Physics of Leap Second

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Contents

1. Newcomb Time and Atomic Watch

Orbital Period of Earth Revolution : Average Regression Year T_0.

$T_0 = 365.242189$ **(day)** $= 3.155692513 \times 10^7$ **(s)**

This defines the Newtonian time.

- **Newcomb time : Orbital Period of Earth Revolution.**
  Newcomb defines a second by Newtonian Orbital Period.

- **Time of Atomic Watch : Orbital Period of Earth Revolution.**
  Measured value by Atomic Watch deviates $\boxed{\textbf{0.62 second}}$ (Leap Second) from Newtonian Orbital Period.

2. Origin of Leap Second

What is leap second ? :
The effect of the additional gravitational force on Orbital Period of Earth Revolution.

The observed period T **is longer than** T_0 **by** $\Delta T = 0.621$ **(s)**
$T = T_0 + \Delta T$

(1) **Newtonian Orbital Period of Earth Revolution**

Orbital Period From perihelion to perihelion T_0 **(s)**

(2) **Orbital Period measured by Atomic Watch**

From perihelion to perihelion : 0.62 (s) longer than T_0

- **Additional Gravitational Force : Relativistic Effect**

Relativistic Effect on Orbital Motion : $\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8}$
Leap Second $\sim 2 \times 10^{-8}$ **: Good agreement**

- **Additional Gravitational Force : Prediction** $\boxed{\Delta T_{(Pred.)} = 0.621(s)}$

Reference :

(1) **"Symmetry and Its Breaking in Quantum Field Theory"**
(T. Fujita, Nova Science Publishers, 2011, 2nd edition)

(2) **"Fundamental Problems in Quantum Field Theory"**
(T. Fujita and N. Kanda, Bentham Publishers, 2013)

3. Leap Second by New Gravity Model

Potential of New Gravity Model

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2$$

- **Non-integrable Potential :**

The additional potential is a non-integrable, and therefore, the treatment should be done by the perturbation theory.

The period T of the earth revolution is written as
$\omega T \simeq 2\pi(1+2\eta)$

where η is given as

$$\eta = \frac{G^2M^2}{c^2R^4\omega^2}.$$

Comparison between Theory and Experiments

• **Leap Second : Predictions of New Gravity Model**

$\frac{\Delta T}{T} = 2\eta$ **where** $\eta = \frac{G^2 M^2}{c^2 R^4 \omega^2}$ **and**

$R = 1.496 \times 10^{11}$ m, $M = 1.989 \times 10^{30}$ kg, $\omega = 1.991 \times 10^{-7}$.

Therefore, the period of the earth revolution per year amounts to

$$\boxed{\Delta T_{N.G.} = 0.621 \quad [\mathrm{s/year}]}$$

• **Leap Second : Observations**

The leap second corrections have been made for more than 40 years. The first leap second correction started from June 1972, and for 40 years, people made corrections of 25 second. Therefore, the average leap second per year becomes

$$\boxed{\Delta T_{N.G.}^{Obs} \simeq 0.625 \pm 0.013 \quad [\mathrm{s/year}]}$$

which agrees with the prediction quite well.

4. Leap Second : General Relativity

It is proved that General Relativity is physically meaningless.
Therefore, there is no effect from the General Relativity.
This is explained in the short note :

”Why Is General Relativity Meaningless?”

https://allphysics.sakura.ne.jp/indexGrelaNSE.html

5. Mercury Perihelion Shifts

There should be no perihelion shifts for one period of revolution.

There should be the Mercury perihelion shifts which should arise from the effects of other planets such as Jupiter.

Concerning the Mercury perihelion shifts, however, the measurements as well as the calculations of the effects from other planets should be carried out more carefully.

After the calculation of Newcomb in the 19 century, no careful calculation on the perihelion shifts has been done until now.
The new calculation of the Planet Effects on Mercury Perihelion can be seen from Chapter 4 of the short note
"Why Is General Relativity Meaningless?"

https://allphysics.sakura.ne.jp/indexGrelaNSE.html

[Appendix A] Earth Rotation and Tidal Force

[Wrong Theory] : Tidal force may affect on Earth's Rotation ?

[Wrong claim] : Tidal force may push up matters to the surface ?
Moment of Inertia may become larger ?
Thus, rotation velocity may become slower ?

- **<u>The reasons why this theory is incorrect</u>**

(1) Tidal Force is conservative and thus does not make Work !
Tidal Force cannot change matter distribution in Earth.

- **<u>Basic Mechanics</u> : Conservative Force and Work**

$\boldsymbol{F} = -\boldsymbol{\nabla} V(r)$ **is called conservative force. Or** $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$.
The gravity is a conservative force. The Work W **is**
$W(A \to B \to A) = \oint \boldsymbol{F} \cdot d\boldsymbol{r} = -\left[V(r)\right]_{A-B-A} = 0$
and thus does not make any Work.

(2) Matter in Earth cannot move to the surface
since this is against the gravity of Earth.

- $\boxed{\text{Period of Earth's Rotation never changes !}}$

[Appendix B] Work of Non-conservative Force

- **Work of non-conservative force :** we consider non-conservative force ($\boldsymbol{\nabla} \times \boldsymbol{F} \neq 0$)

  $$\boxed{\boldsymbol{F} = (-kx + \varepsilon y)\boldsymbol{e}_x - ky\boldsymbol{e}_y}$$

  $F_z = 0$ has no loss of generality.

  When $\varepsilon = 0$, it is a conservative force ($\boldsymbol{\nabla} \times \boldsymbol{F} = 0$).

- **Definition of Work W :** $W = \oint \boldsymbol{F} \cdot d\boldsymbol{r}$

  The motion of a particle is $\boxed{x = a\cos\omega t,\ \ y = a\sin\omega t}$.

  In this case, W is calculated with its period T

  $$\boxed{W = \oint \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^T (-kx\dot{x} + \varepsilon y\dot{x} - ky\dot{y})dt}$$

  where $\omega T = 2\pi$. Thus

  $$\boxed{W = -\varepsilon a^2\omega \int_0^T (\sin\omega t)^2 dt = -\pi\varepsilon a^2}.$$

Therefore, the conservative force cannot make Work.
But the non-conservative force does make Work $\Rightarrow$
Thus, Energy must be consumed.

Chapter 2

New Gravity Model

Quantum field theory is based on the free Dirac fields and four fundamental interactions. These are electromagnetic, weak, strong and gravitational interactions. In terms of coupling constant, the electromagnetic interaction must be a standard, and the strength of the coupling constant which is dimensionless is found to be

$$\alpha = \frac{1}{137}. \tag{2.1}$$

On the other hand, the strong interaction should be stronger by two orders of magnitude than the electromagnetic interaction while weak interaction must be weaker by a few orders of magnitude than the electromagnetic interaction. In this respect, the gravity is, by far, the weakest force among the four interactions. In fact, the gravity is by the order of $\sim 10^{-30}$ smaller than the electromagnetic interaction.

2.1 Introduction

Nevertheless, the gravity is very important in the universe for the formation of stars and galaxies since the force has a very long range, and it is always attractive. In fact, apart from strong interactions that should responsible for nuclear fusion in stars, the basic ingredients of forming stars and galaxies in the universe should be the gravitational interaction.

2.1.1 Why Gravity Has Large Effects on Star Formation?

The gravity is crucially important for the formation of stars even though the interaction strength is quite weak. There are two important aspects in the gravity when the stars should be formed. The first point is connected to the interaction range which is very long since it has the shape of $1/r$. The other point is that the gravity is always attractive and the strength of the force should be proportional to the masses of interacting objects. Therefore, as long as the corresponding body is massive, there should exist the attractive interactions from all other massive objects even though they are far away from each other. Because of the attractive nature, there should be no shielding in contrast to the electromagnetic cases.

2.1.2 Dirac Equation with Gravitational Potential

When the energy of a particle becomes as high as its mass, then we have to consider the relativistic equation of motion under the gravitational potential. In this case, the Newton equation is not appropriate for describing a relativistic motion, and thus, we have to find a new equation of motion. Since we know that the classical mechanics is derived from the Schrödinger equation, we should start from the relativistic equation in quantum mechanics. This is the Dirac equation, and therefore, we have to consider the Dirac equation with the gravitational interaction.

However, the Dirac equation with the gravitational potential has not been determined properly for a long time. This problem is connected to the ambiguity as to whether the gravitational potential should be taken as the fourth component of the vector type interaction or the mass term of scalar type interaction. This problem was not settled until recently, and thus, we should consider the gravitational field theory in some way or other. As will be discussed later, the new gravity model is, indeed, constructed in terms of a massless scalar field theory. Therefore, the corresponding Dirac equation with the gravitational potential is well established by now [2, 5].

2.2 Dirac Equation and Gravity

The Newton equation works very well under the gravitational potential, and indeed, the Kepler problem is best understood by solving the Newton equation.

• Ehrenfest Theorem :
This Newton equation itself is obtained from the Schrödinger equation by making some approximation such as Ehrenfest theorem. In this case, the time development of the expectation values of $\boldsymbol{r}$ and $\boldsymbol{p}$ in quantum mechanics lead to the Newton equation.

• Foldy-Wouthuysen Transformation :
The Schrödinger equation can be derived from the Dirac equation by making the Foldy-Wouthuysen transformation which is a unitary transformation. Therefore, the Dirac equation must be the starting point from which the Newton equation can be derived.

2.2.1 Dirac Equation and Gravitational Potential

As can be seen from the present discussion, it should be crucially important to have the Dirac equation with the gravitational potential properly taken into account. Otherwise, we cannot obtain the Newton equation with the gravitational potential. In other words, we should not start from the Newton equation with the gravitational potential since it is obtained only after some series of approximations should be properly made for quantum mechanics.

• Dirac Equation with Coulomb Potential :
Before going to the discussion of the Dirac equation with the gravity, we should first discuss the Dirac equation with the Coulomb potential of $V_c(r) = -\frac{Ze^2}{r}$. This is well-known and can be written as

$$\left(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\beta - \frac{Ze^2}{r}\right)\Psi = E\Psi. \tag{2.2}$$

On the other hand, we should be careful in which way we put the gravitational potential of $V(r) = -\frac{GmM}{r}$ into the Dirac equation since there are two different ways, either the same way as the Coulomb case or putting the gravitational potential into the mass term.

• **Dirac Equation with Gravitational Potential :**
In fact, the right Dirac equation with the gravitational potential of $V(r) = -\frac{GmM}{r}$ can be written by putting it into the scalar term as

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{G_0 mM}{r}\right)\beta\right]\Psi = E\Psi. \tag{2.3}$$

This is obtained from the field theoretical construction of the gravity model. By now, we see that the scalar type potential of gravity must be the right gravitational potential, and we should discuss it more in detail below.

2.3 New Gravity Model

When we wish to construct the theory of gravity, the first thing we should work out should be to find the framework in which the gravitational potential can be properly taken into account in the Dirac equation. Without doing this procedure, there should be no way to consider the theory of gravity. In fact, the Dirac equation for a particle with its mass m in the gravitational potential can be written as

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{GmM}{r}\right)\beta\right]\Psi = E\Psi \tag{2.4}$$

where M denotes the mass of the gravity center. In addition, if we make the non-relativistic reduction using the Foldy-Wouthuysen transformation, then we find the gravitational potential in classical mechanics

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{2.5}$$

where the second term of the right hand side should be the additional potential which appears as the relativistic effect. This additional potential of gravity is a new gravitational potential, and this must be a new discovery ever since nineteenth century. It turns out that this new potential can explain the problem of leap second of the earth revolution period which will be discussed later.

• **Rough Estimation of Relativistic Effect :**
Historically, the first check of the relativistic effect was done by Michelson-Morley using the velocity of the earth revolution which should be the fastest object relevant to the observed speed on the earth. The result of Michelson-Morley experiment showed that the speed of light is not affected by the earth revolution, and this leads to the concept of the relativity principle. The relativistic effect in this case is

$$\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8} \tag{2.6}$$

where c and v denote the velocities of light and the earth revolution, respectively. It should be interesting to note that the leap second of the earth revolution period is found to be ($\Delta T/T \sim 2 \times 10^{-8}$) which is just the same order of magnitude as the relativistic effect.

2.3.1 Lagrangian Density

When we consider the theory of gravity, we should start from the scalar field theory since it gives always attractive interactions.

• **Lagrangian Density of Gravity :**
Here, we should write the Lagrangian density of a fermion field ψ interacting with the electromagnetic field A_μ and the gravitational field $\mathcal{G}$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m(1+g\mathcal{G})\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{v\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\ \partial^\mu\mathcal{G} \tag{2.7}$$

where m denotes the fermion mass. The gravitational field $\mathcal{G}$ is a massless scalar field. The reason why people did not consider the scalar field for the gravity should be mainly because the scalar field should not be renormalizable. However, there is no necessity of the field quantization of the gravitational field, and thus, there is no divergence at all.

• **Gravity Cannot Be Gauge Theory :**
For a long time, people believed that the basic field theory must be a gauge theory, even though there is no foundation for this belief. Indeed, the gauge theory has both attractive and repulsive interactions, and therefore, it is clear that this model of gauge field theory should not be suitable for the gravity.

By now, it is known that only the gauge theory of quantum electrodynamics using the Feynman propagator should give rise to some divergences in the calculation of physical observables such as vertex corrections. In fact, there is no divergence for the vertex corrections which are calculated from the massive vector field theory [2].

2.3.2 Equation for Gravitational Field

From the Lagrangian density, we can obtain the equation for the gravitational field from the Lagrange equation. Here, we can safely make the static approximation for the equation of motion, and obtain the equation for the gravitational field $\mathcal{G}_0$ as

$$\boldsymbol{\nabla}^2 \mathcal{G}_0 = mg\rho_g \tag{2.8}$$

where $m\rho_g$ corresponds to the matter density. The coupling constant g is related to the gravitational constant G as

$$G = \frac{g^2}{4\pi}.$$

This equation eq.(2.8) is indeed the Poisson equation for gravity.

2.3.3 Dirac Equation with Gravitational Potential

From the Lagrangian density with gravity and electromagnetic interactions, we can derive the Dirac equation

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\beta\left(1 + g\mathcal{G}\right) - \frac{Ze^2}{r} \right] \Psi = E\Psi. \tag{2.9}$$

Further, in case the gravitational force is produced by nucleus with its mass of M, the Dirac equation becomes

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{GmM}{r}\right)\beta - \frac{Ze^2}{r} \right] \Psi = E\Psi \tag{2.10}$$

which is just the equation discussed in the previous section.

2.3.4 Foldy-Wouthuysen Transformation of Dirac Hamiltonian

The Dirac equation with the gravitational interaction

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{GmM}{r} \right)\beta \right]\Psi = E\Psi \qquad (2.11)$$

can be reduced to the non-relativistic equation in quantum mechanics. This can be done in terms of Foldy-Wouthuysen transformation which is a unitary transformation. Therefore, the transformation procedure is very reliable indeed.

• Foldy-Wouthuysen Transformation :
Here, we start from the Hamiltonian with the gravitational potential

$$H = -i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{GmM}{r} \right)\beta. \qquad (2.12)$$

This Hamiltonian can be rewritten in terms of the Foldy-Wouthuysen transformation which is somewhat a complicated and tedious procedure involved, though it can be done in a straightforward way [?]. In this case, the non-relativistic Hamiltonian should be obtained as

$$H = m + \frac{\boldsymbol{p}^2}{2m} - \frac{GmM}{r} + \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2 - \frac{1}{2m^2}\frac{GMm}{r^3}(\boldsymbol{s}\cdot\boldsymbol{L}) \qquad (2.13)$$

which is kept only up to the order of $\left(\frac{\boldsymbol{p}}{m}\right)^2 \frac{GM}{r}$.

2.3.5 Classical Limit of Hamiltonian with Gravity

Here, we should calculate the classical equation of motion from the non-relativistic Hamiltonian in quantum mechanics. In this case, the Hamiltonian which is only relevant to the present discussion can be written as

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{GmM}{r} + \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2. \tag{2.14}$$

This can be reduced to the Newton equation by making the expectation values of operators in quantum theory in terms of the Ehrenfest theorem. In this case, we approximate the products by the factorization in the following way

$$\left\langle \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2 \right\rangle = \left\langle \frac{1}{2m^2}\frac{GmM}{r} \right\rangle \left\langle \boldsymbol{p}^2 \right\rangle \tag{2.15}$$

which must be a good approximation in the classical mechanics application. In addition, we make use of the Virial theorem

$$\left\langle \frac{\boldsymbol{p}^2}{m} \right\rangle = -\left\langle V \right\rangle . \tag{2.16}$$

Therefore, we finally obtain the following additional potential

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{2.17}$$

which is a new gravitational potential in classical mechanics. The derivation of the additional potential is similar to the Zeeman effects in that both interactions appear in the non-relativistic reduction as the higher order terms of coupling constant.

2.4 Predictions of New Gravity Model

By now, a new gravity model is constructed, and as a byproduct, there appears the additional gravitational potential. This is a very small term, but its effect can be measurable. Indeed, this is the relativistic effect which becomes

$$\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8} \tag{2.18}$$

for the earth revolution around the sun. On the other hand, the leap second of the earth revolution is found to be

$$\left(\frac{\Delta T}{T}\right) \sim 2 \times 10^{-8} \tag{2.19}$$

which is just the same order of magnitude as the relativistic effect. Therefore, as we see later, it is natural that the leap second value can be understood by the additional potential of the new gravity model.

2.4.1 Period Shifts in Additional Potential

In the new gravity model, there appears the additional potential in addition to the normal gravitational potential. In the case of the earth revolution around the sun, this potential is written as

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{2.20}$$

where the second term is the additional potential [2]. Here, G and c denote the gravitational constant and the velocity of light, respectively. m and M correspond to the masses of the earth and the sun, respectively.

• Non-integrable Potential :
It should be important to note that the additional potential should be a non-integrable, and therefore, the treatment should be done in terms of the perturbation theory. In this case, the Newton equation with the perturbative procedure of the additional potential can be solved, and the period T of the revolution is written as

$$\omega T \simeq 2\pi(1 + 2\eta) \tag{2.21}$$

where η is given as

$$\eta = \frac{G^2 M^2}{c^2 R^4 \omega^2}. \tag{2.22}$$

Here, R is the average radius of the earth orbit. The angular velocity ω is related to the period T by

$$\omega = \frac{2\pi}{T}. \tag{2.23}$$

The period shift due to the additional potential becomes

$$\frac{\Delta T}{T} = 2\eta \tag{2.24}$$

which is the delay of the period of the revolution [2, 5].

2.4.2 Period Shifts of Earth Revolution (Leap Second)

In the earth revolution, the orbit radius, the mass of the sun and the angular velocity can be written as

$$R = 1.496 \times 10^{11}\ \text{m}, \quad M = 1.989 \times 10^{30}\ \ \text{kg}, \quad \omega = 1.991 \times 10^{-7}. \tag{2.25}$$

In this case, the period shift becomes

$$\frac{\Delta T}{T} = 2\eta \simeq 1.981 \times 10^{-8}. \tag{2.26}$$

Therefore, the period of the earth revolution per year amounts to

$$\Delta T_{N.G.} = 0.621 \quad [\text{s/year}] \tag{2.27}$$

which is a delay. This suggests that the corrections must be necessary in terms of the leap second.

• Leap Second :
In fact, the leap second corrections have been made for more than 40 years. The first leap second correction started from June 1972, and for 40 years, people made corrections of 25 second. Therefore, the average leap second per year becomes

$$\Delta T_{N.G.}^{Obs} \simeq 0.625 \pm 0.013 \quad [\text{s/year}] \tag{2.28}$$

which agrees perfectly with the prediction of eq.(2.27).

• Definition of Newcomb Time :
Newcomb defined the time series of second in terms of the earth revolution period. However, the recent measurement of time in terms of atomic clock turns out to deviate from the Newcomb time [6]. This deviation should be due to the relativistic effects, and indeed this deviation can be understood by the additional potential of gravity.

2.4.3 Mercury Perihelion Shifts

For a long time, people believed that the Mercury perihelion shifts can be understood by the higher order effects of general relativity. However, it is proved that there should be no perihelion shifts for one period of the earth revolution.

Instead, there should be the Mercury perihelion shifts which may arise from the effects of other planets such as Jupiter if we can measure the perihelion shifts for some long period of revolutions. Concerning the Mercury perihelion shifts, however, the measurements as well as the calculations of the effects from other planets should be carried out more carefully. After the calculation of Newcomb in the 19 century, no careful calculation on the perihelion shifts has been done until now.

2.4.4 Retreat of Moon

The moon is also affected by the additional potential of gravity from the earth. The shifts of the moon orbit can be expressed just in the same way as the earth revolution. In this case, η can be written as

$$\eta = \frac{G^2M^2}{c^2R^4\omega^2}. \tag{2.29}$$

Here, R is the radius of the moon orbit. M and ω denote the mass of the earth and the angular velocity, respectively. They are written as

$$R = 3.844 \times 10^8 \quad \text{m}, \quad M = 5.974 \times 10^{24} \quad \text{kg}, \quad \omega = 2.725 \times 10^{-6} \tag{2.30}$$

Therefore, the period shift becomes

$$\frac{\Delta T}{T} = 2.14 \times 10^{-11}. \tag{2.31}$$

Now, we should carry out the calculation as to how the orbit can be shifted, and the shift of the angle can be written as

$$\Delta\theta = 4\pi\eta. \tag{2.32}$$

Thus, the orbit shift $\Delta\ell_m$ can be written as

$$\Delta\ell_m = R\Delta\theta \simeq 0.052 \quad \text{m} \tag{2.33}$$

and therefore, the shift per year becomes

$$\Delta\ell_{m\ (one\ year)} = \Delta\ell_m \times \frac{3.156 \times 10^7}{2.36 \times 10^6} \simeq 69.5 \quad \text{cm}. \tag{2.34}$$

• Calculated Results of Retreat of Moon :
Since the orbit of the moon is ellipse, the orbit shift can be seen as if it were retreated [9]. The orbit is described by

$$r = \frac{R}{1+\varepsilon\cos\theta}. \tag{2.35}$$

In addition, the eccentricity is quite small ($\varepsilon = 0.055$) and therefore, we can rewrite the above equation as

$$r \simeq R(1-\varepsilon\cos\theta). \tag{2.36}$$

Thus, the orbit shift Δr at $\theta \simeq \frac{\pi}{2}$ becomes per year

$$\Delta r \simeq R\Delta\theta\ \varepsilon \simeq \Delta\ell_{m\ (one\ year)}\ \varepsilon\ \simeq\ 3.8 \quad \text{cm} \tag{2.37}$$

On the other hand, the observed value of the retreat shift of the moon orbit is

$$\Delta r_m^{obs} \simeq 3.8 \quad \text{cm} \tag{2.38}$$

which agrees very well with the prediction.

• Retreat Shift is not Real! :
It should be noted that this observation is only possible by making use of the Doppler shift measurement. This is not a direct measurement of the moon orbit distance which is not possible due to the uncertainty of the accuracy of light velocity

$$c = (2.99792458\ \pm 0.000000012) \times 10^8 \quad \text{cm/s}. \tag{2.39}$$

The accuracy of the orbit shift $\Delta r_m^{obs} \simeq 3.8$ cm is at the order of 10^{-10} while the light velocity is measured only up to 10^{-8} accuracy. This means that the shift of the orbit radius is just the instantaneous and apparent effect.

2.5 Summary

The new gravity theory of eq.(2.7)) can naturally lead to the Dirac equation of eq.(2.3). This is very important in modern physics since we have now the Dirac equation with the gravitational potential properly taken into account. This Dirac equation can be reduced to the non-relativistic Hamiltonian which then gives rise to the Newton equation with the gravitational potential, and this new equation should contain a new gravitational potential as the additional potential.

• Massless Scalar Field :
The fact that the gravity is described by the massless scalar field can give rise to some important effects on the non-relativistic reduction. This is in contrast to the Coulomb case, but rather similar to the non-relativistic reduction of the vector potential case. In the non-relativistic reduction of the vector potential term in the Hamiltonian, we find new terms such as Zeeman effects or spin-orbit interactions. In the same way, in the non-relativistic reduction of the scalar potential term in the Hamiltonian, we find the new additional potential. In fact, this new additional potential can reproduce the leap second of the earth revolution.

• Inertial Mass and Gravitational Mass :
From experiments, it is known that the inertial mass and gravitational mass are just the same. This equivalence of two masses is taken to be one of the grounds in constructing the general relativity. On the other hand, this equivalence is derived as a natural consequence in the new gravity model. This is one of the strong reasons why this new gravity model is a correct theory of gravity.

Chapter 3

Non-integrable Potential

When the non-integrable potential appears as the small perturbation on the Newton equation, what should be the best way to take into account this small potential effect?

3.1 Non-integrable Potential

Here we discuss the physical effects of the non-integrable potential. The additional potential from the new gravity model has the shape of $\frac{B_0}{r^2}$, and, therefore, we can write the non-integrable potentials into the simple shape in the following way

$$V_a(r) = \frac{q}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{3.1}$$

where

$$q = \begin{cases} -6 & \textbf{for General Relativity} \\ 1 & \textbf{for New Gravity} \end{cases} . \tag{3.2}$$

In this case, the differential equation for the orbit with the additional potential becomes

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2}. \tag{3.3}$$

This equation can be solved exactly and the effect due to the correction appears in $\cos\varphi$ term and is written as

$$r = \frac{A_g}{1 + \varepsilon\cos\left(\frac{L_g}{\ell}\varphi\right)} \tag{3.4}$$

where A_g and L_g are given as

$$A_g = \frac{L_g^2}{GMm^2}, \quad L_g \equiv \sqrt{\ell^2 + \frac{qG^2M^2m^2}{c^2}} \equiv \ell\sqrt{1+\eta} \simeq \ell\left(1+\frac{1}{2}\eta\right). \tag{3.5}$$

Here, the η is defined as

$$\eta \equiv \frac{qG^2M^2}{c^2R^4\omega^2} \tag{3.6}$$

which is a very small number. It is around 10^{-8} for the planet motion such as the earth or Mercury.

3.1.1 Effects of Non-integrable Potential on Solution

The solution of eq.(3.4) has a serious problem in that the orbit is not closed. This is quite well known that the potential with the non-integrable shape such as $V_c(r) = \frac{C}{r^2}$ gives rise to the orbit which is not closed. It is, of course, clear that this type of orbits should not happen in nature.

The abnormal behavior of the solution eq.(3.4) can also be seen from the following term

$$\cos\left(\frac{L_g}{\ell}\varphi\right) \simeq \cos(\varphi + \frac{1}{2}\eta\varphi). \tag{3.7}$$

It should be interesting to see that this term cannot be described in terms of the cartesian coordinates of $x = r\cos\varphi,\ y = r\sin\varphi$. In fact, $\cos(\varphi + \frac{1}{2}\eta\varphi)$ term becomes

$$\cos(\varphi + \frac{1}{2}\eta\varphi) = \frac{x}{r}\cos\frac{1}{2}\eta\varphi - \frac{y}{r}\sin\frac{1}{2}\eta\varphi \tag{3.8}$$

and there is no way to transform the $\cos\frac{1}{2}\eta\varphi$ term into $x,\ y$ coordinates even though we started from this cartesian coordinate. This is very serious since the solution expressed by polar coordinates cannot be written any more in the cartesian coordinates. This is related to the fact that the orbit is not closed due to the non-integrable potential effects.

3.1.2 Discontinuity of Orbit

The effect of the non-integral potential can be further seen as the discontinuity of the orbit trajectory since the orbit is not closed. In order to see

this discontinuity of the orbit, we first start from the orbit solution with the non-integral potential, which is eq.(3.4)

$$r = \frac{A_g}{1 + \varepsilon \cos\left(1 + \frac{1}{2}\eta\right)\varphi}.$$

In this case, we find the radius r at $\varphi = 0$ and $\varphi = 2\pi$ as

$$r = \frac{A_g}{1+\varepsilon}, \qquad \varphi = 0 \tag{3.9}$$

$$r = \frac{A_g}{1+\varepsilon\cos\pi\eta}, \qquad \varphi = 2\pi. \tag{3.10}$$

Therefore the difference Δr becomes

$$\Delta r \equiv r_{(\varphi=2\pi)} - r_{(\varphi=0)} \simeq \frac{1}{2}A_g\pi^2\eta^2\varepsilon \simeq 0.15 \text{ cm} \tag{3.11}$$

for the Mercury orbit case of the general relativity as an example. This means that the orbit is discontinuous when φ becomes 2π. This is not acceptable for the classical mechanics, and indeed it disagrees with the observation. In addition, eq.(3.4) cannot generate the perihelion shift, and this can be easily seen from the orbit trajectory of eq.(3.4).

3.2 Perturbative Treatment of Non-integrable Potential

Here we should present a perturbative treatment of the non-integrable potential. This must be the only way to reliably treat the non-integrability in classical mechanics.

3.2.1 Integrable Expression

The equation for the orbit determination becomes

$$\begin{aligned} \frac{dr}{d\varphi} &= \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2}+\frac{2m\alpha}{\ell^2 r}-\frac{1}{r^2}-\frac{q}{\ell^2c^2}\left(\frac{GmM}{r}\right)^2} \\ &= r^2\sqrt{1+\eta}\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}}. \end{aligned} \tag{3.12}$$

Therefore, we can rewrite the above equation as

$$\sqrt{1+\eta}\,d\varphi = \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}}}. \tag{3.13}$$

Here we note that $\eta = \frac{q}{\ell^2c^2}\left(GmM\right)^2$ is a very small number which is of the order $\eta \sim 10^{-8}$. Now in order to keep the effect of the non-integrable potential in terms of integrable expression, we should make an approximation as

$$\sqrt{1+\eta}\,d\varphi \simeq d\varphi. \tag{3.14}$$

The reason why we should make this approximation is because we should consider the dynamical effect as the perturbation while the η in the right hand side of eq.(3.13) should only change the value of constants such as E or α in the differential equation. In this way, the equation to determine the orbit becomes

$$\frac{dr}{d\varphi} = r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}} \tag{3.15}$$

which gives the right orbit solution. Now the orbit is closed, and the solution can be written as

$$r = \frac{A_g}{1+\varepsilon\cos\varphi} \tag{3.16}$$

where A_g is given as

$$A_g = \frac{\ell^2}{GMm^2}(1+\eta). \tag{3.17}$$

Note that the ε is also changed due to the η term, but here we can safely neglect this effect since it does not play any role for physical observables. Therefore, the effect of the additional potential is to change the radius A_g of the orbit even though this change is very small indeed. Now eq.(3.16) clearly shows that there is no perihelion shift, and this is very reasonable since the additional potential cannot shift the main axis of the orbit.

3.2.2 Higher Order Effect of Perturbation

Here we should estimate the higher order effect of the perturbation in eq.(3.13). Denoting the solution of eq.(3.16) by $r^{(0)}$

$$r^{(0)} = \frac{A_g}{1+\varepsilon\cos\varphi}$$

and the perturbative part of the radius by r' ($r = r^{(0)} + r'$), we can write the equation for r' as

$$\frac{dr'}{d\varphi} = \frac{1}{2}\eta(r^{(0)})^2\sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r^{(0)}} - \frac{1}{(r^{(0)})^2}} \tag{3.18}$$

where the right side depends only on φ. Here, we should make a rough estimation and only consider the case in which the eccentricity ε is zero. In this case, the right side does not depend on the variable ε, and thus we can prove that the right side is zero. Therefore, the higher order correction of r' should be proportional to the eccentricity ε and can be written as

$$r' \simeq C_0\eta\varepsilon A_g \tag{3.19}$$

where C_0 should be some numerical constant. For the earth revolution, the value of ε is very small ($\varepsilon \simeq 0.0167$) and thus we can safely ignore this higher order perturbative effect.

Chapter 4

Planet Effects on Mercury Perihelion

In this Appendix, we discuss the Mercury perihelion shifts which should come from the gravitational interactions between Mercury and other planets such as Jupiter or Saturn. This calculation can be carried out in the perturbation theory of the Newton dynamics, which is rather new to the classical mechanics. Here, we should compare the numerical results with those calculated by Newcomb in 1898.

4.1 Planet Effects on Mercury Perihelion

The motion of the other planets should affect on the Mercury orbits. However, this is the three body problems, and thus it is not easy to solve the equation of motion in an exact fashion. Here, we develop the perturbative treatment of the other planet motions. Suppose Mercury and the planet (Jupiter) are orbiting around the sun, and in this case, the Lagrangian can be written as

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{GmM}{r} + \frac{1}{2}m_w\dot{\boldsymbol{r}_w}^2 + \frac{Gm_wM}{r_w} + \frac{Gmm_w}{|\boldsymbol{r}-\boldsymbol{r}_w|} \tag{4.1}$$

where $(m,\ \boldsymbol{r})$ and $(m_w,\ \boldsymbol{r}_w)$ denote the mass and coordinate of Mercury and the planet, respectively. The last term in the right side of eq.(4.1) is the gravitational potential between Mercury and the planet, and therefore, it should be much smaller than the gravitational force from the sun.

4.1.1 The Same Plane of Planet Motions

Here, we assume that the motion of Mercury and the planet must be in the same plane, and therefore we rewrite the Lagrangian in terms of polar coordinates in two dimensions

$$\begin{aligned} L &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GmM}{r} + \frac{1}{2}m_w(\dot{r_w}^2 + r_w^2\dot{\varphi_w}^2) + \frac{Gm_wM}{r_w} \\ &+ \frac{Gmm_w}{\sqrt{r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w)}}. \end{aligned} \tag{4.2}$$

In this case, the Lagrange equation for Mercury can be written as

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2} - \frac{Gmm_w(r - r_w\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{4.3}$$

$$\frac{d}{dt}(mr^2\dot{\varphi}) = -\frac{GmMrr_w\sin(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{4.4}$$

$$m_w\ddot{r_w} = m_wr_w\dot{\varphi}^2 - \frac{Gm_wM}{r_w^2} - \frac{Gmm_w(r_w - r\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{4.5}$$

$$\frac{d}{dt}(m_wr_w^2\dot{\varphi}) = -\frac{Gm_wMrr_w\sin(\varphi_w - \varphi))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{4.6}$$

4.1.2 Motion of Mercury

If we ignore the interaction between Mercury and the planet, then the Mercury orbit is just given as the Kepler problem, and the equations of motion become

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2} \tag{4.7}$$

$$\frac{d}{dt}(mr^2\dot{\varphi}) = 0. \tag{4.8}$$

Here, the solution of the orbit trajectory is given as

$$r = \frac{A}{1 + \varepsilon\cos\varphi} \tag{4.9}$$

where A and ε are written as

$$A = \frac{\ell^2}{m\alpha}, \qquad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}} \quad \textbf{with} \quad \alpha = GMm \tag{4.10}$$

which should be taken as the unperturbed solution of the revolution orbit.

4.2 Approximate Estimation of Planet Effects

Now we should make a perturbative calculation of the many body Kepler problem by assuming that the interaction between Mercury and the planet is sufficiently small. In this case, we can estimate the effects of other planets on the Mercury orbit. Here we write again the equation of motion for Mercury including the gravity from the other planet

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(r - r_w\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{4.11}$$

Now we replace $r,\ r_w$ by the average orbit radius $R,\ R_w$ in the last term of the right side, and thus, the equation becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(R - R_w\cos(\varphi - \varphi_w))}{(R^2 + R_w^2 - 2RR_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{4.12}$$

Below we present some approximate solution of eq.(4.12).

4.2.1 Legendre Expansion

First we define the last term of eq.(4.12) by F as

$$F(x) \equiv -\frac{Gm_w(R - R_w x)}{(R^2 + R_w^2 - 2RR_w x)^{\frac{3}{2}}}, \qquad \text{with} \quad x = \cos(\varphi - \varphi_w) \tag{4.13}$$

and we make the Legendre expansion

$$F(x) = -\frac{Gm_w R}{(R^2 + R_w^2)^{\frac{3}{2}}} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}}\, x + \cdots\ . \tag{4.14}$$

Therefore we obtain the equation of motion

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}}\cos(\varphi - \varphi_w) \tag{4.15}$$

where the constant term is irrelevant and thus we do not write it above.

4.2.2 Iteration Method

Now we employ the iteration method in order to solve eq.(4.15). First we make use of the solution of the Kepler problem

$$\varphi = \varphi^{(0)} + \omega t \tag{4.16}$$

$$\varphi_w = \varphi_w^{(0)} + \omega_w t \tag{4.17}$$

and thus eq.(4.15) becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{4.18}$$

where b and β should be given as

$$b = \varphi^{(0)} - \varphi_w^{(0)}, \quad \beta = \omega - \omega_w. \tag{4.19}$$

4.2.3 Particular Solution

In order to solve eq.(4.18), we assume that the last term is sufficiently small and therefore r may be written in the following shape as

$$r = r^{(0)} + K \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{4.20}$$

where $r^{(0)}$ denotes the Kepler solution of $r^{(0)} = \frac{A}{1+\varepsilon\cos\varphi}$. Now we insert the solution of eq.(4.20) into eq.(4.18), and we find the solution of K as

$$K = -\frac{1}{\beta^2}. \tag{4.21}$$

Therefore, we obtain the approximate solution as

$$r = r^{(0)} - \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}} \beta^2} \cos(b + \beta t). \tag{4.22}$$

4.3 Effects of Other Planets on Mercury Perihelion

Therefore we should put the Kepler solution for $r^{(0)}$ and thus the Mercury orbit can be written as

$$\begin{aligned} r &= \frac{A}{1+\varepsilon\cos\varphi} - \frac{Gm_w R_w (R_w^2 - 2R^2)}{(R^2+R_w^2)^{\frac{5}{2}}\beta^2}\cos(b+\beta t) \\ &\simeq \frac{A}{1+\varepsilon\cos\varphi + \frac{Gm_w R_w (R_w^2-2R^2)}{R(R^2+R_w^2)^{\frac{5}{2}}(\omega-\omega_w)^2}\cos(b+\beta t)} \end{aligned} \tag{4.23}$$

where we take $A \simeq R$ and also $\beta = \omega - \omega_w$. Here as for ε_w, we take

$$\varepsilon_w \equiv \frac{Gm_w}{RR_w^2(\omega-\omega_w)^2}\frac{\left(1-\frac{2R^2}{R_w^2}\right)}{\left(1+\frac{R^2}{R_w^2}\right)^{\frac{5}{2}}} \tag{4.24}$$

and using $b + \beta t = \varphi - \varphi_w$, we obtain

$$r \simeq \frac{A}{1+\varepsilon\cos\varphi + \varepsilon_w\cos(\varphi-\varphi_w)}. \tag{4.25}$$

This equation suggests that the Mercury perihelion may well be affected by the planet motions.

4.3.1 Numerical Evaluations

Now we calculate the Mercury perihelion shifts due to the planet motions such as Jupiter or Venus. In order to do so, we first rewrite $\varepsilon\cos\varphi + \varepsilon_w\cos(\varphi-\varphi_w)$ terms as

$$\varepsilon\cos\varphi + \varepsilon_w\cos(\varphi-\varphi_w) = c_1\cos\varphi + c_2\sin\varphi = \sqrt{c_1^2+c_2^2}\cos(\varphi+\delta) \tag{4.26}$$

where c_1 and c_2 are defined as

$$c_1 = \varepsilon + \varepsilon_w\cos\varphi_w \tag{4.27}$$

$$c_2 = \varepsilon_w\sin\varphi_w. \tag{4.28}$$

Here $\cos\delta$ can be written as

$$\cos\delta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}. \tag{4.29}$$

Further, ε_w is much smaller than ε and thus eq.(4.29) becomes

$$\cos\delta = \frac{\varepsilon + \varepsilon_w \cos\varphi_w}{\sqrt{(\varepsilon + \varepsilon_w \cos\varphi_w)^2 + (\varepsilon_w \sin\varphi_w)^2}} \simeq 1 - \frac{1}{2}\left(\frac{\varepsilon_w}{\varepsilon}\right)^2 \sin^2\varphi_w. \tag{4.30}$$

4.3.2 Average over One Period of Planet Motion

Now we should make the average over one period of planet motion and therefore we find

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\varphi_w \, d\varphi_w = \frac{1}{2}. \tag{4.31}$$

Thus, δ becomes

$$\begin{aligned} \delta &\simeq \frac{\varepsilon_w}{\sqrt{2}\,\varepsilon} \simeq \frac{1}{\sqrt{2}\,\varepsilon}\frac{GM}{R_w^2}\frac{1}{R(\omega-\omega_w)^2}\left(\frac{m_w}{M}\right)\frac{\left(1-\frac{2R^2}{R_w^2}\right)}{\left(1+\frac{R^2}{R_w^2}\right)^{\frac{5}{2}}} \\ &\simeq \frac{R_w\,\omega_w^2}{\sqrt{2}\,\varepsilon R\,(\omega-\omega_w)^2}\left(\frac{m_w}{M}\right)\frac{\left(1-\frac{2R^2}{R_w^2}\right)}{\left(1+\frac{R^2}{R_w^2}\right)^{\frac{5}{2}}} \end{aligned} \tag{4.32}$$

where the planet orbits are taken to be just the circle, for simplicity.

4.3.3 Numerical Results

In order to calculate the effects of the planet motions on the δ, we first write the properties of planets in Table 1. Here, numbers are shown in units of the earth.

Table 1

	Mercury	Venus	Mars	Jupiter	Saturn	Earth	Sun
Orbit Radius	0.387	0.723	1.524	5.203	9.55	1.0	
Mass	0.055	0.815	0.107	317.8	95.2	1.0	332946.0
Period	0.241	0.615	1.881	11.86	29.5	1.0	
ω	4.15	1.626	0.532	0.0843	0.0339	1.0	

In Table 2, we present the calculations of the values δ for one hundred years of averaging and the calculations are compared with the calculated results by Newcomb.

Table 2 The values of δ for one hundred years

Planets	Venus	Earth	Mars	Jupiter	Saturn	Sum of Planets
δ by eq.(4.32)	49.7	27.4	0.77	32.1	1.14	111.1
δ by Newcomb	56.8	18.8	0.51	31.7	1.5	109.3

As one sees, the agreement between the present calculation and Newcomb results is surprisingly good [6]. Here we do not verify the calculation of Newcomb for the other planet effects on the Mercury perihelion shifts, and instead we simply employ his calculated results.

4.3.4 Comparison with Experiments

The observed values of the Mercury perihelion shifts are often quoted in some of the old textbooks. However, it should be very difficult to find some reliable numbers of the Mercury perihelion shifts since these values are determined for 100 years of observation period in 19 century. In this respect, the comparison between the calculation and observation should be a homework problem for readers.