = \rho_a(p) + \rho_b(p) \tag{6.7a}$$

$$\tilde{j}_1(p) = \rho_a(p) - \rho_b(p) \tag{6.7b}$$

where $\rho_a(p)$ and $\rho_b(p)$ are defined as

$$\rho_a(p) = \sum_k a^\dagger_{k+p} a_k \tag{6.8a}$$

$$\rho_b(p) = \sum_k b^\dagger_{k+p} b_k. \tag{6.8b}$$

Now, we can easily prove that $\rho_a(p)$ and $\rho_b(p)$ satisfy the following commutation relations,

$$[\rho_a(p), \rho_a(q)]|\text{phys}\rangle = -p\delta_{p,-q}|\text{phys}\rangle \tag{6.9a}$$

$$[\rho_b(p), \rho_b(q)]|\text{phys}\rangle = p\delta_{p,-q}|\text{phys}\rangle. \tag{6.9b}$$

These commutation relations can only be valid when these equations are always supposed to operate on the physical states $|\text{phys}\rangle$. Here, the physical states mean that the negative energy states must be completely occupied if the negative energy levels are sufficiently deep. Further, in this physical state, there should be no particles in the positive energy states if the particle energy is sufficiently high. Under these conditions, eqs.(6.9) hold true as operator equations.

In this case, $\tilde{j}_0(p)$ and $\tilde{j}_1(p)$ are related to the boson field and its conjugate field as

$$\tilde{j}_0(p) = ip\sqrt{\frac{L}{\pi}}\Phi(p) \qquad \text{for} \quad p \neq 0 \tag{6.10a}$$

$$\tilde{j}_1(p) = \sqrt{\frac{L}{\pi}}\Pi(p) \qquad \text{for} \quad p \neq 0 \tag{6.10b}$$

where $\Phi(p)$ and $\Pi(p)$ denote the boson field and its conjugate field, respectively. L denotes the box length.

It is very important to note that $\Pi(0)$ and $\Phi(0)$ are not defined in eqs.(6.10). In the Schwinger model, they are related to the chiral charge and its time derivative as

$$\Pi(0) = \frac{\pi}{g\sqrt{L}}Q_5 \tag{6.11a}$$

$$\Phi(0) = \frac{\pi}{g\sqrt{L}}\dot{Q}_5 \tag{6.11b}$$

where $\dot{Q}_5$ is described by the vector field A^1 due to the anomaly equation

$$\dot{Q}_5 = \frac{Lg}{\pi}\dot{A}^1. \tag{6.12}$$

Here, we briefly discuss how one obtains the chiral anomaly when one regularizes the charge and the energy of the vacuum. The procedure to obtain eqs.(6.12) is shown in [46] in a clear way.

First, we define the charges of the right and the left movers by

$$Q_L = \sum_{n=-\infty}^{N_L} e^{\lambda(n+\frac{LgA^1}{2\pi})} \tag{6.13a}$$

$$Q_R = \sum_{n=N_R}^{\infty} e^{\lambda(-n-\frac{LgA^1}{2\pi})} \tag{6.13b}$$

where the charges are regularized in terms of the ζ function regularization. Here, it is important to note that one should regularize the charge with the gauge invariant way since the Hamiltonian has still the invariance of a large gauge transformation $n \to n + \frac{LgA^1}{2\pi}$. In this case, the charge and the chiral charge of the vacuum state is defined as

$$Q = Q_R + Q_L \tag{6.14a}$$

$$Q_5 = Q_R - Q_L. \tag{6.14b}$$

The regularized charge and chiral charge become

$$Q = \frac{2}{\lambda} + N_L + 1 - N_R + O(\lambda) \tag{6.15a}$$

$$Q_5 = 2N_L + 1 + \frac{LgA^1}{\pi}. \tag{6.15b}$$

Since the charge of the vacuum must be zero, we set $Q = 0$ where we should neglect the $\frac{2}{\lambda}$ term. From eq.(6.15b), we obtain the anomaly equation of eq.(6.12) by making the time derivative of Q_5. Namely, the chiral current is not conserved any more due to the anomaly.

Further, we should regularize the vacuum energy in the same way as the charge. Denoting the left and right movers of the vacuum energy by E_L^{vac} and E_R^{vac}, we can calculate them as

$$E_L^{vac} = \frac{2\pi}{L} \sum_{n=-\infty}^{N_L} \left(n + \frac{LgA^1}{2\pi}\right) e^{\lambda(n+\frac{LgA^1}{2\pi})} \tag{6.16a}$$

$$E_R^{vac} = \frac{2\pi}{L} \sum_{n=N_R}^{\infty} \left(-n - \frac{LgA^1}{2\pi}\right) e^{\lambda(-n-\frac{LgA^1}{2\pi})}. \tag{6.16b}$$

Making use of eqs.(6.13), we obtain

$$E_L^{vac} = \frac{\pi}{L} Q_L^2 \tag{6.17a}$$

$$E_R^{vac} = \frac{\pi}{L} Q_R^2. \tag{6.17b}$$

Therefore, the total vacuum energy can be written as

$$E^{vac} = \frac{\pi}{2L} Q_5^2.$$

(6.18)

Thus, the vacuum energy part of the Hamiltonian eq.(6.6) can be written as

$$H^{vac} = \frac{\pi^2}{2g^2 L} \dot{Q}_5^2 + \frac{\pi}{2L} Q_5^2.$$

(6.19)

If we identify the boson field $\Phi(0)$, $\Pi(0)$ as

$$\Phi(0) = \frac{\pi}{g\sqrt{L}} Q_5$$

(6.20a)

$$\Pi(0) = \frac{\pi}{g\sqrt{L}} \dot{Q}_5$$

(6.20b)

then we can write the vacuum part of the Hamiltonian

$$H^{vac} = \frac{1}{2} \Pi^\dagger(0)\Pi(0) + \frac{g^2}{2\pi} \Phi^\dagger(0)\Phi(0).$$

(6.21)

Further, the positive energy part of the kinetic energy Hamiltonian can be rewritten in terms of the kinetic energy of the boson Hamiltonian.

$$\sum_p \frac{2\pi p}{L} a_p^\dagger a_p - \sum_p \frac{2\pi p}{L} b_p^\dagger b_p = \frac{1}{2} \sum_{p\neq 0} \left\{ \Pi^\dagger(p)\Pi(p) + \left(\frac{2\pi p}{L}\right)^2 \Phi^\dagger(p)\Phi(p) \right\}.$$

(6.22)

Here, we should note that the identification of the kinetic energies between the fermion and boson fields can be considered as operator equations with the condition that all the operations should be done onto physical states in fermion Fock space which are explained above. In this case, eq.(6.22) holds true as operator equations under these conditions.

Together with the Coulomb interaction part, we can write down the Hamiltonian for the Schwinger model as

$$H = \sum_p \left\{ \frac{1}{2} \Pi^\dagger(p)\Pi(p) + \frac{1}{2}\left(\frac{2\pi p}{L}\right)^2 \Phi^\dagger(p)\Phi(p) + \frac{g^2}{2\pi} \Phi^\dagger(p)\Phi(p) \right\}.$$

(6.23)

This is just the free massive boson Hamiltonian.

It should be important to note that the Schwinger model has the right zero mode in the Hamiltonian of the boson field. However, as we saw in section 5, there is no corresponding zero mode in the massless Thirring model, and this leads to the finite gap of the spectrum in the massless Thirring model, which is indeed consistent with the fact that there should exist no physical massless boson in two dimensions.

6.3   QED$_2$ with Massive Fermions

It should be interesting to make some comments on the boson spectrum in QED$_2$ with massive fermions. The Lagrangian density is just the same as eq.(6.1) with the fermion mass term

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu D^\mu - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{6.24}$$

At the massless limit, there is only one boson which is the Schwinger boson. For the finite but small fermion mass m_0 region, the lowest boson mass can be written as [50]

$$\mathcal{M} \simeq \frac{g}{\sqrt{\pi}} + e^\gamma m_0 \tag{6.25}$$

where γ denotes Euler's constant.

As the mass increases, the number of the bosonic bound states increases [32]. When the mass is much larger than the coupling constant $\frac{g}{\sqrt{\pi}}$, then the system becomes just the same as the nonrelativistic quantum mechanics. The particle and antiparticle are interacting with each other through the linear rising potential. In this case, the vacuum state is just like the perturbative vacuum, and there is no fermion condensate.
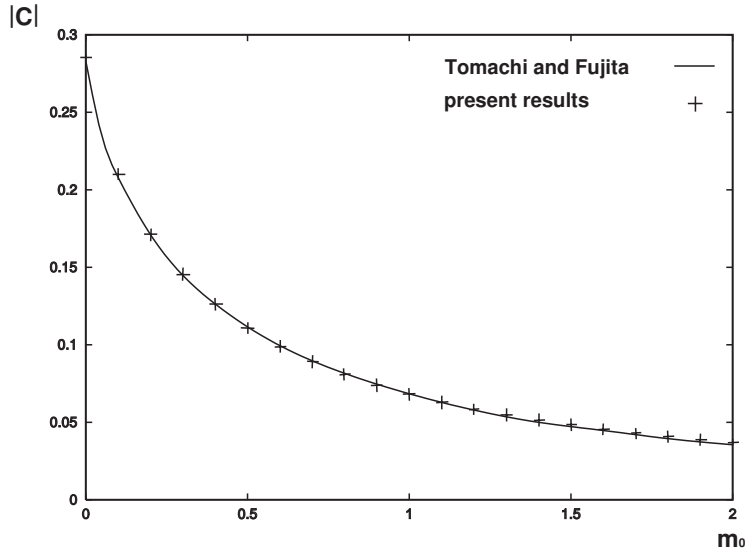

Figure 3: The absolute values of the condensate for massive QED$_2$ are plotted as the function of the fermion mass m_0 in the small mass regions. The solid line is calculated in [32] while the crosses are evaluated in the present paper with the massless fermion basis.

In this respect, the interesting region must be the one in which the fermion mass is much smaller then the coupling constant $\frac{g}{\sqrt{\pi}}$, but it is still finite. If the fermion mass is finite, then there is no chiral symmetry in the Lagrangian density. This means that there cannot be any

symmetry breaking phenomena in the massive fermion QED$_2$. However, if one evaluates the chiral condensates, then one finds a finite condensate value for the massive fermion case. In other words, the fermion condensate value is a smooth function of the fermion mass m_0 [32] as we show in Fig. 3.

The chiral condensate value for the massive fermion QED$_2$ can be written

$$\langle\Omega|\frac{1}{L}\int\bar{\psi}\psi dx|\Omega\rangle \simeq \frac{g}{\sqrt{\pi}}\frac{e^\gamma}{2\pi} + O(m_0) \qquad (6.26)$$

This strongly suggests that the chiral condensate value is not the consequence of the symmetry breaking, but it indicates the vacuum structure how many of the virtual particle pairs can be found in the vacuum. Or in other words, the chiral condensate is related to the change of the momentum distributions of the negative energy particles in comparison with the symmetric distributions of the free particles in the vacuum state.

7   Bosons in Two Dimensional QCD

QCD in two dimensions presents a good example of the strong interaction models since it can be solved to a good accuracy. The basic structure of QCD$_2$ is similar to QED$_2$ in that both of the models are super-renormalizable. In addition, there is no transverse field, and therefore it becomes identical to the four fermion interaction model in the field theory where the interaction is described in terms of the fermion fields only.

The main difference between QCD$_2$ and QED$_2$ is of course due to the color degrees of freedom. Concerning the symmetry breaking phenomena, there is no anomaly in QCD$_2$ since there is no anomaly term which has a color singlet state in two dimensions. Therefore, the axial vector current is conserved after the chiral symmetry is broken in contrast to QED$_2$. In this respect, it should be quite interesting to study whether the vacuum state of QCD$_2$ breaks the chiral symmetry or not. Here, we show that the chiral symmetry is spontaneously broken in QCD$_2$, and the chiral condensate value is finite. However, there is no massless boson and the boson spectrum is just similar to that of QED$_2$.

The boson mass spectrum in QCD$_2$ has been extensively studied by the light cone method [34, 35, 36]. In particular, QCD$_2$ with the $1/N_c$ expansion proposed by 't Hooft has presented interesting results on the boson mass spectrum [37, 38, 39, 40, 41]. The boson mass vanishes when the fermion mass becomes zero. However, this is not allowed since the massless boson cannot *physically* exist in two dimensional field theory [8, 43]. Unfortunately, this problem of the puzzle has never been seriously considered until now, apart from unrealistic physical pictures. People believe that the large N_c limit is special because one takes N_c infinity. But the infinity in physics means simply that the N_c must be sufficiently large, and, in fact, as we show below, physical observables at $N_c = 50$ are just the same as those of $N_c = \infty$.

Further, this boson spectrum of large N_c QCD$_2$ was confirmed by the light cone calculations with $SU(2)$ and $SU(3)$ colors [35]. Indeed, the mass of the boson in the light cone calculations is consistent with the 't Hooft spectrum of the boson even though the latter

is evaluated by the $1/N_c$ approximation. However, the fact that the light cone calculation predicts massless bosons is rather serious since the light cone calculation for $SU(2)$ does not seem to make any unrealistic approximations, apart from the trivial vacuum.

However, there is an interesting indication that the light cone vacuum is not trivial, and indeed there is a finite condensate even for the large N_c QCD$_2$ [28, 29, 30, 44]. What does this mean ? This suggests that one has to consider the effect of the complicated vacuum structure for the boson mass as long as one calculates the boson mass with Fock space expansions. On the other hand, the calculation for the boson spectrum by 't Hooft is based on the trivial vacuum, but, instead he could sum up all of the intermediate fluctuations of the fermion and antifermion pairs. This should be equivalent to considering the true vacuum structure in the Fock space basis. That is, the same spectrum of bosons must be obtained both by the Fock space expansion with the true vacuum and by the sum of all the Feynman diagrams with the trivial vacuum if they are treated properly.

For this argument, people may claim that QED$_2$ is exactly described by the naive light cone calculation with the trivial vacuum, and therefore, QCD$_2$ may well be treated just in the same way as the QED$_2$ case. However, one may well have some uneasy feeling for the fact that the naive light cone calculation cannot reproduce the condensate value of QED$_2$.

In this section, we show that the light cone calculation based on the Fock space expansion with the trivial vacuum is not valid for QCD$_2$. One has to consider properly the effect of the complicated vacuum structure. Here, we present the calculation with the Bogoliubov vacuum in the rest frame, and show that the present calculation reproduces the right condensate values. Indeed, we can compare the present results with the condensate value as predicted by the $1/N_c$ expansion[28, 29, 30, 44],

$$C_{N_c} = -\frac{N_c}{\sqrt{12}}\sqrt{\frac{N_c g^2}{2\pi}}. \tag{7.1}$$

The present calculation of the condensate value for the $SU(2)$ color is $C_2 = -0.495 \frac{g}{\sqrt{\pi}}$ which should be compared with the $-0.577 \frac{g}{\sqrt{\pi}}$ from the $1/N_c$ expansion, and $C_3 = -0.995 \frac{g}{\sqrt{\pi}}$ for the $SU(3)$ color compared with $-1.06 \frac{g}{\sqrt{\pi}}$ of the $1/N_c$ expansion. For the larger value of N_c (up to $N_c = 50$), we obtain the condensate values which perfectly agree with the prediction of C_{N_c} in eq.(7.1).

Further, we show that the boson masses for QCD$_2$ with $SU(2)$ and $SU(3)$ colors are finite even though the fermion mass is set to zero. In fact, the boson mass is found to be $\mathcal{M}_2 = 0.467 \frac{g}{\sqrt{\pi}}$ for the $SU(2)$, and $\mathcal{M}_3 = 0.625 \frac{g}{\sqrt{\pi}}$ for the $SU(3)$ color for the massless fermions. Further, the present calculations of the boson mass up to $N_c = 50$ suggest that the boson mass $\mathcal{M}_{N_c}$ for $SU(N_c)$ can be described for the large N_c by the following phenomenological expression at the massless fermion limit,

$$\mathcal{M}_{N_c} = \frac{2}{3}\sqrt{\frac{N_c g^2}{3\pi}}. \tag{7.2}$$

Also, we calculate the boson mass at the large N_c with the finite fermion mass. From the present calculations, we can express the boson mass in terms of the phenomenological

formula for the small fermion mass m_0 region,

$$\mathcal{M}_{N_c} \approx \left(\frac{2}{3} \sqrt{\frac{2}{3}} + \frac{10}{3} \frac{m_0}{\sqrt{N_c}} \right) \sqrt{\frac{N_c g^2}{2\pi}} \tag{7.3}$$

where m_0 is measured in units of $\frac{g}{\sqrt{\pi}}$.

The above expression (eq.(7.3)) can be compared with the calculation by Li et al. [38] who employed the $1/N_c$ expansion of 't Hooft model in the rest frame [39]. It turns out that their calculated boson mass for their smallest fermion mass case is consistent with the above equation, though their calculated values are slightly smaller than the present results.

In addition, we examine the validity of the light cone calculation for QED$_2$. It is shown that the boson mass for the QED$_2$ case happens to be not very sensitive to the condensate value, and that the spectrum can be reproduced by the light cone calculations with the trivial vacuum as well as with the condensate value only with positive momenta. Therefore, we believe that the QED$_2$ case is accidentally reproduced by the light cone calculation with the trivial vacuum state even though we do not fully understand why this accidental agreement occurs. On the other hand, the QCD$_2$ case is quite different. The boson mass calculated with the trivial vacuum is zero at the massless fermion limit. Further, the calculation in the light cone with the condensate value only with the positive momenta are not stable against the infra-red singularity of the light cone equations.

The present calculations are based on the Fock space expansion, and, in this calculation, we only consider the fermion and anti-fermion (two fermion) space. For QED$_2$, it is shown that the fermion and anti-fermion space reproduces the right Schwinger boson [32]. That is, the four fermion spaces do not alter the lowest boson energy in QED$_2$. However, there is no guarantee that there are finite effects on the lowest boson mass from the four fermion spaces in QCD$_2$. This point is not examined in this paper, and should be worked out in future.

Here, we examine the RPA calculations and show that the boson mass for QED$_2$ with the RPA equations deviates from the Schwinger boson. That means that the agreement achieved by the Fock space expansion is destroyed by the RPA calculation. Further, we calculate the boson mass for QCD$_2$ with $SU(2)$ and the large N_c limit. It turns out that the boson mass vanishes when the fermion mass is equal to the critical value and that it becomes imaginary when the fermion mass is smaller than the critical value. This is obviously unphysical at the massless fermion limit, and is closely related to the fact that the RPA equations are not Hermitian, and therefore we should examine its physical meaning in future.

From the present calculation, we learn that the chiral symmetry in massless QCD$_2$ is spontaneously broken without the anomaly term, in contrast to the Schwinger model. But the boson mass is finite, and therefore there is no Goldstone boson in this field theory model. Thus, the present result confirms that the Goldstone theorem [1, 2] does not hold for the fermion field theory as discussed in section 2. This indicates that the anomaly term has little to do with the chiral symmetry breaking. This is reasonable since the anomaly term arises from the conflict between the gauge invariance and the chiral current conservation when regularizing the vacuum, and this is essentially a kinematical effect. On the other

hand, the symmetry breaking is closely related to the vacuum energy which of the vacuum states should have the lowest energy, and therefore it is the consequence of the dynamical effects in the vacuum.

7.1   Bogoliubov Transformation in QCD$_2$

In this section, we discuss the Bogoliubov transformation in QCD$_2$. The Lagrangian density for QCD$_2$ with $SU(N_c)$ color is described as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m_0)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}, \tag{7.4}$$

where $F_{\mu\nu}$ is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \quad A_\mu = A_\mu^a T^a, \quad T^a = \frac{\tau^a}{2}.$$

Here, m_0 denotes the fermion mass, and at the massless limit, the Lagrangian density has a chiral symmetry.

Now, we first fix the gauge by

$$A_1^a = 0.$$

This gauge fixing has been employed by most of the calculations which have been done up to now. This gauge is simple but cannot describe the zero mode even though the spectrum is properly described by this gauge. In this gauge, the equation of motion for A_0^a becomes

$$\partial_1^2 A_0^a = -g j_0^a, \tag{7.5}$$

where j_0^a is the fermion current defined by

$$j_0^a = \psi^\dagger \frac{\tau^a}{2}\psi.$$

The Hamiltonian can be written as

$$H = \int dx \left[-i\bar{\psi}\gamma^1 \partial_1 \psi + m_0 \bar{\psi}\psi + \frac{g}{2} j_0^a A_0^a \right]. \tag{7.6a}$$

Now, we quantize the Hamiltonian of QCD$_2$ with $SU(N_c)$ color, and it can be written as

$$H = \sum_{n,\alpha} p_n \left(a_{n,\alpha}^\dagger a_{n,\alpha} - b_{n,\alpha}^\dagger b_{n,\alpha} \right) + m_0 \sum_{n,\alpha} \left(a_{n,\alpha}^\dagger b_{n,\alpha} + b_{n,\alpha}^\dagger a_{n,\alpha} \right)$$

$$- \frac{g^2}{4N_c L} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left(\tilde{j}_{1,n,\alpha\alpha} + \tilde{j}_{2,n,\alpha\alpha} \right) \left(\tilde{j}_{1,-n,\beta\beta} + \tilde{j}_{2,-n,\beta\beta} \right)$$

$$+ \frac{g^2}{4L} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left(\tilde{j}_{1,n,\alpha\beta} + \tilde{j}_{2,n,\alpha\beta} \right) \left(\tilde{j}_{1,-n,\beta\alpha} + \tilde{j}_{2,-n,\beta\alpha} \right), \tag{7.6b}$$

where

$$\tilde{j}_{1,n,\alpha\beta} = \sum_m a^\dagger_{m,\alpha} a_{m+n,\beta} \tag{7.7a}$$

$$\tilde{j}_{2,n,\alpha\beta} = \sum_m b^\dagger_{m,\alpha} b_{m+n,\beta}. \tag{7.7b}$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$a_{n,\alpha} = \cos\theta_{n,\alpha} c_{n,\alpha} + \sin\theta_{n,\alpha} d^\dagger_{-n,\alpha} \tag{7.8a}$$

$$b_{n,\alpha} = -\sin\theta_{n,\alpha} c_{n,\alpha} + \cos\theta_{n,\alpha} d^\dagger_{-n,\alpha} \tag{7.8b}$$

where $\theta_{n,\alpha}$ denotes the Bogoliubov angle.

In this case, the Hamiltonian of QCD_2 can be written as

$$H = \sum_{n,\alpha} E_{n,\alpha}(c^\dagger_{n,\alpha} c_{n,\alpha} + d^\dagger_{-n,\alpha} d_{-n,\alpha}) + H' \tag{7.9}$$

where

$$E^2_{n,\alpha} = \left\{ p_n + \frac{g^2}{4N_cL} \sum_{m,\beta} \frac{(N_c \cos 2\theta_{m,\beta} - \cos 2\theta_{m,\alpha})}{(p_m - p_n)^2} \right\}^2$$
$$+ \left\{ m_0 + \frac{g^2}{4N_cL} \sum_{m,\beta} \frac{(N_c \sin 2\theta_{m,\beta} - \sin 2\theta_{m,\alpha})}{(p_m - p_n)^2} \right\}^2. \tag{7.10}$$

H' denotes the interaction Hamiltonian in terms of the new operators but is quite complicated, and therefore it is not given here.

The energy of the single particle state [eq.(7.10)] does not have a proper energy and momentum dispersion relation. This means that the quasi-particle states cannot be a physical state. However, this is quite reasonable since fermions in QCD_2 are confined and they can never be observed. Physical observables are bosonic states which are to be evaluated below.

The conditions that the vacuum energy is minimized give the constraint equations which can determine the Bogoliubov angles

$$\tan 2\theta_{n,\alpha} = \frac{m_0 + \frac{g^2}{4N_cL} \sum_{m,\beta} \frac{(N_c \sin 2\theta_{m,\beta} - \sin 2\theta_{m,\alpha})}{(p_m - p_n)^2}}{p_n + \frac{g^2}{4N_cL} \sum_{m,\beta} \frac{(N_c \cos 2\theta_{m,\beta} - \cos 2\theta_{m,\alpha})}{(p_m - p_n)^2}}. \tag{7.11}$$

The conditions of eq.(7.11) can be also derived from the requirement that the $(c_{n,\alpha} d_{n,\alpha} + h.c.)$ terms should vanish.

In this case, the condensate value C_{N_c} is written as

$$C_{N_c} = \frac{1}{L} \sum_{n,\alpha} \sin 2\theta_{n,\alpha}. \tag{7.12}$$

Now, we can calculate the boson mass for the $SU(N_c)$ color. First, we define the wave function for the color singlet boson as

$$|\Psi_K\rangle = \frac{1}{\sqrt{N_c}} \sum_{n,\alpha} f_n c_{n,\alpha}^\dagger d_{K-n,\alpha}^\dagger |0\rangle. \tag{7.13}$$

In this case, the boson mass can be described as

$$
\begin{aligned}
\mathcal{M} &= \langle \Psi_K | H | \Psi_K \rangle \\
&= \frac{1}{N_c} \sum_{n,\alpha} (E_{n,\alpha} + E_{n-K,\alpha}) |f_n|^2 \\
&\quad + \frac{g^2}{2N_c^2 L} \sum_{l,m,\alpha} \frac{f_l f_m}{(p_l - p_m)^2} \cos(\theta_{l,\alpha} - \theta_{m,\alpha}) \cos(\theta_{l-K,\alpha} - \theta_{m-K,\alpha}) \\
&\quad - \frac{g^2}{2N_c L} \sum_{l,m,\alpha,\beta} \frac{f_l f_m}{(p_l - p_m)^2} \cos(\theta_{l,\alpha} - \theta_{m,\beta}) \cos(\theta_{l-K,\alpha} - \theta_{m-K,\beta}) \\
&\quad + \frac{g^2}{2N_c^2 L} \sum_{l,m,\alpha,\beta} \frac{f_l f_m}{K^2} \sin(\theta_{l-K,\alpha} - \theta_{l,\alpha}) \sin(\theta_{m,\beta} - \theta_{m-K,\beta}) \\
&\quad - \frac{g^2}{2N_c L} \sum_{l,m,\alpha} \frac{f_l f_m}{K^2} \sin(\theta_{l-K,\alpha} - \theta_{l,\alpha}) \sin(\theta_{m,\alpha} - \theta_{m-K,\alpha}). \tag{7.14}
\end{aligned}
$$

This equation can be easily diagonalized together with the Bogoliubov angles, and we obtain the boson mass. Here, we note that the treatment of the last two terms should be carefully estimated since the apparent divergence at $K = 0$ is well defined and finite.

7.2   Condensate and Boson Mass in $SU(2)$ and $SU(3)$

Here, we present the calculated results of the condensate values and the boson mass in QCD$_2$ with the $SU(2)$ and $SU(3)$ colors. Table 3 shows the condensate and the boson mass for the two different vacuum states, one with the trivial vacuum and the other with the Bogoliubov vacuum. As can be seen, the condensate values for the $SU(2)$ and $SU(3)$ are already close to the predictions by the $1/N_c$ expansion of eq.(7.1) [28, 29, 44]. The boson masses for the $SU(2)$ and $SU(3)$ are, for the first time, obtained as the finite value. Unfortunately, we cannot compare our results with any other predictions. But compared with the Schwinger boson, the boson masses for the $SU(2)$ and $SU(3)$ are in the same order of magnitude.

In Fig. 4, we present the fermion mass dependence of the condensate values for the $SU(2)$ and $SU(3)$ cases. As can be seen, the condensate becomes a finite value at the massless limit. It decreases as the function of the fermion mass m_0. This tendency is just the same as the condensate of QED$_2$ [32, 52].

Also, in Fig. 5, we show the calculated results of the boson mass as the function of the m_0 for $SU(2)$ and $SU(3)$. At the massless limit, the boson mass becomes a finite value, and the m_0 dependence is linear. This is exactly the same as the QED_2 case [32, 45].
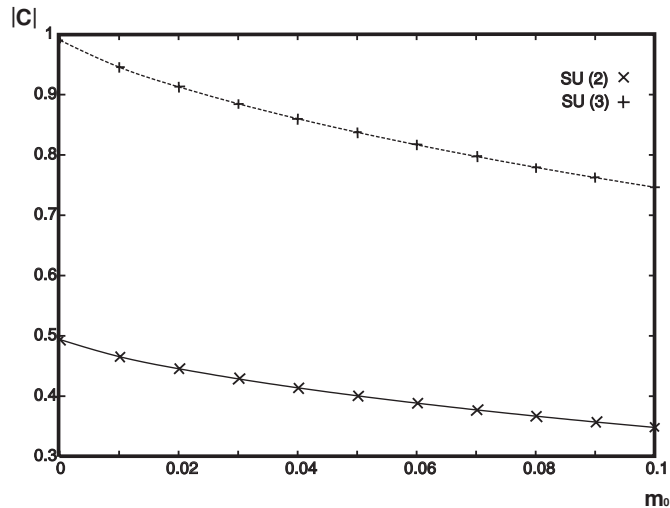

Figure 4: The absolute values of the condensate for $SU(2)$ and $SU(3)$ colors are plotted as the function of the fermion mass m_0 in the very small mass regions. The solid and dashed lines are shown to guide the eyes.

The present calculations show that both of the values (condensate and boson mass) are a smooth function of the fermion mass m_0. This means that the vacuum structure has no singularity at the massless limit. This must be due to the fact that the coupling constant g has the mass dimension and therefore, physical quantities are expressed by the coupling constant g even at the massless limit of the fermion. This is in contrast to the Thirring model where the massless limit is a singular point. In the Thirring model, the coupling constant has no dimension, and therefore, at the massless limit, physical quantities must be described by the cutoff Λ.

Table 3: We show the condensate value C_{N_c} and the boson mass $\mathcal{M}$ of $SU(2)$ and $SU(3)$ QCD_2 in rest frame in units of $g/\sqrt{\pi}$ with $m_0 = 0$

$SU(2)$	Trivial	Bogoliubov	$1/N_c$	$SU(3)$	Trivial	Bogoliubov	$1/N_c$
C_2	0	-0.495	-0.577	C_3	0	-0.995	-1.06
$\mathcal{M}$	$-\infty$	0.467	0	$\mathcal{M}$	$-\infty$	0.625	0

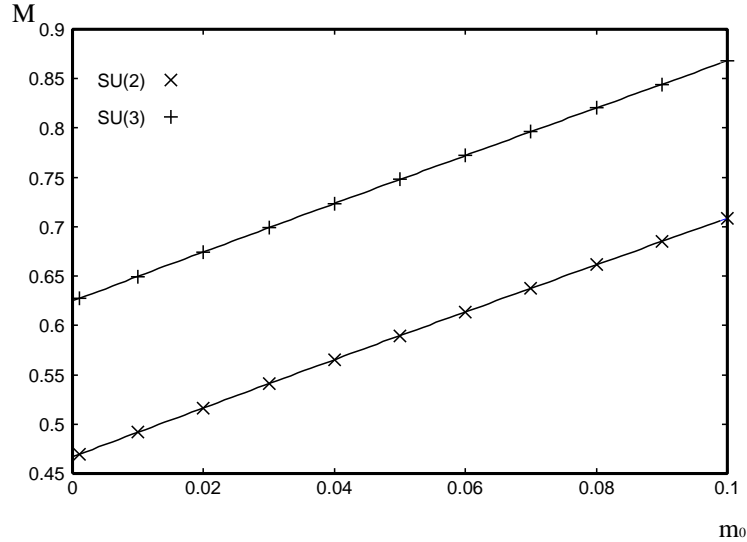In Table 3, we show the condensate values and the boson mass of QCD_2 in the rest

Figure 5: The boson masses for $SU(2)$ and $SU(3)$ colors are plotted as the function of the fermion mass m_0 in the very small mass regions. The solid lines are shown to guide the eyes.

frame. Here, the minus infinity of the boson mass in the trivial vacuum is due to the mass singularity $\ln(m_0)$ as explained in ref. [32]

7.3   Condensate and Boson Mass in $SU(N_c)$

Here, we carry out the calculations of the condensate and the boson mass for the large N_c values of $SU(N_c)$ up to $N_c = 50$. In Fig. 6, we show the calculated condensate values (denoted by crosses) as the function of the N_c together with the prediction of the $1/N_c$ expansion as given in eq.(7.1). As can be seen, the calculated condensate values agree very well with the prediction of the $1/N_c$ expansion if the N_c is larger than 10. Further, the calculated boson masses (denoted by crosses) are shown in Fig. 7 as the function of N_c. It is found that they can be described by the following formula [eq.(7.2)] for the large N_c values,

$$\mathcal{M}_{N_c} = \frac{2}{3}\sqrt{\frac{N_c g^2}{3\pi}}. \tag{7.2}$$

Indeed, the calculated boson masses for N_c larger than $N_c = 10$ perfectly agree with the predicted value of eq.(7.2).

The present calculations show that the second excited state for $SU(N_c)$ colors is higher than the twice of the boson mass at the massless fermions. Therefore, there is only one bound state in QCD$_2$ with the $SU(N_c)$. This indicates that eq.(7.2) must be the full boson spectrum for QCD$_2$ with massless fermions. This indicates that QCD$_2$ with massless
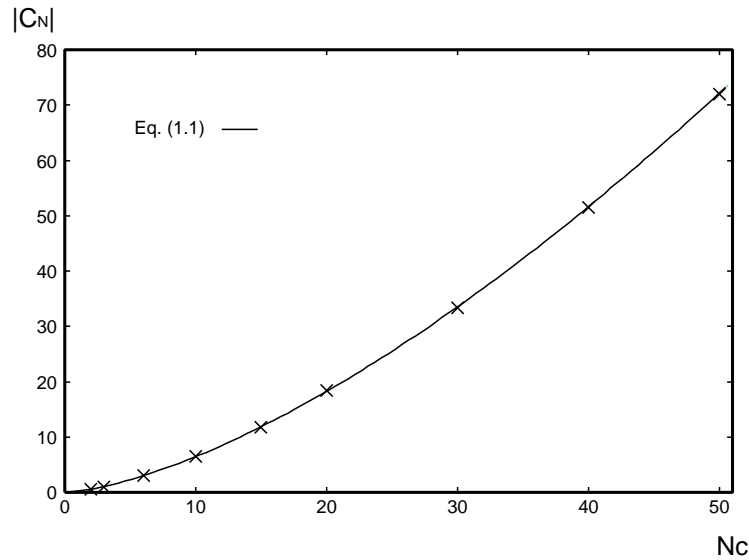
Figure 6: The absolute values of the condensate for $SU(N_c)$ colors are plotted as the function of N_c. The crosses are the calculated values while the solid line is the prediction of eq.(7.1).
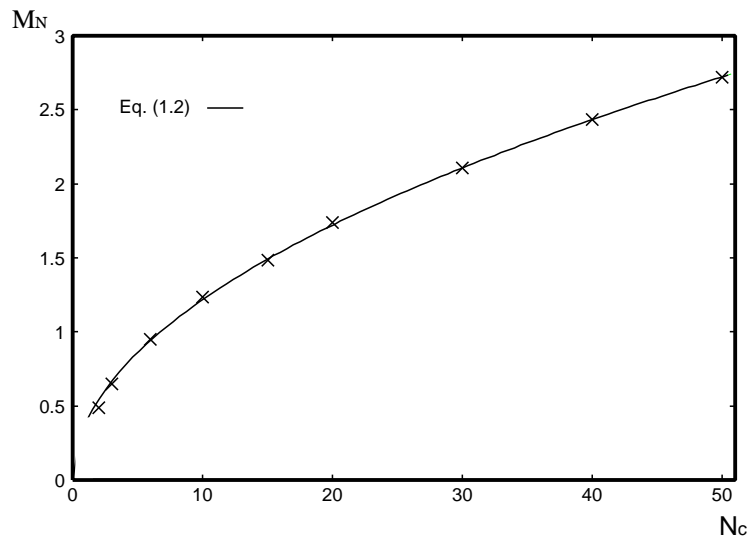

Figure 7: The boson masses for $SU(N_c)$ colors with the massless fermion are plotted as the function of N_c. The crosses are the calculated values while the solid line is the prediction of eq.(7.2).

fermions may well be bosonized like the Schwinger model since the two bosons cannot make any bound states.

Now, we present the calculations of the boson mass for the finite fermion mass m_0 cases. Here, we limit ourselves to the m_0 (in units of $\frac{g}{\sqrt{\pi}}$) which is smaller than unity. In Fig. 8, we show the calculated values of the boson mass as the function of $\frac{m_0}{\sqrt{N_c}}$ for several cases of the fermion mass m_0 and the color N_c. The present calculation is carried out up to the $N_c = 50$ case which is sufficiently large enough for the large N_c limit of the 't Hooft model. The solid line in Fig. 8 is obtained as the following phenomenological formula of the fit to the numerical data

$$\mathcal{M}_{N_c} \approx \left(\frac{2}{3}\sqrt{\frac{2}{3}} + \frac{10}{3}\frac{m_0}{\sqrt{N_c}} \right) \sqrt{\frac{N_c g^2}{2\pi}}. \tag{7.3}$$
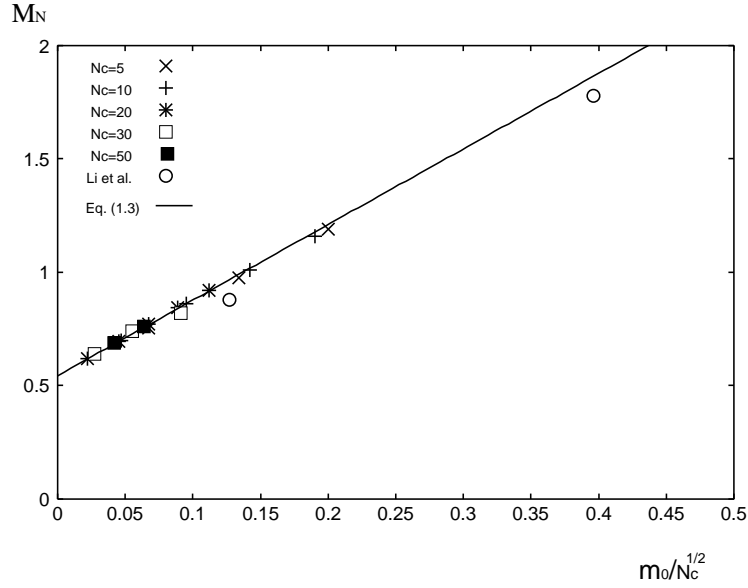

Figure 8: The boson masses in units of $\sqrt{\frac{N_c g^2}{2\pi}}$ for $SU(N_c)$ colors with the massive fermion are plotted as the function of $m_0/\sqrt{N_c}$. The crosses, circles and squares are the calculated values while the solid line is the prediction of eq.(7.3).

Now, we want to compare the present results with the old calculations by Li et al. [37, 38] who obtained the boson mass by solving the 't Hooft equations for QCD$_2$ with the large N_c limit in the rest frame. Li et al. obtained the boson mass for their smallest fermion mass of $m_0 = 0.18\sqrt{\frac{N_c}{2}}$

$$\mathcal{M}_\infty = 0.88\sqrt{\frac{N_c g^2}{2\pi}}. \tag{7.15}$$

There are also a few more points of their calculations with larger fermion mass cases. In Fig. 8, we plot the boson masses calculated by Li et al. by the white circles which should

be compared with the solid line. As can be seen, the boson mass obtained by Li et al. is close to the present calculation. It should be noted that their calculations were carried out with rather small number of the basis functions in the numerical evaluation, and therefore, the accuracy of their calculations may not be very high, in particular, for the small fermion mass regions.

Unfortunately, however, Li et al. made a wrong conclusion on the massless fermion limit since their calculated point of $m_0 = 0.18\sqrt{\frac{N_c}{2}}$ was the smallest fermion mass. Obviously, this value of the fermion mass was by far too large to draw any conclusions on the massless fermion limit.

7.4   QCD$_2$ in Light Cone

Here   we evaluate the boson mass in the light cone. For this, we follow the prescription in terms of the infinite momentum frame [33, 45] since this has a good connection to the rest frame calculation. In this frame, we can calculate the boson mass with and without the condensate in the light cone. But in evaluating the condensate, we only consider the positive momenta. The equation for the boson mass square for the $SU(2)$ case becomes

$$
\begin{aligned}
\mathcal{M}^2 = {}& m_0^2 \int dx f(x)^2 \left(\frac{1}{x} + \frac{1}{1-x} \right) \\
&+ \frac{3g^2}{16\pi} \int dx dy \frac{f(x)^2}{(x-y)^2} \left(\cos 2\theta_{y,1} + \cos 2\theta_{1-y,1} + \cos 2\theta_{y,2} + \cos 2\theta_{1-y,2} \right) \\
&- \frac{g^2}{4\pi} \int dx dy \frac{f(x)f(y)}{(x-y)^2} \Bigg[\frac{1}{2} \cos(\theta_{x,1} - \theta_{y,1}) \cos(\theta_{x-1,1} - \theta_{y-1,1}) \\
&\qquad\qquad\qquad\qquad + \frac{1}{2} \cos(\theta_{x,2} - \theta_{y,2}) \cos(\theta_{x-1,2} - \theta_{y-1,2}) \\
&\qquad\qquad\qquad\qquad + \cos(\theta_{x,1} - \theta_{y,2}) \cos(\theta_{x-1,1} - \theta_{y-1,2}) \\
&\qquad\qquad\qquad\qquad + \cos(\theta_{x,2} - \theta_{y,1}) \cos(\theta_{x-1,2} - \theta_{y-1,1}) \Bigg] \\
&- \frac{g^2}{8\pi} \int dx dy f(x)f(y) \Bigg[\sin(\theta_{x,1} - \theta_{x-1,1}) \sin(\theta_{y-1,1} - \theta_{y,1}) \\
&\qquad\qquad\qquad\qquad + \sin(\theta_{x,2} - \theta_{x-1,2}) \sin(\theta_{y-1,2} - \theta_{y,2}) \\
&\qquad\qquad\qquad\qquad - \sin(\theta_{x,1} - \theta_{x-1,1}) \sin(\theta_{y-1,2} - \theta_{y,2}) \\
&\qquad\qquad\qquad\qquad - \sin(\theta_{x,2} - \theta_{x-1,2}) \sin(\theta_{y-1,1} - \theta_{y,1}) \Bigg] \qquad (7.16)
\end{aligned}
$$

Here, all of the momenta are positive. This can be easily evaluated, and we obtain the condensate values and the boson mass as given in Table 4. We note here that both of the values become smaller as the function of the fermion mass, and finally they vanish to zero. This is exactly what is observed in the light cone calculations. Since the light cone calculations cannot reproduce the condensate values which are finite as predicted in ref.

Table 4: We show the condensate value C_{N_c} and the boson mass $\mathcal{M}$ of $SU(2)$ QCD$_2$ in infinite momentum frame in units of $g/\sqrt{\pi}$ with $m_0 = 0$. Here, $**$ indicates that there is no stable solution.

$SU(2)$	Trivial	Bogoliubov ($p > 0$)	$1/N_c$
C_2	0	$**$	-0.577
$\mathcal{M}$	0	$**$	0

[30], the light cone calculations must have some problems. In Table 4, we also show the calculations of the infinite momentum frame with the positive momenta only. However, the numerical calculations are not stable against the infra-red singularity of the light cone. At the present stage, we do not know how to evaluate them properly, and we do not fully understand what is wrong with the light cone.

7.5   Examination of 't Hooft Model

Here, we discuss the boson mass of QCD$_2$ with $SU(N_c)$ color in the large N_c limit. This model is solved by 't Hooft who sums up all of the Feynman diagrams in the $1/N_c$ expansion and obtains the equations for the boson mass. In principle, the 't Hooft equations must be exact up to the order of $1/N_c$. Therefore, one does not have to consider the effect of the vacuum since the 't Hooft equations take into account all of the fluctuations of the intermediate fermion and antifermion pairs. Therefore, it is expected that the right boson mass can be obtained from the equations at the order of $1/N_c$.

The present calculations of the boson mass with the $SU(N_c)$ colors show that the boson mass can be well described by $\mathcal{M}_{N_c} = \frac{2}{3}\sqrt{\frac{N_c g^2}{3\pi}}$ as the function of N_c for the large values of N_c. In the 't Hooft model, the boson mass should be proportional to $\sqrt{\frac{N_c g^2}{2\pi}}$, and therefore, the present expression of the boson mass is consistent with the 't Hooft evaluation as far as the expansion parameter is concerned. Therefore, the boson mass calculation by the planar diagram evaluations of 't Hooft must be reasonable.

Therefore, the boson mass prediction of 't Hooft should be reexamined from the point of view of the light cone procedure. It seems that the 't Hooft equations in the light cone have lost one important information which is expressed in terms of the θ_p variables both in the paper by Bars and Green [39] and also in the present paper. Since the variables θ_p are closely related to the condensate values, the equations without the θ_p variables should correspond to the trivial vacuum in our point of view. Therefore, if one can recover this constraint in the 't Hooft equations in the light cone, then one may obtain the right boson mass from the 't Hooft model.

7.6 RPA Calculations in QED_2 and QCD_2

Up to this point, we have presented the calculated results of the Fock space expansion with the Bogoliubov vacuum state for QED_2 and QCD_2. The lowest boson mass which is calculated by the Fock space expansion must be exact for the fermion and anti-fermion states if the vacuum is exact. From the present result for the condensate values of QED_2 and QCD_2, it indicates that the Bogoliubov vacuum state should be very good or may well be exact.

On the other hand, there are boson mass calculations by employing the Random Phase Approximation (RPA) method, and some people believe that the RPA calculation should be better than the Fock space expansion.

Therefore, in this section, we present our calculated results of the RPA equations for QED_2 and QCD_2 since there are no careful calculations in the very small fermion mass regions. First, we show that the RPA calculation for QED_2 with the Bogoliubov vacuum state predicts the boson mass which is smaller than the Schwinger boson at the massless fermion limit. This means that the agreement achieved by the Fock space expansion is destroyed by the RPA calculation since it gives a fictitious attraction.

Further, the RPA calculation for QCD_2 with the Bogoliubov vacuum state produces an imaginary boson mass at the massless fermion limit. This is quite interesting, and it strongly suggests that the RPA equation cannot be reliable for fully relativistic cases since the eigenvalue equation of the RPA is not Hermitian, which is, in fact, a well known fact.

Here, we briefly discuss the results of the RPA calculations, but the detailed discussion of the basic physical reason of the RPA problems will be given elsewhere.

The RPA equations are based on the expectation that the backward moving effects of the fermion and anti-fermion may be included if one considers the following operator which contains the $d_{-m}c_m$ term in addition to the fermion and anti-fermion creation term,

$$Q^{\dagger} = \sum_n (X_n c_n^{\dagger} d_{-n}^{\dagger} + Y_n d_{-n} c_n). \tag{7.17}$$

The RPA equations can be obtained by the following double commutations,

$$\langle 0|[\delta Q, [H, Q^{\dagger}]]|0\rangle = \omega \langle 0|[\delta Q, Q^{\dagger}]|0\rangle \tag{7.18}$$

where δQ denotes $\delta Q = d_{-n}c_n$ and $c_n^{\dagger}d_{-n}^{\dagger}$.

Here, the vacuum $|0\rangle$ is assumed to satisfy the following condition,

$$Q|0\rangle = 0. \tag{7.19}$$

However, if the vacuum is constructed properly in the field theory model, it is impossible to find a vacuum that satisfies the condition of eq.(6.3). This fact leads to the RPA equations which are not Hermitian.

For QED$_2$, the RPA equations for X_n and Y_n become

$$\mathcal{M}X_n = 2E_nX_n - \frac{g^2}{L}\sum_m X_m \frac{\cos^2(\theta_n - \theta_m)}{(p_n - p_m)^2}$$

$$- \lim_{\varepsilon \to 0} \frac{g^2}{L}\sum_m X_m \frac{\sin(\theta_{n-\varepsilon} - \theta_n)\sin(\theta_m - \theta_{m-\varepsilon})}{\varepsilon^2} - \frac{g^2}{L}\sum_m Y_m \frac{\sin^2(\theta_n - \theta_m)}{(p_n - p_m)^2}$$

$$- \lim_{\varepsilon \to 0} \frac{g^2}{L}\sum_m Y_m \frac{\sin(\theta_{n-\varepsilon} - \theta_n)\sin(\theta_m - \theta_{m-\varepsilon})}{\varepsilon^2} \qquad (7.20a)$$

$$-\mathcal{M}Y_n = 2E_nY_n - \frac{g^2}{L}\sum_m Y_m \frac{\cos^2(\theta_n - \theta_m)}{(p_n - p_m)^2}$$

$$- \lim_{\varepsilon \to 0} \frac{g^2}{L}\sum_m Y_m \frac{\sin(\theta_{n-\varepsilon} - \theta_n)\sin(\theta_m - \theta_{m-\varepsilon})}{\varepsilon^2} - \frac{g^2}{L}\sum_m X_m \frac{\sin^2(\theta_n - \theta_m)}{(p_n - p_m)^2}$$

$$- \lim_{\varepsilon \to 0} \frac{g^2}{L}\sum_m X_m \frac{\sin(\theta_{n-\varepsilon} - \theta_n)\sin(\theta_m - \theta_{m-\varepsilon})}{\varepsilon^2} \qquad (7.20b)$$

For QCD$_2$, one can easily derive the RPA equations, and at the large N_c limit, they agree with the RPA equations which are obtained by Li et.al [30, 38].

It is important to note that the RPA equations are not Hermitian, and therefore there is no guarantee that the energy eigenvalues are real. In fact, as we see below, the boson mass for QCD$_2$ becomes imaginary at the very small fermion mass.

Table 5: The masses for QED$_2$ and QCD$_2$ with $SU(2)$ are measured by $\frac{g}{\sqrt{\pi}}$. The masses for large N_c QCD$_2$ are measured by $\sqrt{\frac{N_c g^2}{2\pi}}$. $0.104i$ indicates an imaginary eigenvalue.

	QED$_2$		QCD$_2$ SU(2)		Large N_c QCD$_2$	
	$m_0 = 0$	$m_0 = 0.1$	$m_0 = 0$	$m_0 = 0.1$	$m_0 = 0$	$m_0 = 0.1$
Fock Space	1.000	1.180	0.467	0.709	0.543	0.783
RPA	0.989	1.172	0.104i	0.576	0.120i	0.614

In Table 5, we show the calculated values of the boson mass by the RPA equations for QED$_2$ and QCD$_2$ with the Bogoliubov vacuum state. It should be noted that the boson mass for $m_0 = 0$ case with the Fock space in the large N_c limit is obtained from the 't Hooft equation. This equation is exactly the same as eq.(7.14) if we take the large N_c limit. We

note here that the boson mass $(0.543 \sqrt{\frac{N_c g^2}{2\pi}})$ at the large N_c limit with the Fock space expansion just agrees with the value of eq.(7.2).

The behavior of the boson mass of the RPA calculation for QCD_2 is not normal, contrary to the expectation. First, it is not linear as the function of m_0, but nonlinear in the small mass region. Further, the boson mass square becomes zero when the m_0 becomes a critical value, and it becomes negative when the m_0 is smaller than the critical value. In this case, the boson mass is imaginary, and thus this is physically not acceptable. This catastrophe is found to occur for the $SU(2)$ as well as for the large N_c limit, as shown in Table 5.

At this point, we should comment on the belief that the RPA calculation should produce the massless boson at the massless fermion limit in QCD_2. However, if there were physically a massless boson in two dimensions, this would be quite serious since a physical massless boson cannot propagate in two dimensions since it has an infra-red singularity in its propagator. But there is no way to remedy this infra-red catastrophe, and that is related to the theorem of Mermin, Wagner and Coleman [8, 43]. There are some arguments that the large N_c limit is special because one takes the N_c infinity. However, "infinity" in physics means simply that the N_c must be sufficiently large, and in fact, as shown above, physical observables at $N_c = 50$ are just the same as those of $N_c = \infty$. Therefore, it is rigorous that there should not exist any physical massless boson in two dimensions, even though one can write down the free massless boson Lagrangian density and study its mathematical structure. Thus, if one finds a massless boson constructed from the fermion and antifermion in two dimensions, then there must be something wrong in the calculations, and this is exactly what we see in the RPA calculations in QCD_2.

In this respect, the boson mass calculated only by the Fock space expansion with the Bogoliubov vacuum can be reasonable from this point of view since there are some serious problems in the light cone as well as in the RPA calculations at the massless fermion limit.

7.7   Spontaneous Chiral Symmetry Breaking in QCD$_2$

The Lagrangian density of QCD_2 has a chiral symmetry when the fermion mass m_0 is set to zero. In this case, there should be no condensate for the vacuum state if the symmetry is preserved in the vacuum state. However, as we saw above, the physical vacuum state in QCD_2 has a finite condensate value, and thus the chiral symmetry is broken. In contrast to the Schwinger model, there is no anomaly in QCD_2, and therefore the chiral current is conserved. Thus, this symmetry breaking is spontaneous.

However, there appears no massless boson. Even though no appearance of the Goldstone boson is very reasonable in two dimension, this means that the Goldstone theorem does not hold for the fermion field theory. This is just what is proved in section 2 and 4, and the calculations of QCD_2 confirm its claim.

Further, it seems that the chiral anomaly does not play an important role in the symmetry breaking business though it has been believed that the Schwinger model breaks the chiral symmetry due to the anomaly. However, the massless limit in QED_2 is not singular [32]. The condensate value and the boson mass are smooth as the function of the fermion mass

m_0. This means that the vacuum structure is smoothly connected from the massive case to the massless one.

This is just in contrast to the Thirring model [5, 33, 11] where the massless limit is a singular point as we saw in section 4. The structure of the vacuum is completely different from the massive case to the massless one in the Thirring model. Further, the condensate value and the boson mass in the Thirring model are not smooth function of the fermion mass m_0. For the massive Thirring model, there is no condensate, and the boson mass is proportional to the fermion mass m_0 [9, 33, 11, 45]. Indeed, in the massive Thirring model, the induced mass term arising from the Bogoliubov transformation is completely absorbed into the mass renormalization term, and the vacuum stays as it is before the Bogoliubov transformation. But, for the massless Thirring model, the condensate is finite, and the condensate value and the boson mass are both proportional to the cutoff Λ by which all of the physical observables are measured.

On the other hand, QED$_2$ and QCD$_2$ are very different in that the coupling constant of the models have the mass scale dimensions, and all of the physical quantities are described by the coupling constant g even at the massless limit. The super-renormalizability for QED$_2$ and QCD$_2$ must be quite important in this respect, while the Thirring model has no dimensional quantity, and this makes the vacuum structure very complicated when the fermion mass is zero.

In Table 6, we summarize the physical quantities of the chiral symmetry breaking for QED$_2$, QCD$_2$ and Thirring models. All the condensates and the masses are measured in units of $\frac{g}{\sqrt{\pi}}$ for QED$_2$ and QCD$_2$. The Λ and g_0 in the Thirring model denote the cutoff parameter and the coupling constant, respectively. Also, the value of $\alpha(g_0)$ can be obtained by solving the equation for bosons in the Thirring model [5, 11].

For QED$_2$, there is an anomaly, and therefore, the chiral current is not conserved while, for QCD$_2$ and the Thirring model, the chiral current is conserved. From Table 6, one sees that the symmetry breaking mechanism is just the same for QED$_2$ and QCD$_2$. However, the Thirring model has a singularity at the massless fermion limit, and this gives rise to somewhat different behaviors from the gauge theory.

8   Conclusions

The symmetry and its breaking in field theory were considered to be understood and settled down long time ago in terms of the Nambu-Goldstone theorem. In any of the field theory textbooks, the spontaneous symmetry breaking phenomena are well explained and described, and, therefore, it would have appeared somewhat odd to most of the readers to raise questions on the Goldstone boson after the spontaneous symmetry breaking.

In this chapter, however, we have reviewed the recent progress in the spontaneous symmetry breaking and the appearance and non-appearance of a massless boson after the spontaneous symmetry breaking. Indeed, we have shown that the Nambu-Goldstone theorem for the fermion field theory is wrong. At the same time, we have presented examples of the symmetry breaking phenomena in concrete fermion field theory models which clearly

Table 6: We summarize the physical quantities of the chiral symmetry breaking for QED_2, QCD_2 and Thirring models.

	Condensate		Boson Mass		Anomaly
	$m_0 = 0$	$m_0 \neq 0$	$m_0 = 0$	$m_0 \neq 0$	
QED_2	-0.283	$-0.283 + O(m_0)$	1	$1 + O(m_0)$	yes
QCD_2	$-\frac{N_c}{\sqrt{12}}\sqrt{\frac{N_c}{2}}$	$-\frac{N_c}{\sqrt{12}}\sqrt{\frac{N_c}{2}} + O(m_0)$	$\frac{2}{3}\sqrt{\frac{N_c}{3}}$	$\left(\frac{2}{3}\sqrt{\frac{2}{3}} + \frac{10}{3}\frac{m_0}{\sqrt{N_c}}\right)\sqrt{\frac{N_c}{2}}$	no
Thirring	$\frac{\Lambda}{g_0 \sinh\left(\frac{\pi}{g_0}\right)}$	0	$\frac{\alpha(g_0)\Lambda}{\sinh\left(\frac{\pi}{g_0}\right)}$	$\alpha(g_0)m_0$	no

exhibit the essence of the spontaneous symmetry breaking and the non-appearance of the massless boson associated with the symmetry breaking.

The most important of the new aspects in the spontaneous symmetry breaking is to realize the change of the vacuum state after the symmetry breaking. This is of course quite well known since Nambu and Jona-Lasinio showed that the new vacuum is the one that breaks the chiral symmetry and its energy is lower than that of the symmetry preserving vacuum state (perturbative vacuum). In this case, however, one should be very careful for carrying out any field theory calculations since any physical estimations should be based on the physical vacuum state in quantum field theory. Here, it is for sure that one should start from the formalism that is based on the symmetry broken vacuum state since this is indeed a physical vacuum state.

However, almost all of the calculations which were carried out for the NJL model are based on the perturbative vacuum state, and therefore the calculated results were not physical observables. This is rather a serious mistake, and in fact, people found a massless boson in the NJL model without noticing that their calculated boson is an unphysical particle. If one calculates the boson mass by the formulation which is based on the true vacuum (symmetry broken vacuum), then one finds that there is no massless boson. In fact, one obtains a massive boson if one carries out the calculation in terms of the Bogoliubov transformation method. In addition, we show that the Goldstone theorem cannot be applied to the fermion field theory models. This is obvious once we realize that the Goldstone theorem can only give the information on the dispersion relation of the energy and momentum for boson fields, and therefore it cannot tell us anything of the existence of the boson which should be constructed by fermions and antifermions. In fact, the Goldstone theorem had to assume the existence of the massless boson which is the one that should be obtained as the result of the proof. This is of course no proof at all for the fermion field theory models.

Further, one notices that the commutation relation of the conserved charge and the boson field is obtained independently from the interaction Hamiltonian of the field theory model, and this is the basic ingredient of the proof of the Goldstone theorem. Therefore, it is clear that the commutation relation cannot give any information of the existence of the boson in the fermion field theory model.

From these considerations, we can summarize the spontaneous symmetry breaking business in fermion field theory. The chiral symmetry of the fermion field theory models is spontaneously broken, and the chiral symmetry broken vacuum is indeed the physical state of the field theory model. This is all that happens in the chiral symmetry breaking. Even though Nambu and Jona-Lasinio claimed that the original massless fermion should become massive in the NJL model, the massless fermion should stay as it is. Under the Bogoliubov transformation method, it looks that the fermion should acquire the induced mass. But we believe that the fermion cannot change its structure by the spontaneous symmetry breaking phenomena and the massless fermions are still massless. The symmetry breaking is a property of the vacuum state, and the vacuum energy becomes lower than that of the symmetry preserving vacuum state. This energy change is entirely due to the change of the momentum distribution in the negative energy particles and it has nothing to do with the mass of the fermions in the vacuum state. Further, the renormalization procedure cannot give a finite mass to the massless fermions since the concept of the renormalization is just the change of the mass parameter from the infinite number to the finite observable.

In the massless Thirring model, the spectrum with the symmetry preserving vacuum, though it is unphysical, has a gapless spectrum while the physical vacuum which breaks the chiral symmetry has the excitation spectrum with a finite gap. The gap in the excitation spectrum is due to the change in the vacuum structure. Even though this gap can be naturally explained by the massive fermions, it does not mean that the fermion becomes massive. In fact, the Bethe ansatz solutions clearly show that the fermion stays massless even though the momentum distribution of the vacuum state is approximated by the dispersion of the free fermions to a good accuracy.

If one employs the Bogoliubov transformation method in evaluating the boson in the massless Thirring model, then one finds a massive boson. However, the Bethe ansatz calculations show that there is no bosonic state in the massless Thirring model. Therefore, the massive boson which is predicted by the Bogoliubov transformation method in the NJL model should not exist in reality. This is not yet proved, but we believe that the claim must be quite reasonable since the Bogoliubov transformation method obviously overestimates the attraction between fermions and antifermions, and therefore they predict the bound state. In addition, the bound states are more difficult to make in four dimensions than in two dimensions, and this suggests that there should not be any bound states in the NJL model.

The nonexistence of bosons in the NJL and massless Thirring model should be closely related to the observation that the massless fermions and antifermions cannot make any bound states if the interactions are of the $\delta-$function type. This is in contrast to the gauge field theory models in two dimensions. For QED$_2$ and QCD$_2$, the massless fermions and an-

tifermions make the bound states since they are confined. In other words, massless fermions can be either completely confined or cannot make any bound states since they do not have any rest systems.

The physical connection between the chiral symmetry breaking and the chiral condensate is not clarified in this chapter. For the Thirring model, this is relatively simpler. The massive Thirring model has no chiral symmetry and there is no condensate. The vacuum state of the massive Thirring model is trivial. This can be seen at least from the Bogoliubov vacuum. Therefore, the massless limit is a singular point and the massless Thirring model has a chiral symmetry and its broken vacuum. In this chiral symmetry broken vacuum, the chiral condensate is finite. However, the chiral condensate value of the gauge field theory is finite even at the massive fermion case where there is no chiral symmetry. This suggests that the chiral symmetry and the chiral condensate is not strongly connected in the gauge field theory models.

In this chapter, we have not included the recent results on the Heisenberg XXZ model. Therefore, we wish to make some comments on the relation between the massless Thirring model and the Heisenberg XXZ model. It is believed that the two models are equivalent to each other since if one takes the massless limit in the Heisenberg XYZ and massive Thirring models, then they can be reduced to the Heisenberg XXZ model and the massless Thirring model, respectively. This belief is due to the fact that the equivalence between the Heisenberg XYZ and massive Thirring models is well established [51], and there is no problem over there.

However, as we saw, the spectrum of the Thirring model gives a finite gap while the Heisenberg XXZ model predicts always gapless spectrum. This means that, even though the two models are mathematically shown to be equivalent to each other, they are physically very different [48].

What should be the main reason for the difference ? If one makes the field theory into the lattice, then the lattice field theory loses some important continuous symmetry like Lorentz invariance or chiral symmetry. If the lost symmetry plays some important role for the spectrum of the model, then the lattice field theory becomes completely a different model from the continuous field theory model [49]. In general, the way of cutting the continuous space into a discrete one is not unique, and equal cutting of the space which is generally used in physics may not be sufficient for some of the field theory models. For example, if the interaction of the model is centered on the very small region of the space, then it is clear that the equal cutting of the space must be a very bad approximation.

The non-equivalence between the Heisenberg XXZ and massless Thirring models is the only example which shows a mismatch between the mathematical and physical correspondence. In this respect, we have discussed only a specific model in two dimensions, and it is quite different from four dimensional field theory models. However, the result certainly raises a warning on the lattice version of the field theory since clearly there are important continuous symmetries in any of the field theory models in four dimensions, and if these symmetries may be lost in the lattice version, then it is quite probable that the lattice calculations may not be able to reproduce a physically important spectrum of the continuous

field theory models.

Finally, we should like to discuss the symmetry breaking in four dimensional QCD. Since the real nature should be described by QCD in four dimensions, it is of course most important to understand what is the status of the symmetry breaking in QCD. First, let us assume that quarks are massless. In this case, there is the chiral symmetry, and we believe that the chiral symmetry should be broken in the vacuum. In other words, the energy of the symmetry broken vacuum state should be lower than the perturbative vacuum state which preserves the chiral symmetry. Since there is no scale in the four dimensional QCD, all the observables must be measured in terms of the cutoff momentum Λ. The vacuum structure is completely different from the perturbative one. However, we cannot say anything further than this. In a sense, the symmetry breaking phenomena in four dimensional QCD must be more similar to the massless Thirring model than QCD in two dimensions. But the dynamics is so complicated that we cannot build any realistic picture of the vacuum structure of the four dimensional QCD unless we solve the dynamics in a nonperturbative fashion.

Now, in the realistic QCD in four dimensions, quarks have their own mass. In this case, there is no chiral symmetry, and therefore it does not make sense to argue the chiral symmetry breaking of the vacuum state. In this sense, QCD in four dimensions must have a singularity at the massless fermion limit, and the structure of the vacuum states between the massive fermion QCD and massless fermion QCD should be completely different from each other. In this respect, there is some similarity between QCD in four dimensions and Thirring model as far as the chiral symmetry breaking and its vacuum structure are concerned. But the structure of QCD in four dimensions must be much more complicated due to the gauge degree of freedom.

In this sense, we have no idea about the structure of the vacuum in four dimensional QCD. We believe that the QCD vacuum must be quite complicated, but that is not related to the chiral symmetry breaking. All of the baryon and boson masses are measured in terms of the mass of the quarks. In this sense, the mass of pion (around 140 MeV) is quite large compared to the mass of u or d quarks (around 10 MeV) even though pion is lighter than other mesons by a factor of four or five.

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