Appendix A

Introduction to Field Theory

This Appendix is intended for readers who may not be very familiar with the field theory terminology. In particular, the basic notations which are used in this textbook are explained in detail. The notation in physics is important since it is just like the language with which all the communications become possible. Therefore, the notation must be defined well, but readers should remember them all.

In quantum field theory, the basic concept is, for sure, based on quantum mechanics. Therefore, we explain some of the important ingredients in quantum mechanics. The most difficult part of quantum mechanics is the quantization itself, and the quantization procedure which is often called the *first quantization* is consistent with all the experiments even though there is no fundamental principle that may lead to the concept of the quantization. Also, the Schrödinger field is described in terms of the non-relativistic field theory, and apart from the kinematics, the behavior of the Schrödinger field from the field theoretical point of view should be similar to the relativistic field.

The relativistic quantum mechanics of fermions is described such that readers may understand and remember it all. Apart from the creation and annihilation of particles, the Dirac theory can describe physics properly. Some properties of the hydrogen atom are explained.

The relativistic boson fields are also discussed here. However, we present questions rather than descriptions of the boson field properties since unfortunately these questions are not answered well in this textbook. On the other hand, it is by now quite possible that the Klein–Gordon equation may not be derived from the fundamental principle any more [82, 62]. These conceptual problems are still not very well organized in this textbook.

Maxwell equation is reviewed here rather in detail from the point of view of the gauge field theory. In this textbook, the Maxwell equation is considered to be most fundamental for all.

Regularization and renormalization are briefly discussed in terms of physical observables. In addition, we describe the path integral formulation since it is an interesting tool. However, it is not very useful for practical calculations, apart from the derivation of Feynman diagrams. In this regard, the path integral formulation should be taken as an alternative

tool which is expressed in terms of many dimensional integrals rather than the differential equations in order to obtain any physical observables.

In Appendix H, we present a new concept of the first quantization itself. Mathematically, there is nothing new in the new picture, but the understanding of quantum mechanics may well have a conceptual change in future physics. At least, it should be easy to understand why one can replace the energy and momentum by the differential operators in deriving the Schrödinger equation while there is a good reason to believe that the Klein–Gordon equation for elementary fields cannot be derived from the fundamental principle.

Finally, we review briefly the renormalization scheme in QED. This is well explained in the standard field theory textbooks, and there is no special need for the presentation of the renormalization in QED. Here, we stress that the renormalization in QED itself is well constructed since the Fock space of the unperturbed Hamiltonian is prepared in advance.

Notations in Field Theory

In field theory, one often employs special notations which are by now commonly used. In this Appendix, we explain some of the notations which are particularly useful in field theory.

A.1 Natural Units

In this text, we employ the natural units because of its simplicity

$$c = 1, \quad \hbar = 1. \tag{A.1.1}$$

If one wishes to get the right dimensions out, one should use

$$\hbar c = 197.33 \ \ \mathrm{MeV} \cdot \mathrm{fm}. \tag{A.1.2}$$

For example, pion mass is $m_\pi \simeq 140$ MeV/c^2. Its Compton wave length is

$$\frac{1}{m_\pi} = \frac{\hbar c}{m_\pi c^2} = \frac{197 \ \mathrm{MeV} \cdot \ \mathrm{fm}}{140 \ \mathrm{MeV}} \simeq 1.4 \ \mathrm{fm}.$$

The fine structure constant α is expressed by the coupling constant e which is defined in some different ways

$$\alpha = e^2 = \frac{e^2}{\hbar c} = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.036}\,.$$

Some constants:

Electron mass: $m_e = 0.511$ MeV/c^2

Muon mass: $m_\mu = 105.66$ MeV/c^2

Proton mass: $M_p = 938.28$ MeV/c^2

Bohr radius: $a_0 = \dfrac{1}{m_e e^2} = 0.529 \times 10^{-8}$ cm

Magnetic moments: $\left(\begin{array}{lll} \text{Electron :} & \mu_e = 1.00115965219 & \dfrac{e\hbar}{2m_e c} \\ \text{Theory :} & \mu_e = 1.0011596524 & \dfrac{e\hbar}{2m_e c} \\ \text{Muon :} & \mu_\mu = 1.001165920 & \dfrac{e\hbar}{2m_\mu c} \end{array} \right)$.

A.2 Hermite Conjugate and Complex Conjugate

For a complex c-number A

$$A = a + bi \;\; (a, b : \text{ real}). \tag{A.2.1}$$

its complex conjugate A^* is defined as

$$A^* = a - bi. \tag{A.2.2}$$

Matrix A

If A is a matrix, one defines the hermite conjugate $A^\dagger$

$$(A^\dagger)_{ij} = A^*_{ji}. \tag{A.2.3}$$

Differential Operator $\hat{A}$

If $\hat{A}$ is a differential operator, then the hermite conjugate can be defined only when the Hilbert space and its scalar product are defined. For example, suppose $\hat{A}$ is written as

$$\hat{A} = i\frac{\partial}{\partial x}\,. \tag{A.2.4}$$

In this case, its hermite conjugate $\hat{A}^\dagger$ becomes

$$\hat{A}^\dagger = -i\left(\frac{\partial}{\partial x}\right)^T = i\,\frac{\partial}{\partial x} = \hat{A} \tag{A.2.5}$$

which means $\hat{A}$ is Hermitian. This can be easily seen in a concrete fashion since

$$\langle\psi|\hat{A}\psi\rangle = \int_{-\infty}^{\infty} \psi^\dagger(x) i\,\frac{\partial}{\partial x}\,\psi(x)\,dx = -i\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\,\psi^\dagger(x)\right)\psi(x)\,dx = \langle\hat{A}\psi|\psi\rangle, \tag{A.2.6}$$

where $\psi(\pm\infty) = 0$ is assumed. The complex conjugate of $\hat{A}$ is simply

$$\hat{A}^* = -i\,\frac{\partial}{\partial x} \neq \hat{A}. \tag{A.2.7}$$

Field ψ

If the $\psi(x)$ is a c-number field, then the hermite conjugate $\psi^\dagger(x)$ is just the same as the complex conjugate $\psi^*(x)$. However, when the field $\psi(x)$ is quantized, then one should always take the hermite conjugate $\psi^\dagger(x)$. When one takes the complex conjugate of the field as $\psi^*(x)$, one may examine the time reversal invariance as discussed in Chapter 2.

A.3 Scalar and Vector Products (Three Dimensions) :

Scalar Product

For two vectors in three dimensions

$$\boldsymbol{r} = (x, y, z) \equiv (x_1, x_2, x_3), \quad \boldsymbol{p} = (p_x, p_y, p_z) \equiv (p_1, p_2, p_3) \tag{A.3.1}$$

the scalar product is defined

$$\boldsymbol{r} \cdot \boldsymbol{p} = \sum_{k=1}^{3} x_k p_k \equiv x_k p_k, \tag{A.3.2}$$

where, in the last step, we omit the summation notation if the index k is repeated twice.

Vector Product

The vector product is defined as

$$\boldsymbol{r} \times \boldsymbol{p} \equiv (x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3, x_1 p_2 - x_2 p_1). \tag{A.3.3}$$

This can be rewritten in terms of components,

$$(\boldsymbol{r} \times \boldsymbol{p})_i = \epsilon_{ijk} x_j p_k, \tag{A.3.4}$$

where ϵ_{ijk} denotes anti-symmetric symbol with

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \quad \text{otherwise} = 0.$$

A.4 Scalar Product (Four Dimensions)

For two vectors in four dimensions,

$$x^\mu \equiv (t, x, y, z) = (x_0, \boldsymbol{r}), \quad p^\mu \equiv (E, p_x, p_y, p_z) = (p_0, \boldsymbol{p}) \tag{A.4.1}$$

the scalar product is defined

$$x \cdot p \equiv Et - \boldsymbol{r} \cdot \boldsymbol{p} = x_0 p_0 - x_k p_k. \tag{A.4.2}$$

This can be also written as

$$x_\mu p^\mu \equiv x_0 p^0 + x_1 p^1 + x_2 p^2 + x_3 p^3 = Et - \boldsymbol{r}\cdot\boldsymbol{p} = x\cdot p, \qquad (A.4.3)$$

where x_μ and p_μ are defined as

$$x_\mu \equiv (x_0, -\boldsymbol{r}), \quad p_\mu \equiv (p_0, -\boldsymbol{p}). \qquad (A.4.4)$$

Here, the repeated indices of the Greek letters mean the four dimensional summation $\mu = 0, 1, 2, 3$. The repeated indices of the roman letters always denote the three dimensional summation throughout the text.

A.4.1 Metric Tensor

It is sometimes convenient to introduce the metric tensor $g^{\mu\nu}$ which has the following properties

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (A.4.5)$$

In this case, the scalar product can be rewritten as

$$x\cdot p = x^\mu p^\nu g_{\mu\nu} = Et - \boldsymbol{r}\cdot\boldsymbol{p}. \qquad (A.4.6)$$

A.5 Four Dimensional Derivatives ∂_μ

The derivative ∂_μ is introduced for convenience

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \qquad (A.5.1)$$

where the lower index has the positive space part. Therefore, the derivative ∂^μ becomes

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \qquad (A.5.2)$$

A.5.1 $\hat{p}^\mu$ and Differential Operator

Since the operator $\hat{p}^\mu$ becomes a differential operator as

$$\hat{p}^\mu = (\hat{E}, \hat{\boldsymbol{p}}) = \left(\, i\frac{\partial}{\partial t},\ -i\boldsymbol{\nabla}\right) = i\partial^\mu$$

the negative sign, therefore, appears in the space part. For example, if one defines the current j^μ in four dimension as

$$j^\mu = (\rho, \boldsymbol{j}),$$

then the current conservation is written as

$$\partial_\mu j^\mu = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = \frac{1}{i}\,\hat{p}_\mu j^\mu = 0. \qquad (A.5.3)$$

A.5.2 Laplacian and d'Alembertian Operators

The Laplacian and d'Alembertian operators, Δ and $\Box$ are defined as

$$\Delta \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,,$$
$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta.$$

A.6 γ-Matrices

Here, we present explicit expressions of the γ-matrices in two and four dimensions. Before presenting the representation of the γ-matrices, we first give the explicit representation of Pauli matrices.

A.6.1 Pauli Matrices

Pauli matrices are given as

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (A.6.1)$$

Below we write some properties of the Pauli matrices.

Hermiticity

$$\sigma_1^\dagger = \sigma_1, \quad \sigma_2^\dagger = \sigma_2, \quad \sigma_3^\dagger = \sigma_3.$$

Complex Conjugate

$$\sigma_1^* = \sigma_1, \quad \sigma_2^* = -\sigma_2, \quad \sigma_3^* = \sigma_3.$$

Transposed

$$\sigma_1^T = \sigma_1, \quad \sigma_2^T = -\sigma_2, \quad \sigma_3^T = \sigma_3 \quad (\sigma_k^T = \sigma_k^*).$$

Useful Relations

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \qquad (A.6.2)$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \qquad (A.6.3)$$

A.6.2 Representation of γ-matrices

(a) Two dimensional representations of γ-matrices

$$\text{Dirac}: \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\text{Chiral}: \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) Four dimensional representations of gamma matrices

$$\text{Dirac}: \gamma_0 = \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$$

$$\text{Chiral}: \gamma_0 = \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}.$$

where $\mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

A.6.3 Useful Relations of γ-Matrices

Here, we summarize some useful relations of the γ-matrices.

Anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^5, \gamma^\nu\} = 0. \tag{A.6.4}$$

Hermiticity

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \;\; (\gamma_0^\dagger = \gamma_0, \;\; \gamma_k^\dagger = -\gamma_k), \quad \gamma_5^\dagger = \gamma_5. \tag{A.6.5}$$

Complex Conjugate

$$\gamma_0^* = \gamma_0, \quad \gamma_1^* = \gamma_1, \quad \gamma_2^* = -\gamma_2, \quad \gamma_3^* = \gamma_3, \quad \gamma_5^* = \gamma_5. \tag{A.6.6}$$

Transposed

$$\gamma_\mu^T = \gamma_0\gamma_\mu^*\gamma_0, \quad \gamma_5^T = \gamma_5. \tag{A.6.7}$$

A.7 Transformation of State and Operator

When one transforms a quantum state $|\psi\rangle$ by a unitary transformation U which satisfies

$$U^\dagger U = 1$$

one writes the transformed state as

$$|\psi'\rangle = U|\psi\rangle. \tag{A.7.1}$$

The unitarity is important since the norm must be conserved, that is,

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1.$$

In this case, an arbitrary operator $\mathcal{O}$ is transformed as

$$\mathcal{O}' = U\mathcal{O}U^{-1}. \tag{A.7.2}$$

This can be obtained since the expectation value of the operator $\mathcal{O}$ must be the same between two systems, that is,

$$\langle\psi|\mathcal{O}|\psi\rangle = \langle\psi'|\mathcal{O}'|\psi'\rangle. \tag{A.7.3}$$

Since

$$\langle\psi'|\mathcal{O}'|\psi'\rangle = \langle\psi|U^\dagger\mathcal{O}'U|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

one finds

$$U^\dagger\mathcal{O}'U = \mathcal{O}$$

which is just eq.(A.7.2).

A.8 Fermion Current

We summarize the fermion currents and their properties of the Lorentz transformation. We also give their nonrelativistic expressions since the basic behaviors must be kept in the nonrelativistic expressions. Here, the approximate expressions are obtained by making use of the plane wave solutions for the Dirac wave function.

$$\text{Fermioncurrents}: \begin{pmatrix} \text{Scalar}: & \bar{\psi}\psi \simeq 1 \\ \text{Pseudoscalar}: & \bar{\psi}\gamma_5\psi \simeq \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} \\ \text{Vector}: & \bar{\psi}\gamma_\mu\psi \simeq \left(1, \frac{\boldsymbol{p}}{m}\right) \\ \text{Axialvector}: & \bar{\psi}\gamma_\mu\gamma_5\psi \simeq \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}, \boldsymbol{\sigma}\right) \end{pmatrix}. \tag{A.8.1}$$

Therefore, under the parity $\hat{P}$ and time reversal $\hat{T}$ transformation, the above currents behave as

$$\text{Parity } \hat{P}: \quad \begin{pmatrix} \bar{\psi}'\psi' = \bar{\psi}\hat{P}^{-1}\hat{P}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_5\hat{P}\psi = -\bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\hat{P}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi \end{pmatrix}, \tag{A.8.2}$$

$$\text{Timereversal } \hat{T}: \quad \begin{pmatrix} \bar{\psi}'\psi' = \bar{\psi}\hat{T}^{-1}\hat{T}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_5\hat{T}\psi = \bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\hat{T}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\gamma_5\hat{T}\psi = -\bar{\psi}\gamma_k\gamma_5\psi \end{pmatrix}. \tag{A.8.3}$$

A.9 Trace in Physics

A.9.1 Definition

The trace of $N \times N$ matrix A is defined as

$$\text{Tr}\{A\} = \sum_{i=1}^{N} A_{ii}. \tag{A.9.1}$$

This is simply the summation of the diagonal elements of the matrix A. It is easy to prove

$$\text{Tr}\{AB\} = \text{Tr}\{BA\}. \tag{A.9.2}$$

A.9.2 Trace in Quantum Mechanics

In quantum mechanics, the trace of the Hamiltonian H becomes

$$\text{Tr}\{H\} = \text{Tr}\{UHU^{-1}\} = \sum_{n=1} E_n, \tag{A.9.3}$$

where U is a unitary operator that diagonalizes the Hamiltonian, and E_n denotes the energy eigenvalue of the Hamiltonian. Therefore, the trace of the Hamiltonian has the meaning of the sum of all the eigenvalues of the Hamiltonian.

A.9.3 Trace in $SU(N)$

In the special unitary group $SU(N)$, one often describes the element U^a in terms of the generator T^a as

$$U^a = e^{iT^a}. \tag{A.9.4}$$

In this case, the generator must be hermitian and traceless since

$$\det U^a = \exp\left(\mathrm{Tr}\left\{\ln U^a\right\}\right) = \exp\left(i\,\mathrm{Tr}\left\{T^a\right\}\right) = 1 \tag{A.9.5}$$

and thus

$$\mathrm{Tr}\left\{T^a\right\} = 0. \tag{A.9.6}$$

The generators of $SU(N)$ group satisfy the following commutation relations

$$[T^a, T^b] = iC^{abc}T^c, \tag{A.9.7}$$

where C^{abc} denotes a structure constant in the Lie algebra. The generators are normalized in this textbook such that

$$\mathrm{Tr}\left\{T^aT^b\right\} = \frac{1}{2}\,\delta^{ab}. \tag{A.9.8}$$

A.9.4 Trace of γ-Matrices and $\not{p}$

The Trace of the γ-matrices is also important. First, we have

$$\mathrm{Tr}\left\{1\right\} = 4, \quad \mathrm{Tr}\left\{\gamma_\mu\right\} = 0, \quad \mathrm{Tr}\left\{\gamma_5\right\} = 0. \tag{A.9.9}$$

In field theory, one often defines a symbol of $\not{p}$ just for convenience

$$\not{p} \equiv p_\mu\gamma^\mu.$$

In this case, the following relation holds

$$\not{p}\not{q} = pq - i\sigma_{\mu\nu}p^\mu q^\nu. \tag{A.9.10}$$

The following relations may also be useful

$$\mathrm{Tr}\left\{\not{p}\not{q}\right\} = 4pq, \tag{A.9.11}$$

$$\mathrm{Tr}\left\{\gamma_5\not{p}\not{q}\right\} = 0, \tag{A.9.12}$$

$$\mathrm{Tr}\left\{\not{p}_1\not{p}_2\not{p}_3\not{p}_4\right\} = 4\Big\{(p_1p_2)(p_3p_4) - (p_1p_3)(p_2p_4) + (p_1p_4)(p_2p_3)\Big\}, \tag{A.9.13}$$

$$\mathrm{Tr}\left\{\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4\right\} = 4i\epsilon_{\alpha\beta\gamma\delta}\,p_1^\alpha\,p_2^\beta\,p_3^\gamma\,p_4^\delta. \tag{A.9.14}$$

Basic Equations and Principles

A.10 Lagrange Equation

In classical field theory, the equation of motion is most important, and it is derived from the Lagrange equation. Therefore, we review briefly how we can obtain the equation of motion from the Lagrangian density.

A.10.1 Lagrange Equation in Classical Mechanics

Before going to the field theory treatment, we first discuss the Lagrange equation (Newton equation) in classical mechanics. In order to obtain the Lagrange equation by the variational principle in classical mechanics, one starts from the action S as defined

$$S = \int L(q, \dot{q})\, dt, \tag{A.10.1}$$

where the Lagrangian $L(q, \dot{q})$ depends on the general coordinate q and its velocity $\dot{q}$. At the time of deriving equation of motion by the variational principle, q and $\dot{q}$ are independent as the function of t. This is clear since, in the action S, the functional dependence of $q(t)$ is unknown and therefore one cannot make any derivative of $q(t)$ with respect to time t. Once the equation of motion is established, then one can obtain $\dot{q}$ by time differentiation of $q(t)$ which is a solution of the equation of motion.

The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of q and $\dot{q}$.

$$\begin{aligned} \delta S = \int \delta L(q, \dot{q})\, dt &= \int \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q} \right) dt \\ &= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt = 0, \end{aligned} \tag{A.10.2}$$

where the surface terms are assumed to vanish. Therefore, one obtains the Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0. \tag{A.10.3}$$

A.10.2 Hamiltonian in Classical Mechanics

The Lagrangian $L(q, \dot{q})$ must be invariant under the infinitesimal time displacement ϵ of $q(t)$ as

$$q(t+\epsilon) \to q(t) + \dot{q}\epsilon, \quad \dot{q}(t+\epsilon) \to \dot{q}(t) + \ddot{q}\epsilon + \dot{q}\, \frac{d\epsilon}{dt}\,. \tag{A.10.4}$$

Therefore, one finds

$$\delta L(q, \dot{q}) = L(q(t+\epsilon), \dot{q}(t+\epsilon)) - L(q, \dot{q}) = \frac{\partial L}{\partial q}\, \dot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\, \ddot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\, \dot{q} \frac{d\epsilon}{dt} = 0. \tag{A.10.5}$$

Neglecting the surface term, one obtains

$$\delta L(q, \dot{q}) = \left[\frac{\partial L}{\partial q}\, \dot{q} + \frac{\partial L}{\partial \dot{q}}\, \ddot{q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}\, \dot{q} \right) \right] \epsilon = \left[\frac{d}{dt} \left(L - \frac{\partial L}{\partial \dot{q}}\, \dot{q} \right) \right] \epsilon = 0. \tag{A.10.6}$$

Thus, if one defines the Hamiltonian H as

$$H \equiv \frac{\partial L}{\partial \dot{q}}\, \dot{q} - L \tag{A.10.7}$$

then it is a conserved quantity.

A.10.3 Lagrange Equation for Fields

The Lagrange equation for fields can be obtained almost in the same way as the particle case. For fields, we should start from the Lagrangian density $\mathcal{L}$ and the action is written as

$$S = \int \mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt, \tag{A.10.8}$$

where $\psi(x)$, $\dot{\psi}(x)$ and $\frac{\partial \psi}{\partial x_k}$ are independent functional variables.

The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of ψ, $\dot{\psi}$ and $\frac{\partial \dot{\psi}}{\partial x_k}$,

$$\delta S = \int \delta\mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt = \int \left(\frac{\partial \mathcal{L}}{\partial \psi}\,\delta\psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\delta\dot{\psi} + \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,\delta\left(\frac{\partial \psi}{\partial x_k}\right)\right) d^3r\, dt$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\right)\delta\psi\, d^3r\, dt = 0, \tag{A.10.9}$$

where the surface terms are assumed to vanish. Therefore, one obtains

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,, \tag{A.10.10}$$

which can be expressed in the relativistic covariant way as

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right). \tag{A.10.11}$$

This is the Lagrange equation for field ψ, which should hold for any independent field ψ.

A.11 Noether Current

If the Lagrangian density is invariant under the transformation of the field with a continuous variable, then there is always a conserved current associated with this symmetry. This is called *Noether current* and can be derived from the invariance of the Lagrangian density and the Lagrange equation.

A.11.1 Global Gauge Symmetry

The Lagrangian density which is discussed in this textbook should have the following functional dependence in general

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi + \mathcal{L}_I\left[\bar{\psi}\psi, \bar{\psi}\gamma_5\psi, \bar{\psi}\gamma_\mu\psi\right].$$

This Lagrangian density is obviously invariant under the global gauge transformation

$$\psi' = e^{i\alpha}\psi, \quad \psi'^{\dagger} = e^{-i\alpha}\psi^{\dagger}, \tag{A.11.1}$$

where α ia a real constant. Therefore, the Noether current is conserved in this system. To derive the Noether current conservation for the global gauge transformation, one can consider the infinitesimal global transformation, that is, $|\alpha| \ll 1$. In this case, the transformation becomes

$$\psi' = \psi + \delta\psi, \quad \delta\psi = i\alpha\psi. \tag{A.11.2a}$$

$$\psi'^{\dagger} = \psi^{\dagger} + \delta\psi^{\dagger}, \quad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}. \tag{A.11.2b}$$

Invariance of Lagrangian Density

Now, it is easy to find

$$\delta\mathcal{L} = \mathcal{L}(\psi', \psi'^{\dagger}, \partial_\mu\psi', \partial_\mu\psi'^{\dagger}) - \mathcal{L}(\psi, \psi^{\dagger}, \partial_\mu\psi, \partial_\mu\psi^{\dagger}) = 0. \tag{A.11.3a}$$

At the same time, one can easily evaluate $\delta\mathcal{L}$

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\left(\partial_\mu\psi\right) + \frac{\partial\mathcal{L}}{\partial\psi^{\dagger}}\delta\psi^{\dagger} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\delta\left(\partial_\mu\psi^{\dagger}\right) \\
&= i\alpha\left[\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\mu\psi - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\right)\psi^{\dagger} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\partial_\mu\psi^{\dagger}\right] \\
&= i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\psi^{\dagger}\right] = 0,
\end{aligned} \tag{A.11.3b}$$

where the equation of motion for ψ is employed.

Current Conservation

Therefore, if one defines the current j_μ as

$$j^\mu \equiv -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\psi^{\dagger}\right] \tag{A.11.4}$$

then one has

$$\partial_\mu j^\mu = 0. \tag{A.11.5}$$

For Dirac fields with electromagnetic interactions or self-interactions, one can obtain as a conserved current

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{A.11.6}$$

A.11.2 Chiral Symmetry

When the Lagrangian density is invariant under the chiral transformation,

$$\psi' = e^{i\alpha\gamma_5}\psi \tag{A.11.7}$$

then there is another Noether current. Here, $\delta\psi$ as defined in eq.(A.11.2) becomes

$$\delta\psi = i\alpha\gamma_5\psi. \tag{A.11.8}$$

Therefore, a corresponding conserved current for massless Dirac fields with electromagnetic interactions or self-interactions can be obtained

$$j_5^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\gamma_5\psi = \bar{\psi}\gamma^\mu\gamma_5\psi. \tag{A.11.9}$$

In this case, we have

$$\partial_\mu j_5^\mu = 0 \tag{A.11.10}$$

which is the conservation of the axial vector current. The conservation of the axial vector current is realized for field theory models with massless fermions.

A.12 Hamiltonian Density

The Hamiltonian density $\mathcal{H}$ is constructed from the Lagrangian density $\mathcal{L}$. The field theory models which we consider should possess the translational invariance. If the Lagrangian density is invariant under the translation a^μ, then there is a conserved quantity which is the energy momentum tensor $\mathcal{T}^{\mu\nu}$. The Hamiltonian density is constructed from the energy momentum tensor of $\mathcal{T}^{00}$.

A.12.1 Hamiltonian Density from Energy Momentum Tensor

Now, the Lagrangian density is given as $\mathcal{L}\left(\psi_i, \dot{\psi}_i, \frac{\partial\psi_i}{\partial x_k}\right)$. If one considers the following infinitesimal translation a^μ of the field ψ_i and $\psi_i^\dagger$

$$\psi_i' = \psi_i + \delta\psi_i, \quad \delta\psi_i = (\partial_\nu\psi_i)a^\nu,$$

$$\psi_i^{\dagger\prime} = \psi_i^\dagger + \delta\psi_i^\dagger, \quad \delta\psi_i^\dagger = (\partial_\nu\psi_i^\dagger)a^\nu,$$

then the Lagrangian density should be invariant

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi_i', \partial_\mu\psi_i') - \mathcal{L}(\psi_i, \partial_\mu\psi_i)$$

$$= \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) + \frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,\delta\psi_i^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\delta(\partial_\mu\psi_i^\dagger)\right] = 0. \tag{A.12.1}$$

Making use of the Lagrange equation, one obtains

$$\delta\mathcal{L} = \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}(\partial_\nu\psi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}(\partial_\mu\partial_\nu\psi_i) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\partial_\nu\psi_i\right)\right]a^\nu$$

$$+\sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}(\partial_\nu\psi_i^\dagger) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}(\partial_\mu\partial_\nu\psi_i^\dagger) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\partial_\nu\psi_i^\dagger\right)\right]a^\nu$$

$$= \partial_\mu\left[\mathcal{L}g^{\mu\nu} - \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\partial^\nu\psi_i^\dagger\right)\right]a_\nu = 0. \qquad (A.12.2)$$

Energy Momentum Tensor $\mathcal{T}^{\mu\nu}$

Therefore, if one defines the energy momentum tensor $\mathcal{T}^{\mu\nu}$ by

$$\mathcal{T}^{\mu\nu} \equiv \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\partial^\nu\psi_i^\dagger\right) - \mathcal{L}g^{\mu\nu} \qquad (A.12.3)$$

then $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0.$$

This leads to the definition of the Hamltonian density $\mathcal{H}$ in terms of $\mathcal{T}^{00}$

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i)}\partial^0\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i^\dagger)}\partial^0\psi_i^\dagger\right) - \mathcal{L}. \qquad (A.12.4)$$

A.12.2 Hamiltonian Density from Conjugate Fields

When the Lagrangian density is given as $\mathcal{L}(\psi_i, \dot{\psi}_i, \frac{\partial\psi_i}{\partial x_k})$, one can define the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$ as

$$\Pi_{\psi_i} \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_i}, \quad \Pi_{\psi_i^\dagger} \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_i^\dagger}.$$

In this case, the Hamiltonian density can be written as being consistent with eq.(A.12.4)

$$\mathcal{H} = \sum_i\left(\Pi_{\psi_i}\dot{\psi}_i + \Pi_{\psi_i^\dagger}\dot{\psi}_i^\dagger\right) - \mathcal{L}. \qquad (A.12.5)$$

It should be noted that this way of making the Hamiltonian density is indeed easier to remember than the construction starting from the energy momentum tensor.

Hamiltonian

The Hamiltonian is defined by integrating the Hamiltoian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \left[\sum_i (\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger) - \mathcal{L} \right] d^3r.$$

A.12.3 Hamiltonian Density for Free Dirac Fields

For a free Dirac field with its mass m, the Lagrangian density becomes

$$\mathcal{L} = \psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j.$$

Therefore, the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$ are obtained

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = \psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0.$$

Thus, the Hamiltonian density becomes

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L} = \bar{\psi}_i \left[-i\gamma_k \partial_k + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \quad (A.12.6)$$

A.12.4 Hamiltonian for Free Dirac Fields

The Hamiltonian H is obtained by integrating the Hamiltonian density over all space and thus can be written as

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi \, d^3r. \quad (A.12.7)$$

In classical field theory, this Hamiltonian is not an operator but is just the field energy itself. However, this field energy cannot be evaluated unless one knows the shape of the field $\psi(x)$ itself. Therefore, one should determine the shape of the field $\psi(x)$ by the equation of motion in the classical field theory.

A.12.5 Role of Hamiltonian

We should comment on the usefulness of the classical field Hamiltonian itself for field theory models. In fact, the Hamiltonian alone is not useful. This is similar to the classical mechanics case in which the Hamiltonian of a point particle itself does not tell a lot. Instead, one has to derive the Hamilton equations in order to calculate some physical properties of the system and the Hamilton equations are equivalent to the Lagrange equations in classical mechanics.

Classical Field Theory

In classical field theory, the situation is just the same as the classical mechanics case. If one stays in the classical field theory, then one should derive the field equation from the Hamiltonian by the functional variational principle as will be discussed in the next section.

Quantized Field Theory

The Hamiltonian of the field theory becomes important when the fields are quantized. In this case, the Hamiltonian becomes an operator, and thus one has to solve the eigenvalue problem for the quantized Hamiltonian $\hat{H}$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{A.12.8}$$

where $|\Psi\rangle$ is called *Fock state* and should be written in terms of the creation and annihilation operators of fermion and anti-fermion. The space spanned by the Fock states is called *Fock space*.

In normal circumstances of the field theory models such as QED and QCD, it is practically impossible to find the eigenstate of the quantized Hamiltonian. The difficulty of the quantized field theory comes mainly from two reasons. Firstly, one has to construct the vacuum state which is composed of infinite many negative energy particles interacting with each other. The vacuum state should be the eigenstate of the Hamiltonian

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle,$$

where E_Ω denotes the energy of the vacuum and it is in general infinity with the negative sign. The vacuum state $|\Omega\rangle$ is composed of infinitely many negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger(s)}_{\boldsymbol{p}}|0\rangle\rangle,$$

where $|0\rangle\rangle$ denotes the null vacuum state. In the realistic calculations, the number of the negative energy particles must be set to a finite value, and this should be reasonable since physical observables should not depend on the properties of the deep negative energy particles. However, it is most likely that the number of the negative energy particles should be, at least, larger than a few thousand for two dimensional field theory models.

The second difficulty arises from the operators in the Hamiltonian which can change the fermion and anti-fermion numbers and therefore can induce infinite series of the transitions among the Fock states. Since the spectrum of bosons and baryons can be obtained by operating the fermion and anti-fermion creation operators on the vacuum state, the Fock space which is spanned by the creation and annihilation operators becomes infinite. In the realistic calculations, the truncation of the Fock space becomes most important, even though it is difficult to find any reasonable truncation scheme.

In this respect, the Thirring model is an exceptional case where the exact eigenstate of the quantized Hamiltonian is found. This is, however, understandable since the Thirring model Hamiltonian does not contain the operators which can change the fermion and anti-fermion numbers.

A.13 Variational Principle in Hamiltonian

When one has the Hamiltonian, then one can derive the equation of motion by requiring that the Hamiltonian should be minimized with respect to the functional variation of the state $\psi(\boldsymbol{r})$.

A.13.1 Schrödinger Field

When one minimizes the Hamiltonian

$$H = \int \left[-\frac{1}{2m} \psi^\dagger \boldsymbol{\nabla}^2 \psi + \psi^\dagger U \psi \right] d^3r \qquad (A.13.1)$$

with respect to $\psi(\boldsymbol{r})$, then one can obtain the static Schrödinger equation.

Functional Derivative

First, one defines the functional derivative for an arbitrary function $\psi_i(\boldsymbol{r})$ by

$$\frac{\delta \psi_i(\boldsymbol{r}')}{\delta \psi_j(\boldsymbol{r})} = \delta_{ij} \delta(\boldsymbol{r} - \boldsymbol{r}'). \qquad (A.13.2)$$

This is the most important equation for the functional derivative, and once one accepts this definition of the functional derivative, then one can evaluate the functional variation just in the same way as normal derivative of the function $\psi_i(\boldsymbol{r})$.

Functional Variation of Hamiltonian

For the condition on $\psi(\boldsymbol{r})$, one requires that it should be normalized according to

$$\int \psi^\dagger(\boldsymbol{r}) \psi(\boldsymbol{r})\, d^3r = 1. \qquad (A.13.3)$$

In order to minimize the Hamiltonian with the above condition, one can make use of the Lagrange multiplier and make a functional derivative of the following quantity with respect to $\psi^\dagger(\boldsymbol{r})$

$$\begin{aligned} H[\psi] = \int \left[-\frac{1}{2m} \psi^\dagger(\boldsymbol{r}') \boldsymbol{\nabla}'^2 \psi(\boldsymbol{r}') + \psi^\dagger(\boldsymbol{r}') U \psi(\boldsymbol{r}') \right] d^3r' \\ -E \left(\int \psi^\dagger(\boldsymbol{r}') \psi(\boldsymbol{r}')\, d^3r' - 1 \right), \end{aligned} \qquad (A.13.4)$$

where E denotes a Lagrange multiplier and just a constant. In this case, one obtains

$$\frac{\delta H[\psi]}{\delta \psi^\dagger(\boldsymbol{r})} = \int \delta(\boldsymbol{r} - \boldsymbol{r}') \left[-\frac{1}{2m} \boldsymbol{\nabla}'^2 \psi(\boldsymbol{r}') + U \psi(\boldsymbol{r}') - E \psi(\boldsymbol{r}') \right] d^3r' = 0. \qquad (A.13.5)$$

Therefore, one finds

$$-\frac{1}{2m} \boldsymbol{\nabla}^2 \psi(\boldsymbol{r}) + U \psi(\boldsymbol{r}) = E \psi(\boldsymbol{r}) \qquad (A.13.6)$$

which is just the static Schrödinger equation.

A.13.2 Dirac Field

The Dirac equation for free field can be obtained by the variational principle of the Hamiltonian eq.(A.12.7). Below, we derive the static Dirac equation in a concrete fashion by the functional variation of the Hamiltonian.

Functional Variation of Hamiltonian

For the condition on $\psi_i(\boldsymbol{r})$, one requires that it should be normalized according to

$$\int \psi_i^\dagger(\boldsymbol{r})(\gamma^0)_{ij}\psi_j(\boldsymbol{r})\, d^3r = 1. \tag{A.13.7}$$

Now, the Hamiltonian should be minimized with the condition of eq.(A.13.7)

$$\begin{aligned} H[\psi_i] = &\int \psi_i^\dagger(\boldsymbol{r}) \left[-i(\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})_{ij} + m(\gamma^0)_{ij}\right] \psi_j(\boldsymbol{r})\, d^3r \\ &-E\left(\int \psi_i^\dagger(\boldsymbol{r})(\gamma^0)_{ij}\psi_j(\boldsymbol{r})\, d^3r - 1\right), \end{aligned} \tag{A.13.8}$$

where E is just a constant of the Lagrange multiplier. By minimizing the Hamiltonian with respect to $\psi_i^\dagger(\boldsymbol{r})$, one obtains

$$(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi(\boldsymbol{r}) - E\psi(\boldsymbol{r}) = 0 \tag{A.13.9}$$

which is just the static Dirac equation for free field.

Appendix B

Non-relativistic Quantum Mechanics

The quantization has two kinds of the procedure, the first quantization and the second quantization. By the first quantization, we mean that the coordinate $\boldsymbol{r}$ and the momentum $\boldsymbol{p}$ of a point particle do not commute with each other. That is,

$$[x_i, p_j] = i\hbar\delta_{ij}.$$

In terms of this quantization procedure, we can obtain the Schrödinger equation by requiring that the particle Hamiltonian should be an operator and therefore the state ψ should be introduced.

There is another quantization procedure, the second quantization, which is the quantization of fields. From the experimental observations of creations and annihilations of particle pairs or photons, one needs to quantize fields. The field quantization is closely connected to the relativistic field equations which inevitably includes anti-particle states (negative energy states in fermion field case). In this respect, one does not have to quantize the Schrödinger field in the non-relativistic quantum mechanics. Therefore, we discuss only the first quantization procedure and problems related to the quantization.

B.1 Procedure of First Quantization

In the standard procedure of the first quantization, the energy E and momentum $\boldsymbol{p}$ are regarded as operators, and the simplest expressions of $\hat{E}$ and $\hat{\boldsymbol{p}}$ are given as

$$\hat{E} \to i\frac{\partial}{\partial t}\,, \quad \hat{\boldsymbol{p}} \to -i\boldsymbol{\nabla}. \tag{B.1.1}$$

For a free point particle with its mass m, the dispersion relation can be written as

$$E = \frac{\boldsymbol{p}^2}{2m}\,. \tag{B.1.2}$$

If one employs the quantization procedure of eq.(B.1.1), then one should prepare some state which receives the operation of eq.(B.1.1). This state is called wave function and is

often denoted as $\psi(\boldsymbol{r}, t)$. In this case, eq.(B.1.2) becomes

$$i\frac{\partial}{\partial t}\,\psi(\boldsymbol{r}, t) = -\frac{1}{2m}\,\boldsymbol{\nabla}^2\psi(\boldsymbol{r}, t) \qquad (B.1.3)$$

which is the Schrödinger equation.

New Picture of First Quantization

At a glance, one may feel that the procedure of eqs.(B.1.1) and (B.1.2) are more fundamental than eq.(B.1.3) itself. However, this is not so trivial. If one looks into the Maxwell equation, then one realizes that the Maxwell equation is already a quantized equation for classical electromagnetic fields. In this respect, the Maxwell equation does not have any corresponding classical equation of motions like the Newton equation. In this sense, eq.(B.1.3) can be regarded as a fundamental equation for quantum mechanics as well, even though one can derive eq.(B.1.3) from eqs.(B.1.1) and (B.1.2).

In fact, in Appendix H, we treat the derivation of the Dirac equation from the Maxwell equation and the local gauge invariance, and there we see that the first quantization of eq.(B.1.1) is not needed and therefore it is not the fundamental principle any more. Instead, the Schrödinger equation is obtained from the non-relativistic reduction of the Dirac equation. In this respect, the derivation of the Schrödinger equation does not involve the first quantization.

B.2 Mystery of Quantization or Hermiticity Problem?

Here, we present a problem related to the quantization in box with the periodic boundary conditions. We restrict ourselves to the one dimensional case, but the result is easily generalized to three dimensions.

B.2.1 Free Particle in Box

By denoting the wave function as

$$\psi(x) = e^{-iEt}u(x)$$

the static Schrödinger equation without interactions is written

$$-\frac{1}{2m}\,\frac{\partial^2 u(x)}{\partial x^2} = Eu(x). \qquad (B.2.1)$$

The solution can be obtained as

$$u(x) = \left\{ \frac{1}{\sqrt{L}}\,e^{ikx},\ \ \frac{1}{\sqrt{L}}\,e^{-ikx} \right\},\ \ \text{with}\ \ E = \frac{k^2}{2m}\,, \qquad (B.2.2)$$

where one puts the particle into a box with its length L. Now, one requires that the wave function $u(x)$ should satisfy the periodic boundary conditions and should be the eigenstate of the momentum. In this case, one has

$$\hat{p}u_k(x) = ku_k(x), \quad k = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \ldots .$$

Therefore, one can write the eigenstate wave function as

$$u_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x}, \quad n = 0, \pm 1, \ldots .$$

B.2.2 Hermiticity Problem

Now, the quantization relation is written

$$\hat{p}x - x\hat{p} = -i \tag{$B.2.3$}$$

and one takes the expectation value of the quantization relation with the wave function $u_n(x)$ and obtains

$$\langle u_n|\hat{p}x - x\hat{p}|u_m\rangle = -i\delta_{nm}. \tag{$B.2.4$}$$

If one makes use of the hermiticity of $\hat{p}$, then one obtains

$$(n-m)\frac{2\pi}{L}\langle u_n|x|u_m\rangle = -i\delta_{nm}. \tag{$B.2.5$}$$

However, the above equation does not hold for $n = m$ since the left hand side is zero while the right hand side is $-i$.

What is wrong with the calculation? The answer is simple, and one should not make use of the hermiticity of the momentum $\hat{p}$ because the surface term at the boundary does not vanish for the periodic boundary condition. In fact, in the above evaluation, the surface term just gives the missing constant of $-i$ for $n = m$. In other words, one can easily show the following equation

$$\langle u_n|\hat{p}x|u_n\rangle = -i + \langle \hat{p}u_n|x|u_n\rangle. \tag{$B.2.6$}$$

It should be noted that the hermiticity of the momentum in the following sense is valid

$$\langle u_n|\hat{p}|u_m\rangle = \langle \hat{p}u_n|u_m\rangle. \tag{$B.2.7$}$$

From this exercise, one learns that the quantization condition of eq.($B.2.3$) should be all right, but the hermiticity of the momentum operator cannot necessarily be justified as long as one employs the periodic boundary conditions for the wave functions. Also, the periodic boundary conditions must be physically acceptable. Therefore, one should be careful for treating the momentum operator and it should be operated always on the right hand side as it is originally meant. In this case, one does not make any mistakes.

This argument must be valid even for a very large L as long as one keeps the periodic boundary conditions. This may look slightly odd, but the free particle should be present

anywhere in the physical space, and therefore one should give up the vanishing of the surface term in the plane wave case. The criteria of right physics must be given from the observation that physical observables should not depend on L. In other words, one should take the value of L much larger than any other scales in the model, and this is called *thermodynamic limit*. In order to obtain any physical observables, one should always take the thermodynamic limit.

B.3 Schrödinger Fields

The Schrödinger equation with a potential $U(\boldsymbol{r})$ is written as

$$i\,\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} = \left(-\frac{1}{2m}\,\boldsymbol{\nabla}^2 + U(\boldsymbol{r})\right)\psi(\boldsymbol{r},t). \tag{B.3.1}$$

From this equation, one can derive the vector current conservation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{B.3.2}$$

where ρ and $\boldsymbol{j}$ are defined as

$$\rho = \psi^\dagger\psi, \quad \boldsymbol{j} = -\frac{i}{2m}\left(\psi^\dagger\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^\dagger)\psi\right). \tag{B.3.3}$$

B.3.1 Currents of Bound State

Now, it is interesting to observe how the currents from this Schrödinger field behave in the realistic physical situations. Since the time dependence of the Schrödinger field ψ is factorized as

$$\psi(\boldsymbol{r},t) = e^{-iEt}u(\boldsymbol{r})$$

the basic properties of the field are represented by the field $u(\boldsymbol{r})$. When the field $u(\boldsymbol{r})$ represents a bound state, then $u(\boldsymbol{r})$ becomes a real field. In this case, the current density $\boldsymbol{j}$ vanishes to zero,

$$\boldsymbol{j} = -\frac{i}{2m}\Big(u(\boldsymbol{r})\boldsymbol{\nabla}u(\boldsymbol{r}) - (\boldsymbol{\nabla}u(\boldsymbol{r}))u(\boldsymbol{r})\Big) = 0. \tag{B.3.4}$$

On the other hand, the probability density of $\rho \equiv |u(\boldsymbol{r})|^2$ is always time-independent, and since the bound state wave function is confined within a limited area of space, the ρ is also limited within some area of space.

B.3.2 Free Fields (Static)

When there is no potential, that is

$$U(\boldsymbol{r}) = 0$$

then the field can be described as a free particle solution. This solution is obtained when the theory is put into a box with its volume V,

$$\psi(\boldsymbol{r},t) = \frac{1}{\sqrt{V}}\, e^{-iEt} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \quad \frac{1}{\sqrt{V}}\, e^{-iEt} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{$B.3.5$}$$

where E is the energy of the particle and $\boldsymbol{k}$ denotes a quantum number which should correspond to the momentum of a particle. In this case, the probability density is finite and constant

$$\rho = \frac{1}{V} \tag{$B.3.6$}$$

while the current $\boldsymbol{j}$ is also a constant and can be written as

$$\boldsymbol{j} = \frac{\boldsymbol{k}}{m} \tag{$B.3.7$}$$

which just corresponds to the velocity of a particle.

Real Field Condition

It is important to note that the Schrödinger field ψ should be a complex function. If one imposes the condition that the field should be real,

$$\psi(\boldsymbol{r},t) = \psi^{\dagger}(\boldsymbol{r},t)$$

then one obtains from the Schrödinger equation eq.(B.1.3)

$$\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} = 0.$$

Therefore, the Schrödinger field ψ should be time independent. In this case, one sees immediately that the energy E must be zero since

$$\left(-\frac{1}{2m}\,\boldsymbol{\nabla}^2 + U(\boldsymbol{r})\right)\psi(\boldsymbol{r},t) = 0.$$

Therefore, the real field condition of ψ is too strong and it should not be imposed before solving the Schrödinger equation. This concept should always hold in the Schrödinger field, and therefore it is most likely true that the same concept should hold for relativistic boson fields as well. However, this statement may not be justified if the Klein–Gordon field should not have any correspondence with the Schrödinger field in the non-relativistic limit.

B.3.3 Degree of Freedom of Schrödinger Field

The Schrödinger field is a complex field. However, the Schrödinger field ψ itself should correspond to one particle. It is clear that one cannot make the following separation of the field into real and imaginary parts

$$\psi(\boldsymbol{r},t) = \rho(\boldsymbol{r},t) e^{i\xi(\boldsymbol{r},t)} \tag{$B.3.8$}$$

and claim that $\rho(\boldsymbol{r},t)$ and $\xi(\boldsymbol{r},t)$ describe two independent fields (particles). When ψ is a complex field, it has right properties as a field, and the current density of the ψ field has a finite value.

B.4 Hydrogen-like Atoms

When the potential $U(\boldsymbol{r})$ is a Coulomb type,

$$U(\boldsymbol{r}) = -\frac{Ze^2}{r} \tag{B.4.1}$$

then the Schrödinger equation can be solved exactly and the Schrödinger field ψ together with the energy eigenvalue E can be obtained as

$$\psi(\boldsymbol{r}) = R_{n\ell}(r)Y_{\ell m}(\theta,\varphi), \tag{B.4.2}$$

$$E_n = -\frac{mZ^2e^4}{2n^2} = -\frac{m}{2n^2}\left(\frac{Z}{137}\right)^2, \tag{B.4.3}$$

where $R_{n\ell}(r)$ and $Y_{\ell m}$ denote the radial wave function and the spherical harmonics, respectively. The principal quantum number n runs as $n = 1, 2, \ldots, \infty$, and ℓ runs as $\ell = 0, 1, 2, \ldots, \infty$, satisfying the condition

$$\ell \leq n-1. \tag{B.4.4}$$

It should be worth writing the explicit shape of the wave functions for a few lowest states of $1s$, $2p$ and $2s$ with the Bohr radius $a_0 = \frac{1}{me^2}$.

$$1s\text{−statee} \ : \quad R_{1s}(r) = \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} 2e^{-\frac{Zr}{a_0}}, \quad Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}},$$

$$2p\text{−state} \ : \quad R_{2p}(r) = \left(\frac{Z}{2a_0}\right)^{\frac{3}{2}} \frac{Zr}{\sqrt{3}\,a_0} e^{-\frac{Zr}{2a_0}},$$

$$\begin{cases} Y_{11}(\theta,\varphi) = -\sqrt{\dfrac{3}{8\pi}}\,\sin\theta e^{i\varphi} \\ Y_{10}(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\,\cos\theta \\ Y_{1-1}(\theta,\varphi) = \sqrt{\dfrac{3}{8\pi}}\,\sin\theta e^{-i\varphi} \end{cases},$$

$$2s\text{−state} \ : \quad R_{2s}(r) = \frac{1}{2\sqrt{2}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\left(2-\frac{Zr}{a_0}\right)e^{-\frac{Zr}{2a_0}}, \quad Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}.$$

B.5 Harmonic Oscillator Potential

The harmonic oscillator potential $U(x)$

$$U(x) = \frac{1}{2}\, m\omega^2 x^2$$

is a very special potential which is often used in quantum mechanics exercise problems since the Schrödinger equation with the harmonic oscillator potential can be solved exactly. However, the harmonic oscillator potential is not realistic since it does not have a free field-like solution. The Schrödinger field in the harmonic oscillator potential is always confined and there is no scattering state solution.

Nevertheless, it should be worth writing solutions of the Schrödinger field ψ with its mass m in the one dimensional harmonic oscillator potential. The Schrödinger equation can be written as

$$\left(-\frac{1}{2m}\,\frac{\partial^2}{\partial x^2} + \frac{1}{2}\, m\omega^2 x^2\right)\psi(x) = E\psi(x). \qquad (B.5.1)$$

In this case, the solution of the Schrödinger field ψ together with the energy eigenvalue E can be obtained as

$$\psi_n(x) = \left(\frac{\alpha^2}{4^n \pi (n!)^2}\right)^{\frac{1}{4}} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}, \qquad (B.5.2a)$$

$$E_n = \omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \qquad (B.5.2b)$$

where α is given as

$$\alpha = \sqrt{m\omega}\,. \qquad (B.5.3)$$

$H_n(\xi)$ denotes the hermite polynomial and is given as

$$H_n(\xi) = (-)^n e^{\xi^2}\,\frac{d^n}{d\xi^n}\, e^{-\xi^2}$$

since $H_n(\xi)$ can be expressed in terms of the generating function as

$$e^{-x^2+2\xi x} = e^{-(x-\xi)^2+\xi^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\, H_n(\xi) x^n.$$

Some of them are given below

$$H_0(\xi) = 1, \quad H_1(\xi) = 2\xi, \quad H_2(\xi) = 4\xi^2 - 2, \qquad (B.5.4a)$$

$$H_3(\xi) = 8\xi^3 - 12\xi, \quad H_4(\xi) = 16\xi^4 - 48\xi^2 + 12. \qquad (B.5.4b)$$

B.5.1 Creation and Annihilation Operators

The Hamiltonian of the one dimensional harmonic oscillator potential

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\, m\omega^2 x^2 \tag{B.5.5}$$

can be rewritten in terms of creation $a^\dagger$ and annihilation a operators

$$a^\dagger = \sqrt{\frac{m\omega}{2}}\, x - \frac{i}{\sqrt{2m\omega}}\, \hat{p}, \quad a = \sqrt{\frac{m\omega}{2}}\, x + \frac{i}{\sqrt{2m\omega}}\, \hat{p} \tag{B.5.6}$$

as

$$\hat{H} = \omega \left(a^\dagger a + \frac{1}{2} \right). \tag{B.5.7}$$

$a^\dagger$ and a satisfies the following commutation relation

$$[a, a^\dagger] = 1 \tag{B.5.8}$$

because of the definition of $a^\dagger$ and a in eq.(B.5.6).

Number Operator $\hat{N}$

By introducing the number operator $\hat{N}$ as

$$\hat{N} = a^\dagger a \tag{B.5.9}$$

one finds

$$[\hat{N}, a] = -a, \quad [\hat{N}, a^\dagger] = a^\dagger. \tag{B.5.10}$$

With the eigenstate $|\phi_n\rangle$ of the number operator $\hat{N}$ and its eigenvalue n

$$\hat{N}|\phi_n\rangle = n|\phi_n\rangle \tag{B.5.11}$$

one can easily prove the following equations

$$a^\dagger|\phi_n\rangle = \sqrt{n+1}\,|\phi_{n+1}\rangle, \quad a|\phi_n\rangle = \sqrt{n}\,|\phi_{n-1}\rangle. \tag{B.5.12}$$

This indicates that $a^\dagger$ operator increases the quantum number n by one unit while a decreases it in the same way. Therefore, $a^\dagger$ and a are called *creation* and *annihilation* operators, respectively. In addition, one can evaluate the expectation value of the number operator $\hat{N}$ with the state $|\phi_n\rangle$ as

$$n = \langle \phi_n | a^\dagger a | \phi_n \rangle = \|a\phi_n\|^2 \geq 0$$

which shows that the n must be a non-negative value. Therefore, one finds from eq.(B.5.12)

$$a|\phi_0\rangle = 0. \tag{B.5.13}$$

Therefore, one sees that the smallest value of n is $n = 0$. This leads to the constraint for n as

$$n = 0, 1, 2, \ldots . \qquad (B.5.14)$$

Operating the Hamiltonian $\hat{H}$ on $|\phi_n\rangle$, one finds

$$\hat{H}|\phi_n\rangle = \omega\left(\hat{N} + \frac{1}{2}\right)|\phi_n\rangle = \omega\left(n + \frac{1}{2}\right)|\phi_n\rangle = E_n|\phi_n\rangle. \qquad (B.5.15)$$

Thus, the energy E_n can be written as

$$E_n = \omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

which agrees with the result given in eq.(B.5.2). The state $|\phi_n\rangle$ can be easily constructed by operating $a^\dagger$ operator onto the $|\phi_0\rangle$

$$|\phi_n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|\phi_0\rangle. \qquad (B.5.16)$$

It should be noted that the state $|\phi_n\rangle$ is specified by the quantum number n. If one wishes to obtain an explicit expression of the wave function, then one should project the state $|\phi_n\rangle$ onto the $|x\rangle$ or $|p\rangle$ representation as given below.

Explicit Wave Function in x-representation

The wave function $\psi_n(x) \equiv \langle x|\phi_n\rangle$ in the x-representation can be obtained in the following way. First, one solves the differential equation from eq.(B.5.13)

$$\langle x|a|x\rangle\langle x|\phi_0\rangle = \left(\sqrt{\frac{m\omega}{2}}\,x + \frac{1}{\sqrt{2m\omega}}\,\frac{\partial}{\partial x}\right)\psi_0(x) = 0 \qquad (B.5.17)$$

which leads to the ground state wave function

$$\psi_0(x) = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\alpha^2x^2}, \quad \text{with} \;\; \alpha = \sqrt{m\omega}. \qquad (B.5.18)$$

From eq.(B.5.16), one obtains the wave function for an arbitrary state $\psi_n(x)$

$$\begin{aligned}\psi_n(x) &= \frac{1}{\sqrt{n!}}\langle x|\left(a^\dagger\right)^n|x\rangle\langle x|\phi_0\rangle \\ &= \frac{1}{\sqrt{n!}}\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}}\left(\sqrt{\frac{m\omega}{2}}\,x - \frac{1}{\sqrt{2m\omega}}\,\frac{\partial}{\partial x}\right)^n e^{-\frac{1}{2}\,\alpha^2x^2}\end{aligned}$$

which can be shown to be just identical to eq.(B.5.2a).

Appendix C

Relativistic Quantum Mechanics of Bosons

When a particle motion becomes comparable to the velocity of light, then one has to consider the relativistic effects. The dispersion relation between the energy and momentum of the particle with its mass m becomes

$$E = \sqrt{\boldsymbol{p}^2 + m^2}\,. \qquad (C.0.1)$$

If one employs the quantization procedure of eq.(B.1.1),

$$\hat{E} \to i\frac{\partial}{\partial t}\,, \qquad \hat{\boldsymbol{p}} \to -i\boldsymbol{\nabla}$$

then one obtains the following equation from eq.(C.0.1)

$$i\,\frac{\partial}{\partial t}\,\phi = \sqrt{-\boldsymbol{\nabla}^2 + m^2}\,\phi. \qquad (C.0.2)$$

However, it is easy to realize that the differential operator in square root cannot be defined well.

C.1 Klein–Gordon Equation

Therefore, it is essential to rewrite eq.(C.0.1) for the quantization procedure

$$E^2 = \boldsymbol{p}^2 + m^2. \qquad (C.1.1)$$

From this dispersion relation, one obtains

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right)\phi = 0 \qquad (C.1.2)$$

which is the Klein–Gordon equation. Eq.(C.1.2) can describe bosons which must be a spinless particle. This is clear since the field ϕ has only one component, and therefore it should correspond to one component particle which is the spin zero state.

It should be interesting to compare eq.(C.1.2) with the equation for the electromagnetic field eq.(E.5.1) which is the resulting equation from the Maxwell equation. One sees that the Maxwell equation has just the same structure as the Klein–Gordon equation. In this respect, the Maxwell equation does correspond already to the relativistic quantum mechanics. Indeed, in Appendix H, we will see that the Dirac equation can be obtained from the assumption that the Maxwell equation and the local gauge invariance are the most fundamental principles. In this case, one sees that the Klein–Gordon equation cannot be derived from the new principle which shows that the first quantization procedure is not the fundamental principle but is only the result of the Dirac equation as a consequence. Therefore, if one employs this standpoint, then it may well be difficult to justify that the Klein–Gordon equation can be derived as the fundamental equation.

C.2 Scalar Field

Here, we examine whether a real scalar field with a finite mass can exist as a physical observable or not in the Klein–Gordon equation. Normally, one finds that pion with the positive charge is an anti-particle of pion with the negative charge. This can be easily understood if we look into the structure of the pion in terms of quarks. $\pi^{\pm}$ are indeed anti-particle to each other by changing quarks into anti-quarks. Since pion is not an elementary particle, their dynamics must be governed by the complicated quark dynamics. Under some drastic approximations, the motion of pion may be governed by the Klein–Gordon like equation.

C.2.1 Physical Scalar Field

It looks that eq.(C.1.2) contains the negative energy state. However, one sees that eq.(C.1.2) is only one component equation and, therefore the eigenvalue of E^2 can be obtained as a physical observable. There is no information from the Klein–Gordon equation for the energy E itself, but only E^2 as we see it below,

$$\left(-\boldsymbol{\nabla}^2 + m^2\right)\phi = E^2\phi. \qquad (C.2.1)$$

Therefore, if one obtains the eigenvalue of E^2 as

$$E^2 = \alpha, \quad \alpha > 0$$

then one cannot say which one of $E = \sqrt{\alpha}$ or $E = -\sqrt{\alpha}$ should be taken. Both solutions can be all right, but one should take only one of the solutions. Here, we take a positive value of E. In this case, the solution of eq.(C.2.1) should be described just in the same way as the Schrödinger field

$$\phi_{\boldsymbol{k}}(x) = A(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad (C.2.2)$$

which should be an eigenfunction of the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$. The coefficient $A(t)$ can be determined by putting eq.(C.2.2) into eq.(C.1.1). One can easily find that $A(t)$ should be written as

$$A(t) = \frac{1}{\sqrt{V\omega_{\boldsymbol{k}}}}\, a_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t}, \tag{C.2.3}$$

where $a_{\boldsymbol{k}}$ is a constant, and $\omega_{\boldsymbol{k}}$ is given as

$$\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}\,. \tag{C.2.4}$$

The shape of $A(t)$ in eq.(C.2.3) can be also determined from the Lorentz invariance. Now, one sees that eq.(C.2.2) has a right non-relativistic limit. This is indeed a physical scalar field solution of the Klein–Gordon equation.

C.2.2 Current Density

Now, we discuss the current density of the Klein–Gordon field which is defined as

$$\rho(x) = i\left(\phi^\dagger(x)\,\frac{\partial\phi(x)}{\partial t} - \frac{\partial\phi^\dagger(x)}{\partial t}\,\phi(x)\right), \tag{C.2.5a}$$

$$\boldsymbol{j}(x) = -i\left(\phi^\dagger(x)(\boldsymbol{\nabla}\phi(x)) - (\boldsymbol{\nabla}\phi^\dagger(x))\phi(x)\right). \tag{C.2.5b}$$

It should be noted that the current density must be hermitian and therefore the shape of eqs.(C.2.5) is uniquely determined. One cannot change the order between

$$\frac{\partial\phi^\dagger(x)}{\partial t} \;\text{ and }\; \phi(x)$$

in the second term of eq.(C.2.5a).

Classical Real Scalar Field

Now, we come to an important observation that a real scalar field should have a serious problem. The real scalar field $\phi(x)$ can be written as

$$\phi(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2V\omega_k}}\left[a_k e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} + a_k^* e^{i\omega_k t - i\boldsymbol{k}\cdot\boldsymbol{r}}\right], \tag{C.2.6}$$

where V denotes the box. We assume that the $\phi(x)$ is still a classical field, that is, a_k^* and a_k are not operators, but just the c-number.

In this case, it is easy to prove that the current density of $(\rho(x), \boldsymbol{j}(x))$ which is constructed from the real scalar field $\phi(x)$ vanishes to zero

$$\rho(x) = 0, \quad \boldsymbol{j}(x) = 0.$$

This means that the real scalar field cannot propagate classically. This is clear since a real wave function even in the Schrödinger equation cannot propagate. That is, this real state should have the energy $E = 0$.

The basic problem is that one cannot impose the condition that the scalar field should be a real, and this is too strong as a physical condition. Within the classical field theory, the scalar Klein–Gordon field should be reduced to the Schrödinger field when it moves much more slowly than the velocity of light. Therefore, it is quite difficult to make any physical pictures of the real scalar Klein–Gordon field since it is not compatible with the Schrödinger field in the non-relativistic limit.

Quantized Real Scalar Field

Now, we come to the current density when the field is quantized. Below it is shown that the current density of the real scalar field has some problem even if quantized, contrary to a common belief [99].

When one quantizes the boson field of ϕ, then $a_k^\dagger$ and a_k become creation and annihilation operators

$$\hat{\phi}(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2V\omega_k}} \left[a_k e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} + a_k^\dagger e^{i\omega_k t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right]. \qquad (C.2.7)$$

In this case, the current density of eq.(C.2.4) becomes

$$\hat{\rho} = i\left(\hat{\phi}(x)\hat{\Pi}(x) - \hat{\Pi}(x)\hat{\phi}(x)\right) = i\big[\hat{\phi}(x), \hat{\Pi}(x)\big], \qquad (C.2.8)$$

where $\hat{\Pi}(x)$ is a conjugate field of $\hat{\phi}(x)$, that is, $\Pi(x) = \dot{\phi}(x)$. However, the quantization condition of the boson fields with eq.(C.2.7) becomes

$$\big[\hat{\phi}(\boldsymbol{r}), \hat{\Pi}(\boldsymbol{r}')\big]_{t=t'} = i\delta(\boldsymbol{r} - \boldsymbol{r}'). \qquad (C.2.9)$$

Therefore, eq.(C.2.8) becomes

$$\hat{\rho} = i\big[\hat{\phi}(x), \hat{\Pi}(x)\big] = -\delta(\boldsymbol{0}) \qquad (C.2.10)$$

which is infinity. Thus, the current density of the quantized real boson field is divergent after the quantization! Therefore, it is by now obvious that the current density of the real scalar field has an improper physical meaning. This should be related to the fact that the quantized field ϕ is assumed to be a real field which must be a wrong condition in nature.

Physical Scalar Field

The physical scalar field $\phi(x)$ can be written as in eqs.(C.2.2) and (C.2.3)

$$\phi(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{V\omega_{\boldsymbol{k}}}} a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega_{\boldsymbol{k}} t}.$$

In this case, we can calculate the current density for the scalar field with the fixed momentum of $\boldsymbol{k}$ and obtain

$$\rho = \frac{|a_{\boldsymbol{k}}|^2}{V}$$

which is positive definite and finite. Therefore, the physical scalar field does not have any basic problems.

C.2.3 Complex Scalar Field

Since a real scalar field has a difficulty not only in the classical case but also in the quantized case, it should be worth considering a complex scalar field. In this case, the complex scalar field can be written as

$$\hat{\phi}(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2V\omega_k}} \left[a_k e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} + b_k^\dagger e^{i\omega_k t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right]. \qquad (C.2.11)$$

In this case, the current density is well defined and has no singularity when quantized. It is commonly believed that the complex scalar field should describe charged bosons, one which has a positive charge and the other which has a negative charge. However, a question may arise as to where this degree of freedom comes from? By now, one realizes that there is no negative energy solution in the Klein–Gordon equation. If one took into account the negative energy solution, then one should have had field equations of two components. On the other hand, eq.(C.2.11) assumes a scalar field with two components, and not the result of the field equations. It is therefore most important to seek for the two component Klein–Gordon like equation which should be somewhat similar to the Dirac equation.

Boson Number

Since the vector current of the boson field is conserved, one can define the boson number N_B

$$N_B = \int \rho(x)\, d^3r \qquad (C.2.12)$$

which is a conserved quantity. In the Schrödinger field, the N_B is positive definite, but in the Klein–Gordon field of eq.(C.2.11), the N_B is not necessarily positive definite due to the negative energy solutions of the Klein–Gordon equation.

Charge vs. Boson Number

One cannot interpret the boson number as the charge. This is clear since the charge is associated with the coupling constant g of the gauge field, and the negative charge of the positron in QED is associated with the quantum number of the negative energy degree of freedom. Therefore, unless the Klein–Gordon field could take into account the negative energy degree of freedom in a proper manner, there is no way to interpret the negative value of the boson number N_B in terms of proper physics terminology. This cannot be remedied

even if one quantizes the boson field. The quantum number has nothing to do with the field quantization, but it is determined from the equation of motion as well as the symmetry of the Lagrangian density.

C.2.4 Composite Bosons

Pions and ρ-mesons are composed of quark and anti-quark fields. Suppressing the isospin variables, we can describe the boson fields in terms of the Dirac field $\psi_q(x)\chi_{\frac{1}{2}}$ only with the large components, for simplicity

$$\Psi^{(B)} = \psi_q(x_1)\psi_{\bar{q}}(x_2)\big(\chi^{(1)}_{\frac{1}{2}} \otimes \chi^{(2)}_{\frac{1}{2}}\big) = \Phi^{(Rel)}(x)\Phi^{(CM)}(X)\xi_{s,s_z}, \qquad (C.2.13)$$

where $x = x_1 - x_2$, $X = \frac{1}{2}$. Here, $\psi_{\bar{q}}(x)\chi_{\frac{1}{2}}$ denotes the anti-particle field and ξ_{s,s_z} is the spin wave function of the boson. $\Phi^{(Rel)}(x)$ denotes the internal structure of the boson and $\Phi^{(CM)}(X)$ corresponds to the boson field. Now, it is clear that the boson field $\Phi^{(CM)}(X)$ is a complex field.

Neutral scalar field

In the field theory textbooks, the real scalar field is interpreted as a boson with zero charge. But this is not the right interpretation. The charge is a property of the field in units of the coupling constant. The positive and negative charges are connected to the flavor of the scalar fields. For example, a chargeless Schrödinger field, of course, has a finite current density of $\rho(\boldsymbol{r})$. The charge Q of the Schrödinger field is given as $Q = e_0 \int \rho(\boldsymbol{r})\, d^3r$ and for the chargeless field, we simply have $e_0 = 0$, which means that it does not interact with the electromagnetic field due to the absence of the coupling constant.

Schwinger Boson

There is a good example of the physical composite boson which is described in terms of the fermion and anti-fermion operators, and it is a boson in the Schwinger model. The Hamiltonian of the Schwinger model can be bosonized, and it is given in eq.(5.74)

$$H = \frac{1}{2}\sum_p \left\{ \tilde{\Pi}^\dagger(p)\tilde{\Pi}(p) + p^2\tilde{\Phi}^\dagger(p)\tilde{\Phi}(p) + \mathcal{M}^2\tilde{\Phi}^\dagger(p)\tilde{\Phi}(p) \right\}. \qquad (5.74)$$

The boson fields $\tilde{\Phi}(p)$ and its conjugate field $\tilde{\Pi}(p)$ satisfy the bosonic commutation relation

$$\big[\tilde{\Phi}(p), \tilde{\Pi}^\dagger(p')\big] = i\delta_{p,p'}.$$

One can see that the boson field $\tilde{\Phi}(p)$ in the Schwinger model is indeed a complex field.

C.2.5 Gauge Field

The electromagnetic field A_μ is a real vector field which is required from the Maxwell equation, and therefore it has zero current density. However, the gauge field itself is gauge dependent and therefore it is not directly a physical observable. In this case, the energy flow in terms of the Poynting vector becomes a physical quantity. After the gauge fixing and the field quantization, the vector field $\hat{\boldsymbol{A}}(x)$ can be written as

$$\hat{\boldsymbol{A}}(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \boldsymbol{\epsilon}(\boldsymbol{k},\lambda) \left[c_{\boldsymbol{k},\lambda} e^{-ikx} + c^{\dagger}_{\boldsymbol{k},\lambda} e^{ikx} \right], \tag{C.2.14}$$

where $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. Here, $\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)$ denotes the polarization vector. In this case, one-photon state with $(\boldsymbol{k},\lambda)$ becomes

$$\boldsymbol{A}_{\boldsymbol{k},\lambda}(x) = \langle \boldsymbol{k},\lambda | \hat{\boldsymbol{A}}(x) | 0 \rangle = \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \boldsymbol{\epsilon}(\boldsymbol{k},\lambda) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega_{\boldsymbol{k}}t} \tag{C.2.15}$$

which is the eigenstate of the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$. In this respect, the gauge field A_μ is completely different from the Klein–Gordon scalar field. Naturally, the gauge field does not have any corresponding non-relativistic field.

C.3 Degree of Freedom of Boson Fields

The Klein–Gordon equations are obtained by just replacing the momentum and energy by the differential operators. Now, we should examine the number of the degrees of freedom. If the positive and negative energy states should be taken into account, there should be two degrees of freedom. Therefore, the states should be described by a spinor with two components. However, the Klein–Gordon equation has only one component for a real scalar field. This should not be sufficient for describing the boson state if it should contain the negative energy state.

Below we discuss the Klein–Gordon equation with a spinor of two components and present some attempt to obtain a Hamiltonian in the two by two matrix form. One example of the Hamiltonian for a free boson is given in the textbook by Gross [63]

$$H = \begin{pmatrix} m + \dfrac{\boldsymbol{p}^2}{2m} & \dfrac{\boldsymbol{p}^2}{2m} \\ -\dfrac{\boldsymbol{p}^2}{2m} & -\left(m + \dfrac{\boldsymbol{p}^2}{2m} \right) \end{pmatrix}. \tag{C.3.1}$$

In this case, the wave equation becomes

$$i\,\frac{\partial \phi}{\partial t} = \begin{pmatrix} m + \dfrac{\boldsymbol{p}^2}{2m} & \dfrac{\boldsymbol{p}^2}{2m} \\ -\dfrac{\boldsymbol{p}^2}{2m} & -\left(m + \dfrac{\boldsymbol{p}^2}{2m} \right) \end{pmatrix} \phi. \tag{C.3.2}$$

If one writes ϕ as

$$\phi^{\pm} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{r}\pm iEt}$$

then one has for the energy eigenvalue as $E^2 = m^2 + \boldsymbol{k}^2$ which is a right dispersion relation for a boson. However, one notices that the Hamiltonian is not hermite, and in principle this should give rise to some problems when one wishes to extend it to interacting systems.

There are other attempts to write Klein–Gordon equation in the spinor form with two components. However, one can easily prove that, if one has E, $\boldsymbol{p}$, and m with which one wishes to make a Lorentz scalar as

$$H = E\beta_0 + \boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta_1 \qquad (C.3.3)$$

then one needs to have 5 independent elements of the hermite matrices. If the hermite matrices are 2×2, then there are only 4 independent components which in fact correspond to the Pauli matrices. Thus, the two dimensional representation of 5 independent hermite matrices are not possible, and this strongly suggests that the Klein–Gordon equation cannot be rewritten in a physically plausible form.

Appendix D

Relativistic Quantum Mechanics of Fermions

Electron has a spin and its magnitude is $1/2$. The relativistic equation of eq.(C.1.2) has only one component of the field ϕ, and therefore it cannot describe the spin one half particle such as electron. Therefore, one has to consider some other equations for electron, and this equation is discovered by Dirac as we describe below.

D.1 Derivation of Dirac Equation

The procedure of obtaining Dirac equation is rather simple. One starts from eq.(C.1.1) and tries to factorize it into a linear equation for E and $\boldsymbol{p}$. This can be realized as

$$E^2 - \boldsymbol{p}^2 - m^2 = (E - \boldsymbol{p}\cdot\boldsymbol{\alpha} - m\beta)(E + \boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta) = 0, \qquad (D.1.1)$$

where $\boldsymbol{\alpha}$ and β are four by four matrices, and they satisfy the following anti-commutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1,$$

where i and j run $i, j = x, y, z$. Some of the representations are given in Appendix A.6. The relativistic quantum mechanical equation for a free electron discovered by Dirac is given as

$$\left(i\,\frac{\partial}{\partial t} + i\,\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta\right)\psi = 0, \qquad (D.1.2)$$

where ψ denotes the wave function and should have four components since $\boldsymbol{\alpha}$ and β are four by four matrices,

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}. \qquad (D.1.3)$$

In the four components of the wave function, two of them correspond to the spin degrees of freedom and the other two components represent the positive and negative energy states. The static Dirac equation becomes

$$(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+m\beta)\,\psi=E\psi. \tag{D.1.4}$$

If the energy eigenvalue contains the negative energy states, which is indeed the case, then the negative energy states must be physical, in contrast to the boson case. For fermions, the negative energy states play a very important role in quantum field theory.

D.2 Negative Energy States

The negative energy states in the Dirac equation are essential for describing the vacuum of quantum field theory, and Dirac interpreted that the vacuum of quantum field theory must be occupied completely by the negative energy states. Because of the Pauli principle, the vacuum is stable as long as the states are full. If one creates a hole in the vacuum, then this corresponds to a new particle with the same mass of the particle which one considers. This can be easily seen since, from the vacuum energy

$$E_v=-\sum_n\sqrt{\boldsymbol{p}_n^2+m^2}$$

one extracts the state n_0 which is one hole state

$$E_v^{1hole}=-\sum_{n\neq n_0}\sqrt{\boldsymbol{p}_n^2+m^2}\,.$$

Therefore one obtains a hole state energy

$$E^h\equiv E_v^{1hole}-E_v=\sqrt{\boldsymbol{p}_{n_0}^2+m^2} \tag{D.2.1}$$

which means that the hole state has the same mass as the original fermion. Further, if the fermion has a charge e, then the hole state must have an opposite charge which can be seen in the same way as the energy case.

$$\text{hole charge}\quad e_h\equiv\left(\sum_{n\neq n_0}e-\sum_n e\right)=-e. \tag{D.2.2}$$

D.3 Hydrogen Atom

The Dirac equation is most successful for describing the spectrum of hydrogen atom. One writes the Dirac equation for the hydrogen-like atoms as

$$\left(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+m\beta-\frac{Ze^2}{r}\right)\psi=E\psi, \tag{D.3.1}$$

where Z denotes the charge of nucleus. Before presenting the solution of the above equation, we should make some comments on this equation. It is of course clear that there is no system which is composed of one electron, except a free electron state. Therefore, even though eq.(D.3.1) shows a simple one body problem for an electron with the potential which is measured from the coordinate center, the realistic problem must be at least two body problem, a system composed of electron and proton. In addition, there must be some contributions from photons and virtual pairs of electron and positron in the intermediate states. If one wishes to discuss the problem of hydrogen atom in the field theoretical treatment, it becomes extremely difficult. This is clear since, in this case, one has to evaluate the system as a many body problem. Even a reliable treatment of the center of mass effects is a non-trivial issue if one starts from the Dirac equation.

D.3.1 Conserved Quantities

The Dirac Hamiltonian for electron in hydrogen atom is written as

$$H = -i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\beta - \frac{Ze^2}{r}\,. \tag{$D.3.2$}$$

Now, one defines the total angular momentum $\boldsymbol{J}$ and an operator K by

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\,\boldsymbol{\Sigma}, \tag{$D.3.3a$}$$

$$K = \beta(\boldsymbol{\Sigma}\cdot\boldsymbol{L} + 1), \tag{$D.3.3b$}$$

where $\boldsymbol{\Sigma}$ is extended to 4×4 matrix of $\boldsymbol{\sigma}$ and is defined as

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}.$$

In this case, it is easy to prove that $\boldsymbol{J}$ and K commute with the Hamiltonian H

$$[H, \boldsymbol{J}] = 0, \quad [H, K] = 0. \tag{$D.3.4$}$$

Therefore, the energy eigenvalues can be specified by the eigenvalues of $\boldsymbol{J}^2$ and K

$$\boldsymbol{J}^2\psi^{\kappa}_{j,j_m} = j(j+1)\psi^{\kappa}_{j,j_m}, \tag{$D.3.5a$}$$

$$K\psi^{\kappa}_{j,j_m} = \kappa\psi^{\kappa}_{j,j_m}, \tag{$D.3.5b$}$$

where κ takes values according to

$$\kappa = \mp\left(j + \frac{1}{2}\right) \quad \text{for} \quad j = \ell \pm \frac{1}{2}\,. \tag{$D.3.6$}$$

D.3.2 Energy Spectrum

In this case, the energy eigenvalue of the Dirac Hamiltonian in hydrogen atom is given as

$$E_{n,j} = m\left[1 - \frac{(Z\alpha)^2}{n^2 + 2\left(n - (j+\frac{1}{2})\right)\left[\sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2} - (j+\frac{1}{2})\right]}\right]^{\frac{1}{2}}, \quad (D.3.7)$$

where α denotes the fine structure constant and is given as

$$\alpha = \frac{1}{137}\,.$$

The quantum number n runs as $n = 1, 2, \ldots$. The energy $E_{n,j}$ can be expanded to order α^4

$$E_{n,j} - m = -\frac{m(Z\alpha)^2}{2n^2} - \frac{m(Z\alpha)^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \mathcal{O}\left((Z\alpha)^6\right). \quad (D.3.8)$$

The first term in the energy eigenvalue is the familiar energy spectrum of the hydrogen atom in the non-relativistic quantum mechanics.

D.3.3 Ground State Wave Function ($1s_{\frac{1}{2}} - \text{state}$)

It may be worth to write the Dirac wave function of the lowest state in a hydrogen-like atom. We denote the ground state wave function by

$$\psi^{(-1)}_{\frac{1}{2},m_s}(\boldsymbol{r}) \;\text{ for }\; 1s_{\frac{1}{2}}\text{−state}$$

since $\kappa = -1$ and $s = \frac{1}{2}$. The energy eigenvalue $E_{1s_{\frac{1}{2}}}$ is simply written as

$$E_{1s_{\frac{1}{2}}} = m\sqrt{1 - (Z\alpha)^2}\,.$$

The ground state wave function is explicitly given as

$$\psi^{(-1)}_{\frac{1}{2},m_s}(\boldsymbol{r}) = \begin{pmatrix} f^{(-1)}(r) \\ -i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,g^{(-1)}(r)\end{pmatrix}\frac{\chi_{m_s}}{\sqrt{4\pi}}\,, \quad (D.3.9)$$

where χ_{m_s} denotes the two component spinor and is given as

$$\chi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}.$$

$\hat{\boldsymbol{r}}$ is defined as

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}\,.$$

The radial wave functions $f^{(-1)}(r)$ and $g^{(-1)}(r)$ can be analytically written

$$f^{(-1)}(r) = N\rho^{\sqrt{1-(Z\alpha)^2}}\,\frac{e^{-\rho}}{\rho}\,, \tag{D.3.10a}$$

$$g^{(-1)}(r) = N\rho^{\sqrt{1-(Z\alpha)^2}}\,\frac{e^{-\rho}}{\rho}\left[\frac{\epsilon\left(\sqrt{1-(Z\alpha)^2}-1\right)-Z\alpha}{\sqrt{1-(Z\alpha)^2}+1-Z\alpha\epsilon}\right]\,, \tag{D.3.10b}$$

where ρ and ϵ are defined as

$$\rho = \sqrt{m^2 - E^2_{1s_{\frac{1}{2}}}}\; r = Z\alpha m r,$$
$$\epsilon = \sqrt{\frac{m - E_{1s_{\frac{1}{2}}}}{m + E_{1s_{\frac{1}{2}}}}} = \frac{1-\sqrt{1-(Z\alpha)^2}}{Z\alpha}\,.$$

N is a normalization constant and should be determined from

$$\int_0^\infty \left[\left(f^{(-1)}(r)\right)^2 + \left(g^{(-1)}(r)\right)^2\right] r^2\, dr = 1.$$

D.4 Lamb Shifts

The important consequence of the Dirac equation in hydrogen atom is that the energy eigenvalues are specified by the total angular momentum j apart from the principal quantum number n, and indeed this is consistent with experimental observations.

There is one important deviation of the experiment from the Dirac prediction, that is, the degeneracy of $2p_{\frac{1}{2}}$-state and $2s_{\frac{1}{2}}$-state is resolved. The $2p_{\frac{1}{2}}$-state is lower than the $2s_{\frac{1}{2}}$-state in hydrogen atom. This splitting is originated from the second order effect of the vector field $\boldsymbol{A}$ which affects only on the $2s_{\frac{1}{2}}$-state. Intuitively, the second order effects must be always attractive. However, in the calculation of the Lamb shift, one has to consider the renormalization of the mass term, and due to this renormalization effect, the second order contribution of the vector field $\boldsymbol{A}$ becomes repulsive, and therefore the $2s_{\frac{1}{2}}$-state becomes higher than the $2p_{\frac{1}{2}}$-state.

D.4.1 Quantized Vector Field

Here, we briefly explain how to evaluate the Lamb shift in the non-relativistic kinematics. In this calculation, the quantized electromagnetic field $\boldsymbol{A}$ should be employed

$$\hat{\boldsymbol{A}}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\,\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c^\dagger_{\boldsymbol{k},\lambda}e^{ikx}\right], \tag{D.4.1}$$

where $c_{\boldsymbol{k},\lambda}$ and $c^{\dagger}_{\boldsymbol{k},\lambda}$ denote the creation and annihilation operators which satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda}, c^{\dagger}_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'}$$

and all other commutation relations vanish.

D.4.2 Non-relativistic Hamiltonian

We start from the Hamiltonian for electron in the hydrogen atom with the electromagnetic interaction

$$H = \frac{\hat{\boldsymbol{p}}^2}{2m_0} - \frac{Ze^2}{r} - \frac{e}{m_0}\,\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{A}}, \tag{D.4.2}$$

where the $\hat{\boldsymbol{A}}^2$ term is ignored in the Hamiltonian.

D.4.3 Second Order Perturbation Energy

Now, the second order perturbation energy due to the electromagnetic interaction for a free electron state can be written as

$$\delta E = -\sum_{\lambda}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{p}'}\left(\frac{e}{m_0}\right)^2 \frac{1}{2V\omega_{\boldsymbol{k}}}\,\frac{|\langle \boldsymbol{p}'|\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\hat{\boldsymbol{p}}|\boldsymbol{p}\rangle|^2}{E_{\boldsymbol{p}'}+k-E_{\boldsymbol{p}}}, \tag{D.4.3}$$

where $|\boldsymbol{p}\rangle$ and $|\boldsymbol{p}'\rangle$ denote the free electron state with its momentum. Since the photon energy ($\omega_{\boldsymbol{k}} = k$) is much larger than the energy difference of the electron states ($E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}$)

$$|E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}| \ll k \tag{D.4.4}$$

one obtains

$$\delta E = -\frac{1}{6\pi^2}\,\Lambda\left(\frac{e}{m_0}\right)^2 \boldsymbol{p}^2, \tag{D.4.5}$$

where Λ is the cutoff momentum of photon. This divergence is proportional to the cutoff Λ which is not the logarithmic divergence. However, this is essentially due to the non-relativistic treatment, and if one carries out the relativistic calculation of quantum field theory, then the divergence becomes logarithmic.

D.4.4 Mass Renormalization and New Hamiltonian

Defining the effective mass δm as

$$\delta m = \frac{1}{3\pi^2}\,\Lambda e^2 \tag{D.4.6}$$

the free energy of electron can be written as

$$E_F = \frac{\boldsymbol{p}^2}{2m_0} - \frac{\boldsymbol{p}^2}{2m_0^2}\,\delta m \simeq \frac{\boldsymbol{p}^2}{2(m_0+\delta m)}\,, \tag{D.4.7}$$

where one should keep only the term up to order of e^2 because of the perturbative expansion. Now, one defines the renormalized (physical) electron mass m by

$$m = m_0 + \delta m \tag{D.4.8}$$

and rewrites the Hamiltonian H in terms of the renormalized electron mass m

$$H = \frac{\hat{\boldsymbol{p}}^2}{2m} - \frac{Ze^2}{r} + \frac{\hat{\boldsymbol{p}}^2}{2m^2}\,\delta m - \frac{e}{m}\,\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{A}}. \tag{D.4.9}$$

Here, the third term ($\frac{\hat{\boldsymbol{p}}^2}{2m^2}\delta m$) corresponds to the counter term which cancels out the second order perturbation energy [eq.(D.4.3)].

D.4.5 Lamb Shift Energy

Using eq.(D.4.3), one can calculate the second order perturbation energy due to the electromagnetic interaction for the $2s_{1/2}$ electron state in hydrogen atom [12]

$$\begin{aligned}\Delta E_{2s_{1/2}} = {} & \frac{1}{6\pi^2}\,\Lambda\left(\frac{e}{m}\right)^2\langle 2s_{1/2}|\hat{\boldsymbol{p}}^2|2s_{1/2}\rangle \\ & -\sum_{\lambda}\sum_{\boldsymbol{k}}\sum_{n\ell}\left(\frac{e}{m}\right)^2\frac{1}{2V\omega_{\boldsymbol{k}}}\,\frac{|\langle n\ell|\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\hat{\boldsymbol{p}}|2s_{1/2}\rangle|^2}{E_{n,\ell}+k-E_{2s_{1/2}}}\,,\end{aligned} \tag{D.4.10}$$

where the first term comes from the counter term. This energy can be rewritten as

$$\Delta E_{2s_{1/2}} = \frac{1}{6\pi^2}\left(\frac{e}{m}\right)^2\sum_{n,\ell}|\langle n,\ell|\hat{\boldsymbol{p}}|2s_{1/2}\rangle|^2\int_0^{\Lambda}dk\,\frac{E_{n,\ell}-E_{2s_{1/2}}}{E_{n,\ell}+k-E_{2s_{1/2}}}\,, \tag{D.4.11}$$

where the following identity equation is employed

$$\sum_{n,\ell}|\langle n,\ell|\hat{\boldsymbol{p}}|2s_{1/2}\rangle|^2 = \langle 2s_{1/2}|\hat{\boldsymbol{p}}^2|2s_{1/2}\rangle. \tag{D.4.12}$$

Further, neglecting the dependence of $E_{n,\ell}$ in the energy denominator because of eq.(D.4.4), one can carry out the summation of $\sum_{n,\ell}$ in eq.(D.4.11) as

$$\begin{aligned}&\sum_{n,\ell}|\langle n,\ell|\hat{\boldsymbol{p}}|2s_{1/2}\rangle|^2(E_{n,\ell}-E_{2s_{1/2}}) \\ &= \frac{1}{2}\langle 2s_{1/2}|\left[[\hat{\boldsymbol{p}},\hat{H}_0],\hat{\boldsymbol{p}}\right]|2s_{1/2}\rangle = 2\pi Ze^2\langle 2s_{1/2}|\delta(\boldsymbol{r})|2s_{1/2}\rangle,\end{aligned} \tag{D.4.13}$$

where $\hat{H}_0$ is the unperturbed Hamiltonian of hydrogen atom

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2m} - \frac{Ze^2}{r}\,.$$

Choosing the value of the cutoff Λ, though there is no justification, as

$$\Lambda \sim m \qquad (D.4.14)$$

one obtains the energy shift for the $2s_{1/2}$ electron state

$$\Delta E_{2s_{1/2}} \simeq 1040 \text{ MHz}$$

which is close to a right Lamb shift for the $2s_{1/2}$ state

$$\Delta E^{exp}_{2s_{1/2}} = 1057.862 \ \pm 0.020 \text{ MHz}.$$

It should be noted that there is no energy shift for the $2p_{1/2}$ state.

D.4.6 Lamb Shift in Muonium

The Lamb shift energy of $2s_{\frac{1}{2}}$ state in muonium (μ^+e^- system) presents an important QED test [7, 79, 97, 113]. This Lamb shift energy of muonium $\Delta E^{(\mu)}_{2s_{1/2}}$ can be related to that of hydrogen atom $\Delta E^{(H)}_{2s_{1/2}}$ as [85]

$$\Delta E^{(\mu)}_{2s_{1/2}} = \left(\frac{m_r^{(\mu)}}{m_r^{(H)}} \right)^3 \Delta E^{(H)}_{2s_{1/2}} \qquad (D.4.15)$$

where $m_r^{(\mu)}$ and $m_r^{(H)}$ are given as

$$m_r^{(\mu)} = \frac{m_e}{1 + \frac{m_e}{m_\mu}}, \quad m_r^{(H)} = \frac{m_e}{1 + \frac{m_e}{M_p}}.$$

Here, m_μ and M_p denote the masses of muon and proton, respectively. Using the experimental value of the hydrogen Lamb shift energy

$$\Delta E^{(H)}_{2s_{1/2}}(exp) = 1057.845 \ \pm 0.009 \text{ MHz}$$

we can predict the Lamb shift energy of muonium

$$\Delta E^{(\mu)}_{2s_{1/2}}(th) = 1044 \text{ MHz}.$$

This value should be compared to the observed value [113]

$$\Delta E^{(\mu)}_{2s_{1/2}}(exp) = 1042 \ \pm 22 \text{ MHz}$$

which perfectly agrees with the prediction. The important point is that the new relation does not depend on the cutoff Λ. If the observed accuracy of the Lamb shift energy in muonium is improved, then the Lamb shift of muonium should present a very good test of the QED renormalization scheme.

D.4.7 Lamb Shift in Anti-hydrogen Atom

The structure of the negative energy state should be examined if one can measure the Lamb shift of anti-hydrogen atom [4]. In this case, positron should feel the effect of the vacuum state which should be somewhat different from the case in which electron may feel in the same situation. The Lamb shift energy of the $2s_{1/2}$ state in anti-hydrogen atom can be written as

$$\Delta E_{2s_{1/2}} = \frac{2m_r^3\alpha^5}{3m_e^2}\ln\left(\frac{\bar{\Lambda}}{< E_{n,\ell} >}\right) \qquad (D.4.16)$$

where $\bar{\Lambda}$ denotes the effective cutoff value of the anti-hydrogen atom. If the observed value of the $\bar{\Lambda}$ differs from the Λ value, then it should mean that there is some chance of understanding the structure of the negative energy state in the interacting field theory model.

D.4.8 Physical Meaning of Cutoff Λ

The calculated result of the Lamb shift energy depends on the cutoff Λ, which is not satisfactory at all. However, there is no way to avoid the presence of the cutoff Λ as long as we treat the Lamb shift in the non-relativistic field equations. The important point is that we should understand the origin of the value of the cutoff Λ. This should, of course, be understood if one treats it relativistically.

In the non-relativistic treatment, the mass counter term is linear divergent. However, if one treats it relativistically, the divergence is logarithmic. This reason of the one rank down of the divergence is originated from the fact that the relativistic treatment considers the negative energy states which in fact reduce the divergence rank due to the cancellation. Now, we consider the renormalization effect in hydrogen atom, and if we calculate the Lamb shift energy in the non-relativistic treatment, then it has the logarithmic divergence as we saw above, and this is the one rank down of the divergence. This is due to the fact that the evaluation of the Lamb shift energy is based on the cancellation between the counter term and the perturbation energy in hydrogen atom. In the same way, if one can calculate the Lamb shift energy relativistically, then one should obtain the one rank down of the divergence, and this means that it should be finite.

Appendix E

Maxwell Equation and Gauge Transformation

Fundamental equations for electromagnetic fields are the Maxwell equation, and they are written for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \tag{E.0.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{E.0.2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{E.0.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}\,, \tag{E.0.4}$$

where ρ and $\boldsymbol{j}$ denote the charge and current densities, respectively. These are already equations for the fields and therefore they are quantum mechanical equations. In this respect, it is important to realize that the first quantization procedure ($[x_j, p_i] = i\hbar\delta_{ij}$) is already done in the Maxwell equation.

E.1 Gauge Invariance

The Maxwell equation is written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. Now, if one introduces the vector potential $\boldsymbol{A}$ as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{E.1.1}$$

then eq.(E.0.2) can be always satisfied since

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0.$$

Therefore, one often employs the vector potential in order to solve the Maxwell equation. However, one notices in this case that the number of the degrees of freedom is still 3, that is,

A_x, A_y, A_z in spite of the fact that we made use of one equation [eq.($E.0.2$)]. This means that there must be a redundancy in the vector potential. This is the gauge freedom, that is, if one transforms

$$\boldsymbol{A} = \boldsymbol{A}' + \boldsymbol{\nabla}\chi \qquad (E.1.2a)$$

then the magnetic field $\boldsymbol{B}$ does not depend on χ

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A}' + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\chi = \boldsymbol{\nabla} \times \boldsymbol{A}',$$

where χ is an arbitrary function that depends on $(\boldsymbol{r}, t)$. Now, Faraday's law [eq.($E.0.3$)] can be rewritten by using the vector potential,

$$\boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = 0. \qquad (E.1.3)$$

This means that one can write the electric field $\boldsymbol{E}$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t}\,, \qquad (E.1.4)$$

where A_0 is an arbitrary function of $(\boldsymbol{r}, t)$ and is called electrostatic potential. Since $\boldsymbol{E}$ in eq.($E.1.4$) must be invariant under the gauge transformation of eq.($E.1.2$a), it suggests that A_0 should be transformed under the gauge transformation as

$$A_0 = A_0' - \frac{\partial \chi}{\partial t}\,. \qquad (E.1.2b)$$

In this case, the electric field is invariant under the gauge transformation of eqs.($E.1.2$)

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{\nabla}\left(A_0' - \frac{\partial \chi}{\partial t} \right) - \frac{\partial}{\partial t}\left(\boldsymbol{A}' + \boldsymbol{\nabla}\chi \right) = -\boldsymbol{\nabla} A_0' - \frac{\partial \boldsymbol{A}'}{\partial t}$$

and eq.(E.1.4) can automatically reproduce Faraday's law since

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} A_0 - \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{B}}{\partial t}\,.$$

E.2 Derivation of Lorenz Force in Classical Mechanics

The interaction of electrons with the electromagnetic forces in nonrelativistic kinematics can be determined from the gauge invariance. This is remarkable and therefore we explain the derivation below since it is indeed interesting to learn the basic mechanism of the interaction. First, one starts from a free electron Lagrangian in classical mechanics

$$L = \frac{1}{2}\, m\dot{\boldsymbol{r}}^2. \qquad (E.2.1)$$

When one wishes to add any interaction of electron with $\boldsymbol{A}$ and A_0 to the above Lagrangian, one sees that the Lagrangian must be linear functions of $\boldsymbol{A}$ and A_0. This is clear since the

Lagrangian must be gauge invariant under eqs.(E.1.2). From the parity and time reversal invariance, one can write down the new Lagrangian

$$L = \frac{1}{2}\, m\dot{\boldsymbol{r}}^2 + g(\dot{\boldsymbol{r}} \cdot \boldsymbol{A} - A_0), \tag{E.2.2}$$

where g is a constant which cannot be determined from the gauge condition. When one makes the gauge transformation

$$\boldsymbol{A} = \boldsymbol{A}' + \boldsymbol{\nabla}\chi, \quad A_0 = A_0' - \frac{\partial \chi}{\partial t}$$

one obtains

$$L = \frac{1}{2}\, m\dot{\boldsymbol{r}}^2 + g(\dot{\boldsymbol{r}} \cdot \boldsymbol{A}' - A_0') + g\,\frac{d\chi}{dt}\,. \tag{E.2.3}$$

Since the total derivative in the Lagrangian does not have any effects on the equation of motion, eq.(E.2.2) is invariant under the gauge transformation. It is amazing that the shape of the Lagrangian for electrons interacting with the electromagnetic fields is determined from the gauge invariance.

It is now easy to calculate the equation of motion for electron,

$$m\ddot{\boldsymbol{r}} = g\dot{\boldsymbol{r}} \times \boldsymbol{B} + g\boldsymbol{E}, \tag{E.2.4}$$

where the first term in the right hand side corresponds to the Lorenz force.

E.3 Number of Independent Functional Variables

The Maxwell equations are described in terms of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. Once the charge density ρ and the current density $\boldsymbol{j}$ are given, then one can determine the fields $\boldsymbol{E}$, $\boldsymbol{B}$. It should be important to count the number of the unknown functional variables and the number of equations.

E.3.1 Electric and Magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$

In terms of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$, it is easy to count the number of the functional variables. The number is six since one has

$$E_x,\ \ E_y,\ \ E_z,\ \ B_x,\ \ B_y,\ \ B_z. \tag{E.3.1}$$

On the other hand, the number of equations looks eight since the Gauss law [eq.(E.0.1)] and no magnetic monopole [eq.(E.0.2)] give two equations, and Faraday's law [eq.(E.0.3)] and Ampere's law [eq.(E.0.4)] seem to have six equations. However, Faraday's law gives only two equations since there is one constraint because

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \longrightarrow \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} \right) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \cdot \boldsymbol{E} + \frac{\partial (\boldsymbol{\nabla} \cdot \boldsymbol{B})}{\partial t} = 0. \tag{E.3.2}$$

In addition, Ampere's law has two equations since there is one constraint due to the continuity equation because

$$\boldsymbol{\nabla}\times\boldsymbol{B}-\boldsymbol{j}-\frac{\partial\boldsymbol{E}}{\partial t}=0\longrightarrow\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{B}-\boldsymbol{j}-\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

$$=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{B}-\boldsymbol{\nabla}\cdot\boldsymbol{j}-\frac{\partial\rho}{\partial t}=0. \qquad (E.3.3)$$

Therefore, the number of the Maxwell equations is six which agrees with the number of the independent functional variables as expected.

Integrated Gauss's Law

In the electro-static exercise problems, one often employs the integrated Gauss law

$$\int_S \boldsymbol{E}\cdot d\boldsymbol{S}=\int_V \boldsymbol{\nabla}\cdot\boldsymbol{E}\,d^3r=\int_V \rho\,d^3r=Q. \qquad (E.3.4)$$

For the spherical charge distribution of ρ, for example, one can determine the electric field E_r in spite of the fact that one has employed only one equation of the Gauss law. This is of course clear because the symmetry makes it possible to adjust the number of the independent functional variable E_r which is one and the number of equation which is also one.

E.3.2 Vector Field A_μ and Gauge Freedom

When one introduces the vector field A_μ as

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A},\quad \boldsymbol{E}=-\boldsymbol{\nabla}A_0-\frac{\partial\boldsymbol{A}}{\partial t} \qquad (E.3.5)$$

then the number of the independent fields is four since

$$A_0,\ \ A_x,\ \ A_y,\ \ A_z. \qquad (E.3.6)$$

On the other hand, the number of equations is three since the Gauss law [eq.(E.0.1)] gives one equation

$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}A_0+\frac{\partial\boldsymbol{A}}{\partial t}\right)=\rho \qquad (E.3.7)$$

and Ampere's law gives two equations as discussed above due to the continuity equation

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{j}-\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}A_0+\frac{\partial\boldsymbol{A}}{\partial t}\right). \qquad (E.3.8)$$

It is of course easy to see that no magnetic monopole

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{A}=0 \qquad (E.3.9)$$

and Faraday's law

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} A_0 \right) = -\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (E.3.10)$$

are automatically satisfied in terms of the vector potential A_μ.

Gauge Freedom

Therefore, the number of the unknown functional variables is four, but the number of equations is three, and they are not the same. This redundancy of the vector field is just related to the gauge freedom, and if one wishes to solve the Maxwell equations in terms of the vector potential A_μ, then one should reduce the number of the functional variables of the vector potential by fixing the gauge freedom.

Electromagnetic Wave

As an example, if there is no source term present [$\rho = 0$ and $\boldsymbol{j} = 0$], then the solution of the Maxwell equations with the Coulomb gauge fixing gives the electromagnetic wave which is composed of the transverse field only

$$A_0 = 0, \quad A_z = 0, \quad (A_x, A_y) \neq 0, \qquad (E.3.11)$$

where the direction of $\boldsymbol{k}$ is chosen to be z-direction.

E.4 Lagrangian Density of Electromagnetic Fields

For the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the total energy of the system becomes

$$\begin{aligned} E &= \frac{1}{2} \int \left(E_k E_k + B_k B_k \right) d^3 r \\ &= \frac{1}{2} \int \left[\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k} \right)^2 + \left(\frac{\partial A_k}{\partial x_j} \frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j} \frac{\partial A_j}{\partial x_k} \right) \right] d^3 r. \end{aligned} \qquad (E.4.1)$$

Now, one introduces the field strength $F_{\mu\nu}$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (E.4.2)$$

which is gauge invariant. In this case, one sees that $F_{\mu\nu}$ just corresponds to the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$F_{0k} = F^{k0} = -F_{k0} = -F^{0k} = E_k, \quad F_{ij} = F^{ij} = -F_{ji} = -F^{ji} = -\epsilon_{ijk} B_k.$$

The Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}(E_k E_k - B_k B_k) = \frac{1}{2}\left[\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right)^2 - \left(\frac{\partial A_k}{\partial x_j}\frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}\right)\right] \quad (E.4.3)$$

which leads to the following Lagrangian density

$$\mathcal{L} = \frac{1}{4}\left(-F_{0k}F^{0k} - F_{k0}F^{k0} - F_{jk}F^{jk}\right) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (E.4.4)$$

The Lagrange equation for A_ν is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{A}_0} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial A_\nu}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial A_\nu} \quad (E.4.5)$$

which becomes

$$[\nu = 0] \longrightarrow \frac{\partial}{\partial x_k}\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right) = 0 \longrightarrow \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0,$$
$$[\nu = k] \longrightarrow \frac{\partial}{\partial t}\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right) + \frac{\partial}{\partial x_j}(\epsilon_{jki}B_i) = 0 \longrightarrow \frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{\nabla}\times\boldsymbol{B} = 0.$$

They are just the Maxwell equations [eqs.($E.0.1$) and ($E.0.4$)] without any source terms. Since no magnetic monopole and Faraday's law [eqs.($E.0.2$) and ($E.0.43$] can be automatically satisfied in terms of the vector potential A_μ, the Lagrangian density of eq.($E.4.4$) is the right one that reproduces the Maxwell equations.

E.5 Boundary Condition for Photon

When there is no source term present ($\rho = 0$, $\boldsymbol{j} = 0$), then eq.(E.3.8) becomes

$$\left(\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = 0, \quad (E.5.1)$$

where the Coulomb gauge fixing condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$$

is employed. In this case, one sees that eq.(E.5.1) is a quantum mechanical equation for photon. Yet, one does not discuss the bound state of photon. This is clear since photon cannot be confined. There is no bound state of photon in quantum mechanics and eq.(E.5.1) has always the plane wave solution

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\boldsymbol{\epsilon}_\lambda\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c^\dagger_{\boldsymbol{k},\lambda}e^{ikx}\right], \quad (E.5.2)$$

where the polarization vectors $\boldsymbol{\epsilon}_\lambda$ has two components

$$\boldsymbol{\epsilon}_1 = (1, 0, 0), \quad \boldsymbol{\epsilon}_2 = (0, 1, 0) \tag{$E.5.3$}$$

when the direction of $\boldsymbol{k}$ is chosen to be z-direction.

This is basically due to the fact that photon is massless and therefore one cannot specify the system one measures. It always propagates with the speed of light! But still the equation derived from the Maxwell equation is a quantum mechanical equation of motion, though relativistic.

Appendix F

Regularizations and Renormalizations

In quantum field theory, one often faces to the regularization which is not very easy to understand. Mathematically it is straightforward, but the connection of the regularization with physics is not at all simple.

F.1 Euler's Regularization

Here, we first explain Euler's regularization which has nothing to do with physics, at least, at the time of derivation. However, we can learn the essence of the regularization from this mathematical example.

F.1.1 Abelian Summation

Let us define the following abelian summation N_0

$$N_0 = \sum_{n=0}^{\infty} (-)^n. \tag{F.1.1}$$

This quantity has no definite value as long as one makes the summation as it is.

F.1.2 Regularized Abelian Summation

Now, one regularizes this quantity as

$$N_\lambda = \sum_{n=0}^{\infty} (-)^n e^{-n\lambda}, \tag{F.1.2}$$

where λ is a positive value which is eventually set to zero. The summation of eq.(F.1.2) can be carried out in a straightforward way, and one obtains

$$N_0 = \lim_{\lambda \to 0} \frac{1}{1+e^{-\lambda}} = \frac{1}{2}\,. \qquad (F.1.3)$$

In this way, one obtains the finite value for the summation of $(-)^n$ if one regularizes the summation in a proper way. It is no doubt that quantities defined in eqs.(F.1.1) and (F.1.2) are mathematically different from each other. If there is any model in physics which can realize the procedure described above, then one should employ the above procedure of eq.(F.1.2) as the regularization. Therefore, the regularization is closely connected with field theory models in physical world.

F.2 Chiral Anomaly

In the massless QED$_2$, one finds that the axial vector current is classically conserved due to Noether's theorem in the Lagrangian density

$$\partial_\mu j_5^\mu = 0.$$

However, the conservation of the axial vector current is violated when the vacuum state is regularized in a gauge invariant fashion. Here, we discuss how the anomaly appears when regularized and show it in an explicit calculation.

F.2.1 Charge and Chiral Charge of Vacuum

In the vacuum of the Dirac fields, all the negative energy states are filled with the negative energy particles. In this case, one often asks what is the charge and the chiral charge of the vacuum states.

Here, we consider the Schwinger model which is the two dimensional field theory model. In this case, the charge and the chiral charge of the vacuum states can be defined as

$$Q = \sum_{n=-\infty}^{n_0} + \sum_{n=n_0+1}^{\infty}\,, \qquad (F.2.1a)$$

$$Q^5 = \sum_{n=-\infty}^{n_0} - \sum_{n=n_0+1}^{\infty}\,. \qquad (F.2.1b)$$

It is obvious that the summation of the Q and Q^5 does not make sense.

F.2.2 Large Gauge Transformation

Now, one wishes to regularize the above charge and chiral charge. Here, one makes the regularization so that the regularization procedure is consistent with the so called *large gauge transformation*.

$$Q_\lambda = \sum_{n=-\infty}^{n_0} e^{\lambda(n+\frac{Lg}{2\pi}A_1)} + \sum_{n=n_0+1}^{\infty} e^{-\lambda(n+\frac{Lg}{2\pi}A_1)}, \qquad (F.2.2a)$$

$$Q_\lambda^5 = \sum_{n=-\infty}^{n_0} e^{\lambda(n+\frac{Lg}{2\pi}A_1)} - \sum_{n=n_0+1}^{\infty} e^{-\lambda(n+\frac{Lg}{2\pi}A_1)}, \qquad (F.2.2a)$$

where L, g and A_1 denote the box length, the gauge coupling constant and the vector potential, respectively. Here, the vector potential A_1 depends only on time. The reason why the term $(n + \frac{Lg}{2\pi}A_1)$ appears is because the Hamiltonian of the Schwinger model has the invariance under the large gauge transformation

$$A_1 \to A_1 + \frac{2\pi}{Lg}N, \quad N \text{ integer}.$$

Therefore, it is natural to assume that the regularization should keep this invariance.

F.2.3 Regularized Charge

In this case, the regularized charge Q_λ and the chiral charge Q_λ^5 become

$$Q_\lambda = \frac{2}{\lambda} + O(\lambda), \qquad (F.2.3a)$$

$$Q_\lambda^5 = 2n_0 + 1 + \frac{Lg}{\pi} A_1 + O(\lambda). \qquad (F.2.3b)$$

Now, one should let λ very small, and then the charge Q becomes infinity. But this is of course expected since one counts the number of the particles in the negative energy states which must be infinity. Since this number has no meaning in the vacuum, one can reset the charge Q to zero.

On the other hand, the chiral charge Q^5 is finite when the value of λ is set to zero. This means that the chiral charge of the vacuum is not conserved since A_1 depends on time! This is somewhat strange since the Schwinger model has the chiral symmetry in the classical Lagrangian density. But the regularization induces the non-conservation of the chiral charge and this is called *anomaly*. In four dimensional QED, the same phenomena occur and the presence of the anomaly is indeed confirmed by experiments.

F.2.4 Anomaly Equation

If one makes a time derivative of Q^5, then one obtains

$$\dot{Q}_5 = \frac{Lg}{\pi}\,\dot{A}_1. \tag{F.2.4}$$

Since Q_5 is described as

$$Q_5 = \int j_5^0(x)\,dx \tag{F.2.5}$$

one can rewrite eq.(F.2.4) as

$$\partial_\mu j_5^\mu = \frac{g}{2\pi}\,\epsilon_{\mu\nu}F^{\mu\nu}, \tag{F.2.6}$$

where $\epsilon_{\mu\nu}$ denotes the anti-symmetric symbol in two dimensions. That is,

$$\epsilon_{01} = -\epsilon_{10} = 1, \quad \epsilon_{00} = \epsilon_{11} = 0.$$

Eq.(F.2.6) clearly shows that the conservation of the axial vector current does not hold any more due to the anomaly.

Physics of Anomaly

Physics of the anomaly is closely related to the gauge invariant regularization of the energy and charge of the vacuum state. Since we respect most the local gauge invariance in the regularization procedure, the axial vector current conservation is violated. The regularization of the charge is originated from the field quantization, and therefore one may say that the quantization and the local gauge invariance induce the violation of the axial vector current conservation.

F.3 Index of Renormalizability

Field theory models have infinite degrees of freedom, and therefore some of the quantities which are evaluated perturbatively become infinity. When the infinite quantity can be renormalized into some constants such as the coupling constant or fermion mass, this field theory model is called "renormalizable". When one calculates a dimensionless quantity $\hat{O}$ in the perturbation theory, then one can expand the expectation value of $\hat{O}$ in terms of the coupling constant g as

$$\langle\hat{O}\rangle = c_0 + c_1 g + c_2 g^2 + \cdots + c_n g^n + \cdots. \tag{F.3.1}$$

F.3.1 Renormalizable

If the coupling constant g is dimensionless, then one sees that c_1 should depend on the cut-off momentum Λ, at most,

$$c_1 \sim \ln(\Lambda/m). \tag{F.3.2}$$

The logarithmic dependence of the cut-off momentum Λ is not at all serious. This is clear since even if one makes Λ very large, $\ln(\Lambda/m)$ is just some number. In addition, c_n should be described in terms of some function of $\ln(\Lambda/m)$ as

$$c_n \sim f_n\left(\ln(\Lambda/m)\right). \tag{F.3.3}$$

Therefore, one can expect that this field theory model must be renormalizable since the infinity only comes from $\ln(\Lambda/m)$.

F.3.2 Unrenormalizable

On the other hand, if the coupling constant g has a mass inverse dimension, then one sees that

$$c_1 \sim \Lambda. \tag{F.3.4}$$

This is difficult since the $\langle O \rangle$ becomes quickly infinity when Λ becomes very large. In addition, c_n should behave as

$$c_n \sim \Lambda^n. \tag{F.3.5}$$

In this case, it is impossible to adjust the parameters to renormalize the theory since the n-th order perturbative calculations diverge at the different level of the Λ dependence. Namely, to the n-th order perturbation evaluation, one needs a new parameter to adjust a new level of infinity. Therefore, one sees that the field theory model with the coupling constant of the mass inverse dimension cannot be renormalized perturbatively.

If one solves it non-perturbatively, then it should be a different story. However, at the present stage, we cannot say more what should happen to this field theory model if it is solved exactly.

F.3.3 Summary of Renormalizability

Here, we summarize the renormalizability conditions. The coupling constant g is defined in the interaction Lagrangian density in the following shape.

$$\begin{aligned}
\text{QED} &: \mathcal{L}_I = g\bar{\psi}A_\mu\gamma^\mu\psi, \\
\text{QCD} &: \mathcal{L}_I = g\bar{\psi}A^a_\mu T^a\gamma^\mu\psi, \\
\text{Thirring} &: \mathcal{L}_I = g(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi), \\
\text{NJL} &: \mathcal{L}_I = g\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right].
\end{aligned}$$

Renormalizability conditions

$g \sim M^1$	super-renormalizable	QED_2, QCD_2
$g \sim M^0$	renormalizable	QED_4, QCD_4, Thirring
$g \sim M^{-1}$	unrenormalizable	
$g \sim M^{-2}$	unrenormalizable	NJL

QED_2: Quantum electrodynamics in two dimensions D.

QCD_2: Quantum chromodynamics in two dimensions D.

QED_4: Quantum electrodynamics in four dimensions D.

QCD_4: Quantum chromodynamics in four dimensions D.

Thirring: current- current interaction model in two dimensions D.

NJL: current- current interaction model in four dimensions D.

F.4 Infinity in Physics

In physics, there appear many kinds of infinities. Mathematically, infinity is simply denoted as ∞. For example, when one has to take the thermodynamic limit, one says that the box length L must be set to infinity $L \to \infty$. However, in physics, the above equation does not make sense since there is no infinity in reality. In other words, if one calculates some quantity in atomic system and has to take the thermodynamic limit, then $L = 6.4 \times 10^6$ m (the earth radius) is of course infinity since the atomic scale is of the order 10^{-6} cm, at most.

In this respect, the physical meaning of $L \to \infty$ is that the box length L must be much larger than any of the relevant scales in the system of the model. If one has to take an infinity in the mathematical sense, then it indicates that the model one considers is too much simplified and should be improved to introduce some relevant scale such that the new scale can play some role in the model.

Also, when one evaluates the momentum integral in the perturbation theory, then the integral ranges from $-\Lambda$ to Λ where Λ is set to infinity. However, the infinity of this cutoff momentum Λ means again that it should be much larger than any of the relevant mass scale m

$$\Lambda \gg m.$$

For example, if the value of Λ is

$$\Lambda \sim 10^8 \, m$$

then it is physically sufficiently larger than m. The important point is that one should build a scheme in which physical observables should not depend on the choice of the Λ value if it is sufficiently larger than any of the mass scales.

In this respect, the logarithmic divergence $\ln(\frac{\Lambda}{m})$ is just a number for the infinity of the cutoff momentum Λ. For example, the value of $\Lambda \sim 10^8 \, m$ gives

$$\ln\left(\frac{\Lambda}{m}\right) \simeq \ln 10^8 \simeq 18$$

which is not at all a large value. This indicates that one should not worry about the logarithmic divergence $\ln(\frac{\Lambda}{m})$ when one renormalizes the infinity of the self-energy diagrams as discussed in Appendix I.

Appendix G

Path Integral Formulation

The path integral in quantum mechanics can be obtained by rewriting the quantum mechanical amplitude $K(x, x' : t)$ into many dimensional integrations over discretized coordinates x_n. We discuss some examples which can be related to physical observables. However, the path integral expression cannot be connected to the dynamics of classical mechanics, even though, superficially, there is some similarity between them. Then, we show Feynman's formulation of the path integral in quantum electrodynamics (QED), which is based on many dimensional integrations over the parameters $q_{\boldsymbol{k},\lambda}$ appearing in the vector potential. This should be indeed connected to the second quantization in field theory models. However, the field theory path integral formulation in most of the textbooks is normally defined in terms of many dimensional integrations over fields. In this case, the path integral does not correspond to the field quantization. Here, we clarify what should be the problems of the path integral formulation over the field variables and why the integrations over fields do not correspond to the field quantization.

G.1 Path Integral in Quantum Mechanics

The path integral formulation in one dimensional quantum mechanics starts from the amplitude $K(x, x' : t)$ which is defined by

$$K(x, x' : t) = \langle x'|e^{-iHt}|x\rangle, \tag{G.1.1}$$

where the system is specified by the Hamiltonian H. In the field theory textbooks, one often finds the expression of the amplitude $K(x, x' : t)$ in terms of the transition between the state $|x, t\rangle$ and $|x', t'\rangle$ as

$$\langle x', t'|x, t\rangle \longrightarrow \langle x'|e^{-iH(t'-t)}|x\rangle. \tag{G.1.2}$$

However, the state $|x, t\rangle$ is not an eigenstate of the Hamiltonian and therefore, one cannot prove the rightarrow of eq.(G.1.2). Instead, we should rewrite the amplitude $K(x, x' : t)$ so

as to understand its physical meaning

$$K(x, x' : t) = \langle x'|e^{-iHt}|x\rangle = \sum_n \psi_n(x')\psi_n^\dagger(x)e^{-iE_n t}, \tag{G.1.3}$$

where $\psi_n(x)$ and E_n should be the eigenstate and the eigenvalue of the Hamiltonian H. We note that eq.(G.1.3) is not yet directly related to physical observables.

G.1.1 Path Integral Expression

We start from the amplitude $K(x', x : t)$

$$K(x', x : t) = \langle x'|e^{-iHt}|x\rangle,$$

where the Hamiltonian in one dimension is given as

$$H = \frac{\hat{p}^2}{2m} + U(x) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + U(x). \tag{G.1.4}$$

Here, a particle with its mass m is bound in the potential $U(x)$. Now, one can make n partitions of t and $x' - x$, and therefore we label the discretized coordinate x as

$$x = x_0, x_1, x_2, \ldots, x' = x_n. \tag{G.1.5}$$

In this case, we assume that each x_i and p should satisfy the following completeness relations

$$\int_{-\infty}^{\infty} dx_i\, |x_i\rangle\langle x_i| = 1, \quad \langle x_i|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx_i}, \quad \int_{-\infty}^{\infty} dp|p\rangle\langle p| = 1 \;\; (i = 1, \ldots, n), \tag{G.1.6}$$

Therefore, $K(x', x : t)$ becomes

$$\begin{aligned} &K(x', x : t) \\ &= \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1}\, \langle x'|e^{-iH\Delta t}|x_{n-1}\rangle\langle x_{n-1}|e^{-iH\Delta t}|x_{n-2}\rangle \cdots \langle x_1|e^{-iH\Delta t}|x\rangle, \end{aligned} \tag{G.1.7}$$

where Δt is defined as $\Delta t = \frac{t}{n}$. Further, one can calculate the matrix elements, for example, as

$$\begin{aligned} \langle x_1|e^{-iH\Delta t}|x\rangle &= \langle x_1| \exp\left[-i\left(-\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2} + U(x)\Delta t\right)\right]|x\rangle \\ &\simeq \exp(-iU(x)\Delta t)\langle x_1| \exp\left(i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}\right)|x\rangle + O((\Delta t)^2). \end{aligned} \tag{G.1.8}$$

In addition, $\langle x_1|e^{i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}}|x\rangle$ can be evaluated by inserting a complete set of momentum states

$$\langle x_1| \exp\left(i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}\right)|x\rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-i\frac{p^2}{2m}\Delta t}\, e^{-ip(x-x_1)} = \sqrt{\frac{m}{2i\pi\Delta t}}\, e^{-i\frac{m(x-x_1)^2}{2\Delta t}} \,. \qquad (G.1.9)$$

Therefore, one finds now the path integral expression for $K(x', x:t)$

$$K(x', x:t) = \lim_{n\to\infty} \left(\frac{m}{2i\pi\Delta t}\right)^{\frac{n}{2}} \times$$

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \exp\left\{ i \sum_{k=1}^{n} \left(\frac{m(x_k - x_{k-1})^2}{2\Delta t} - U(x_k)\Delta t \right) \right\}, \qquad (G.1.10a)$$

where $x_0 = x$ and $x_n = x'$, respectively. Since the classical action S is given as

$$S = \int_0^t dt \left(\frac{1}{2}\, m\dot{x}^2 - U(x) \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta t \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\Delta t} \right)^2 - U(x_k) \right\} \quad (G.1.11)$$

the amplitude can be symbolically written as

$$K(x', x:t) = \mathcal{N} \int [\mathcal{D}x] \exp\left\{ i \int_0^t \left(\frac{1}{2}\, m\dot{x}^2 - U(x) \right) dt \right\}, \qquad (G.1.10b)$$

where $\mathcal{N} \int [\mathcal{D}x]$ is defined as

$$\mathcal{N} \int [\mathcal{D}x] \equiv \lim_{n\to\infty} \left(\frac{m}{2i\pi\Delta t}\right)^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1}.$$

This is indeed amazing in that the quantum mechanical amplitude is connected to the Lagrangian of the classical mechanics for a particle with its mass m in the same potential $U(x)$. Since the procedure of obtaining eq.(G.1.10) is just to rewrite the amplitude by inserting the complete set of the $|x_n\rangle$ states, there is no mathematical problem involved in evaluating eq.(G.1.10).

G.1.2 Physical Mmeaning of Path Integral

However, the physical meaning of the procedure is not at all easy to understand.

It is clear that eq.(G.1.10a) is well defined and there is no problem since it simply involves mathematics. However, there is a big jump from eq.(G.1.10a) to eq.(G.1.10b), even though it looks straightforward. Eq.(G.1.10b) indicates that the first term of eq.(G.1.10b) in the curly bracket is the kinetic energy of the particle in classical mechanics. In this case, however, x_k and x_{k-1} cannot be varied independently as one sees it from classical mechanics since it is related to the time derivative. On the other hand, they must be varied independently in the original version of eq.(G.1.10a) since it has nothing to do with the

time derivative in the process of the evaluation. This is clear since, in quantum mechanics, time and coordinate are independent from each other. Therefore, it is difficult to interpret the first term of eq.(G.1.10a) as the kinetic energy term in classical mechanics. Secondly, the procedure of rewriting the amplitude is closely connected to the fact that the kinetic energy of the Hamiltonian is quadratic in p, that is, it is described as $\frac{p^2}{2m}$. In this respect, the fundamental ingredients of the path integral formulation must lie in eq.(G.1.9) which relates the momentum operator p^2 to the time derivative of the coordinate x under the condition that x_k and x_{k-1} are sufficiently close to each other. In this sense, if the kinetic energy operator were linear in p like the Dirac equation, then there is no chance to rewrite the amplitude since the Gaussian integral is crucial in evaluating the integral.

Therefore, it should be difficult to claim that eq.(G.1.10a) can correspond to the dynamics of classical mechanics, even though, superficially, there is some similarity between them. In other words, it is hard to prove that the quantum mechanical expression of $K(x', x\ :\ t)$ is related to any dynamics of classical mechanics. One may say that $K(x', x : t)$ happens to have a similar shape to classical Lagrangian, mathematically, but, physically it has nothing to do with the dynamics of classical mechanics.

No summation of Classical Path

In some of the path integral textbooks, one finds the interpretation that the quantum mechanical dynamics can be obtained by summing up all possible paths in the classical mechanical trajectories. However, if one starts from eq.(G.1.10b) and tries to sum up all the possible paths, then one has to find the functional dependence of the coordinates on time and should integrate over all the possible coordinate configurations as the function of time. This should be quite different from the expression of eq.(G.1.10a). If one wishes to sum up all the possible paths in eq.(G.1.10b), then one may have to first consider the following expression of the coordinate x as the function of time

$$x = x_{cl}(t') + \sum_{n=1}^{\infty} y_n \sin\left(\frac{2\pi n}{t}\right) t',$$

where $x_{cl}(t')$ denotes the classical coordinate that satisfies the Newton equation of motion with the initial conditions of $x_{cl}(0) = x$ and $x_{cl}(t) = x'$. The amplitude y_n is the expansion coefficient. Therefore, the integrations over all the paths should mean that one should integrate over

$$\prod_{n=1}^{\infty} \int_{-\infty}^{\infty} dy_n$$

and, in this case, one can easily check that the calculated result of the integration over all the paths cannot reproduce the proper quantum mechanical result for the harmonic oscillator case. This simply means that the integration over $dx_1 \cdots dx_{n-1}$ in eq.(G.1.10a) and the integration over classical paths are completely different from each other, which is a natural result. This fact is certainly known to some careful physicists, but most of the path integral

textbooks are reluctant to putting emphasis on the fact that the classical trajectories should not be summed up in the path integral formulation. Rather, they say that the summation of all the classical paths should correspond to the quantization by the path integral, which is a wrong and misleading statement.

G.1.3 Advantage of Path Integral

What should be any merits of the path integral formulation? Eqs.(G.1.10) indicate that one can carry out the first quantization of the classical system once the Lagrangian is given where the kinetic energy term should have a quadratic shape, that is, $c\dot{x}^2$ with c some constant. In this case, one can obtain the quantized expression just by tracing back from eq.(G.1.10b) to eq.(G.1.10a).

There is an advantage of the path integral formulation. That is, one does not have to solve the differential equation. Instead, one should carry out many dimensional integrations. It is, of course, not at all clear whether the many dimensional integrations may have some advantage over solving the differential equation or not. However, one can, at least, claim that the procedure of the many dimensional integrations is indeed an alternative method to solving the Schrödinger equation.

G.1.4 Harmonic Oscillator Case

When the potential $U(x)$ is a harmonic oscillator

$$U(x) = \frac{1}{2}\, m\omega^2 x^2$$

then one can evaluate the amplitude analytically after some lengthy calculations

$$K(x', x : t) = \sqrt{\frac{m\omega}{2i\pi \sin \omega t}}\, \exp\left\{ i\frac{m\omega}{2}\left[(x'^2 + x^2)\cot \omega t - \frac{2x'x}{\sin \omega t}\right]\right\}. \qquad (G.1.12)$$

On the other hand, one finds

$$K(x, x : t) = \langle x|e^{-iHt}|x\rangle = \sum_n e^{-iE_n t}|\psi_n(x)|^2. \qquad (G.1.13)$$

Therefore, if one integrates $K(x, x : t)$ over all space, then one obtains

$$\int_{-\infty}^{\infty} dx K(x, x : t) = \sum_{n=0}^{\infty} e^{-iE_n t} = \int_{-\infty}^{\infty} dx \sqrt{\frac{m\omega}{2i\pi \sin \omega t}}\, e^{-im\omega x^2 \tan \frac{\omega t}{2}}$$

$$= \frac{1}{2i \sin \frac{\omega t}{2}}\,. \qquad (G.1.14)$$

Since the last term can be expanded as

$$\frac{1}{2i\sin\frac{\omega t}{2}} = \frac{e^{-\frac{i}{2}\omega t}}{1-e^{-i\omega t}} = e^{-\frac{i}{2}\omega t}\sum_{n=0}^{\infty}(e^{-i\omega t})^n = \sum_{n=0}^{\infty} e^{-i\omega t(n+\frac{1}{2})} \qquad (G.1.15)$$

one obtains by comparing two equations

$$E_n = \omega\left(n+\frac{1}{2}\right) \qquad (G.1.16)$$

which is just the right energy eigenvalue of the harmonic oscillator potential in quantum mechanics.

It should be important to note that the evaluation of eq.(G.1.12) is entirely based on the expression of eq.(G.1.10a) which is just the quantum mechanical equation. Therefore, this example of the harmonic oscillator case shows that the rewriting of the amplitude $K(x', x : t)$ is properly done in obtaining eq.(G.1.10a). This does not prove any connection of the $K(x', x : t)$ to the classical mechanics.

G.2 Path Integral in Field Theory

The basic notion of the path integral was introduced by Feynman [56, 57, 58], and the formulation of the path integral in quantum mechanics is given in terms of many dimensional integrations of the discretized coordinates x_n. As one sees from eq.(G.1.10), the amplitude is expressed in terms of many dimensional integrations with the weight factor of e^{iS} where S is the action of the classical mechanics. This was, of course, surprising and interesting. However, as Feynman noted in his original papers, the path integral expression is not more than the ordinary quantum mechanics.

When the classical particle interacts with the electromagnetic field $\boldsymbol{A}$, the amplitude of the particle can be expressed in terms of the many dimensional integrations of the action of the classical particle. However, the electromagnetic field $\boldsymbol{A}$ is already a quantum mechanical object, and therefore, there is no need for the first quantization in the Maxwell equation. However, when one wishes to treat physical processes which involve the absorption or emission of photon, then one has to quantize the electromagnetic field $\boldsymbol{A}$ which is called field quantization or second quantization.

G.2.1 Field Quantization

The field quantization of the electromagnetic field $\boldsymbol{A}$ can be done by expanding the field $\boldsymbol{A}$ in terms of the plane wave solutions

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c^{\dagger}_{\boldsymbol{k},\lambda}e^{ikx}\right]. \qquad (G.2.1)$$

The field quantization requires that $c_{\boldsymbol{k},\lambda}$ and $c^{\dagger}_{\boldsymbol{k},\lambda}$ should be operators which satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda}, c^{\dagger}_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} \tag{$G.2.2$}$$

and all other commutation relations should vanish. This is the standard way of the second quantization procedure even though it is not understood well from the fundamental principle. However, it is obviously required from experiments since electron emits photon when it decays from the $2p_{\frac{1}{2}}$ state to the $1s_{\frac{1}{2}}$ state in hydrogen atom.

G.2.2 Field Quantization in Path Integral (Feynman's Ansatz)

In his original paper, Feynman proposed a new method to quantize the electromagnetic field $\boldsymbol{A}$ in terms of the path integral formulation [56, 57, 58]. Here, we should first describe his formulation of the path integral. For the fermion part, he employed the particle expression, and therefore the path integral is defined in terms of quantum mechanics.

For the gauge field, Feynman started from the Hamiltonian formulation of the electromagnetic field. The Hamiltonian of the electromagnetic field can be expressed in terms of the sum of the harmonic oscillators

$$H_{el} = \frac{1}{2}\sum_{\boldsymbol{k},\lambda}\left(p^2_{\boldsymbol{k},\lambda} + k^2 q^2_{\boldsymbol{k},\lambda}\right), \tag{$G.2.3$}$$

where $p_{\boldsymbol{k},\lambda}$ is a conjugate momentum to $q_{\boldsymbol{k},\lambda}$. Here, it should be noted that the $q_{\boldsymbol{k},\lambda}$ corresponds to the amplitude of the vector potential $\boldsymbol{A}(x)$. The classical $c_{\boldsymbol{k},\lambda}$ and $c^{\dagger}_{\boldsymbol{k},\lambda}$ can be expressed in terms of $p_{\boldsymbol{k},\lambda}$ and $q_{\boldsymbol{k},\lambda}$ as

$$c_{\boldsymbol{k},\lambda} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(p_{\boldsymbol{k},\lambda} - i\omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda}\right), \tag{$G.2.4a$}$$

$$c^{\dagger}_{\boldsymbol{k},\lambda} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(p_{\boldsymbol{k},\lambda} + i\omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda}\right). \tag{$G.2.4b$}$$

In this case, the Hamiltonian can be written in terms of $c_{\boldsymbol{k},\lambda}$ and $c^{\dagger}_{\boldsymbol{k},\lambda}$ as

$$H_{el} = \frac{1}{2}\sum_{\boldsymbol{k},\lambda}\omega_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + c_{\boldsymbol{k},\lambda}c^{\dagger}_{\boldsymbol{k},\lambda}\right). \tag{$G.2.5$}$$

It should be important to note that the Hamiltonian of eq.(G.2.3) is originated from the Hamiltonian of field theory and it has nothing to do with the classical Hamiltonian of Newton dynamics. For the electromagnetic field, there is no corresponding Hamiltonian of the Newton dynamics.

Feynman's Ansatz

Feynman proposed a unique way of carrying out the field quantization [56, 57, 58]. Since the Hamiltonian of the electromagnetic field can be written as the sum of the harmonic oscillators, he presented the path integral formulation for the electromagnetic field

$$K(q_{\boldsymbol{k},\lambda}, q'_{\boldsymbol{k},\lambda}, t) \equiv \mathcal{N} \int [\mathcal{D}q_{\boldsymbol{k},\lambda}] \exp\left\{ \frac{i}{2} \int_0^t \sum_{\boldsymbol{k},\lambda} \left(\dot{q}^2_{\boldsymbol{k},\lambda} - k^2 q^2_{\boldsymbol{k},\lambda} \right) dt \right\} \qquad (G.2.6)$$

which should correspond to the quantization of the variables $q_{\boldsymbol{k},\lambda}$, and this corresponds to the quantization of the $c_{\boldsymbol{k},\lambda}$ and $c^\dagger_{\boldsymbol{k},\lambda}$. Thus, it is the second quantization of the electromagnetic field.

In this expression, there is an important assumption for the coordinates $q_{\boldsymbol{k},\lambda}$ which are the parameters appearing in the vector potential. That is, the states $|q_{\boldsymbol{k},\lambda}\rangle$ should make a complete set. Only under this assumption, one can derive the quantization of the harmonic oscillators. Since this is the parameter space, it may not be easy to prove the completeness of the states $|q_{\boldsymbol{k},\lambda}\rangle$. Nevertheless, Feynman made use of the path integral expression to obtain the Feynman rules in the perturbation theory for QED. It may also be important to note that the path integral in Feynman's method has nothing to do with the integration of the configuration space. It is clear that one should not integrate out over the configuration space in the path integral since the field quantization should be done for the parameters $c_{\boldsymbol{k},\lambda}$ and $c^\dagger_{\boldsymbol{k},\lambda}$.

G.2.3 Electrons Interacting through Gauge Fields

When one treats the system in which electrons are interacting through electromagnetic fields, one can write the whole system in terms of the path integral formulation. In this case, however, we treat electrons in the non-relativistic quantum mechanics. The electromagnetic fields are treated just in the same way as the previous section.

$$K(q_{\boldsymbol{k},\lambda}, q'_{\boldsymbol{k},\lambda}, \boldsymbol{r}, \boldsymbol{r}', t)$$

$$\equiv \mathcal{N} \int [\mathcal{D}\boldsymbol{r}][\mathcal{D}q_{\boldsymbol{k},\lambda}] \exp\left\{ i \int_0^t \left(\frac{1}{2} m\dot{\boldsymbol{r}}^2 - g\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}) + \frac{1}{2} \sum_{\boldsymbol{k},\lambda} \left(\dot{q}^2_{\boldsymbol{k},\lambda} - k^2 q^2_{\boldsymbol{k},\lambda} \right) \right) dt \right\}, \quad (G.2.7)$$

where the vector potential $\boldsymbol{A}$ is given in eq.(G.2.1)

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{\boldsymbol{\epsilon}(\boldsymbol{k}, \lambda)}{\sqrt{V}\omega_{\boldsymbol{k}}} \left[\dot{q}_{\boldsymbol{k},\lambda} \cos(\boldsymbol{k} \cdot \boldsymbol{r}) + \omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda} \sin(\boldsymbol{k} \cdot \boldsymbol{r}) \right] \qquad (G.2.8)$$

which is now rewritten in terms of the variables $q_{\boldsymbol{k},\lambda}$.

It should be noted that the path integral formulation works only for the electromagnetic field since its field Hamiltonian can be described by the sum of the harmonic oscillators.

This is a very special case, and there is only little chance that one can extend his path integral formulation to other field theory models. In particular, it should be hopeless to extend the path integral formulation to the field quantization of quantum chromodynamics (QCD) since QCD includes fourth powers of $q_{\boldsymbol{k},\lambda}$. In this case, one cannot carry out the Gaussian integral in the parameter space of $q_{\boldsymbol{k},\lambda}$.

G.3 Problems in Field Theory Path Integral

In this section, we discuss the problems in the standard treatment of the path integral formulation in field theory models. Normally, one starts from writing the path integral formulation in terms of the many dimensional integrations over field variables [33, 34].

G.3.1 Real Scalar Field as Example

For simplicity, we take a real scalar field in 1+1 dimensions. In most of the field theory textbooks, the amplitude Z is written as

$$Z = \mathcal{N} \int [\mathcal{D}\phi(t,x)] \exp\left[i \int \mathcal{L}(\phi, \partial_\mu \phi)\, dt\, dx \right], \qquad (G.3.1)$$

where the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ is given as

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{2}\, m^2 \phi^2. \qquad (G.3.2)$$

If we rewrite the path integral definition explicitly in terms of the field variable integrations like eq.(G.1.10), we find

$$Z = \mathcal{N} \lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k,\ell=1}^{n} d\phi_{k,\ell}$$

$$\times \exp\left[i \sum_{k,\ell=1}^{n} \Delta t \Delta x \left(\frac{(\phi_{k,\ell} - \phi_{k-1,\ell})^2}{2(\Delta t)^2} - \frac{(\phi_{k,\ell} - \phi_{k,\ell-1})^2}{2(\Delta x)^2} - \frac{1}{2} m^2 \phi_{k,\ell}^2 \right) \right], \qquad (G.3.3)$$

where $\phi_{k,\ell}$ is defined as

$$\phi_{k,\ell} = \phi(t_k, x_\ell), \quad \text{with} \quad t_1 = t, \cdots, t_n = t' \quad \text{and} \quad x_1 = x, \cdots, x_n = x'. \qquad (G.3.4)$$

Also, Δt and Δx are defined as

$$\Delta t = \frac{(t - t')}{n}, \quad \Delta x = \frac{(x - x')}{n}. \qquad (G.3.5)$$

Now, we should examine the physical meaning of the expression of the amplitude Z in eq.(G.3.3), and clarify as to what are the problems in eq.(G.3.3) in connection to the field

quantization. The first problem is that eq.(G.3.3) does not contain any quantity which is connected to the initial and final states. This is clear since, in eq.(G.3.3), one should integrate over fields as defined and therefore no information of $\phi(t,x)$ and $\phi(t',x')$ is left while, in quantum mechanics version, the amplitude is described by the quantities $\psi_n(x)$ and $\psi_n^\dagger(x')$ which are the eigenstates of the Hamiltonian. The second problem is that the calculated result of the amplitude Z must be only a function of $m, \Delta t, \Delta x$, that is

$$Z = f(m, \Delta t, \Delta x). \tag{G.3.6}$$

This shows that the formulation which is started from many dimensional integrations over the field $\phi(t,x)$ has nothing to do with the second quantization. In addition, Z depends on the artificial parameters Δt and Δx, and this clearly shows that it cannot be related to any physical observables.

This is in contrast with the formulation of eq.(G.2.7) where the amplitude K is specified by the quantum number of the parameter space $q_{\boldsymbol{k},\lambda}$ which is connected to the state with a proper number of photons, and it must be a function of $m, g, q_{\boldsymbol{k},\lambda}$

$$K = f(m, g, q_{\boldsymbol{k},\lambda}, q'_{\boldsymbol{k},\lambda}). \tag{G.3.7}$$

In addition, the K does not depend on the parameters Δt and Δx, which is a natural result as one can see it from the quantum mechanical path integral formulation.

Finally, we note that the treatment of Feynman is based on the total QED Hamiltonian which is a conserved quantity. On the other hand, eq.(G.3.3) is based on the action which is obtained by integrating the Lagrangian density over space and time. As one knows, the Lagrangian density is not directly related to physical observables. Therefore, unless one can confirm that the path integral is reduced to the quantum mechanical amplitude like eq.(G.1.10), one cannot make use of the field theory path integral formulation. In fact, one cannot rewrite the expression of eq.(G.3.3) in terms of the field theory Hamiltonian density $\mathcal{H}$, contrary to the path integral formulation in quantum mechanics. This shows that the amplitude Z has nothing to do with the amplitude K in eq.(G.2.7). In this respect, the amplitude Z has no physical meaning, and therefore one cannot calculate any physical quantities from the path integral formulation of Z in field theory.

G.3.2 Lattice Field Theory

Most of the numerical calculations in the lattice field theory are based on the path integral formulation of eq.(G.3.3). Unfortunately, the path integral formulation of eq.(G.3.3) has lost its physical meaning, and therefore there is little chance that one can obtain any physics out of numerical simulations of the lattice field theory. In this respect, it is, in a sense, not surprising that the calculated result of Wilson's area law [112] in QED is unphysical with incorrect dimensions as we saw in Chapter 8.

Since Wilson's calculation is presented analytically, it may be worth writing again the

result of the Wilson loop calculation in QED as given in Chapter 8

$$W \equiv \mathcal{N} \prod_m \prod_\mu \int_{-\infty}^{\infty} dB_{m\mu} \exp\left(i \sum_P B_{n\mu} + \frac{1}{2g^2} \sum_{n\mu\nu} e^{i f_{n\mu\nu}} \right). \tag{8.49}$$

In the strong coupling limit, one can evaluate eq.(8.49) analytically as

$$W \simeq (g^2)^{-\Delta S/a^2}, \tag{G.3.8}$$

where ΔS should be an area encircled by the loop. This has the same behavior as that of eq.(G.3.6) since the lattice constant a is equal to $a = \Delta x = \Delta t$, that is,

$$W = f(g, \Delta S, \Delta x, \Delta t). \tag{G.3.9}$$

This amplitude W has the dependence of the artificial parameters Δt and Δx. This is completely different from eq.(G.3.7), and therefore one sees that the calculation of eq.(8.49) has no physical meaning, contrary to Feynman's treatment which has a right behavior as the function of the field parameters $q_{\boldsymbol{k},\lambda}$.

G.3.3 Physics of Field Quantization

The quantization of fields is required from experiment. Yet, it is theoretically quite difficult to understand the basic physics of the field quantization. The fundamental step of the quantization is that the Hamiltonian one considers becomes an operator after the quantization. The reason why one considers the Hamiltonian is because it is a conserved quantity. In this respect, one cannot quantize the Hamiltonian density since it is not a conserved quantity yet. This is an important reason why one must quantize the field in terms of the creation and annihilation operator $c_{\boldsymbol{k},\lambda}$ and $c^\dagger_{\boldsymbol{k},\lambda}$ in QED. In this respect, it is clear that the field quantization must be done in terms of $c_{\boldsymbol{k},\lambda}$ and $c^\dagger_{\boldsymbol{k},\lambda}$ with which the Hamiltonian in classical QED can be described.

G.3.4 No Connection between Fields and Classical Mechanics

Here, we should make a comment on the discretized coordinates and fields. The discretized space is, of course, artificial, and there is no physics in the discretized fields and equations. In some textbooks, the field equation is derived from the picture that the field is constructed by the sum of springs in which the discretized coordinates of neighboring sites are connected by the spring. This picture can reproduce the field equation for a massless scalar field by adjusting some parameters, even though one started from a non-relativistic classical mechanics. However, this is obviously a wrong picture for a scalar field theory since the field equation has nothing to do with classical mechanics. It is somewhat unfortunate that it may have had some impact on the picture concerning lattice gauge field calculations as an excuse to make use of the discretized classical fields. As we saw in the previous section, the path integral formulation has nothing to do with the dynamics of classical mechanics, and it is, of course, clear that the field theory path integral is never connected to any dynamics of the classical field theory.

G.4 Path Integral Function Z in Field Theory

As we saw in Section G.2, Feynman's path integral formulation in terms of many dimensional integrations in parameter space is indeed a plausible method to quantize the fields in QED. However, people commonly use the expression of the amplitude Z as defined in eq.(G.3.1).

G.4.1 Path Integral Function in QCD

The "new formulation" of the path integral in QCD was introduced by Faddeev and Popov who wrote the S-matrix elements as [32]

$$\langle \text{out}|\text{in}\rangle \equiv Z = \mathcal{N} \int [\mathcal{D}A^a_\mu(x)] \exp\left[i \int \mathcal{L}_{\text{QCD}} d^4x \right], \qquad (G.4.1)$$

where the definition of the path integral volume $[\mathcal{D}A^a_\mu(x)]$ is just the same as the one explained in the previous section

$$\int [\mathcal{D}A^a_\mu(x)] \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{\mu,a} \prod_{(i,j,k,\ell)}^{n} dA^a_\mu(x^i_0, x^j_1, x^k_2, x^\ell_3). \qquad (G.4.2)$$

The Lagrangian density $\mathcal{L}_{\text{QCD}}$ for QCD is given as

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2} \text{Tr}\left(G_{\mu\nu} G^{\mu\nu} \right), \qquad (G.4.3)$$

where $G_{\mu\nu}$ denotes the field strength in QCD as given in Chapter 6

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \qquad (G.4.4)$$

However, the expression of the path integral in QCD in eq.(G.4.1) does not correspond to the field quantization. One can also understand why the path integral formulation of QCD cannot be done, in contrast to Feynman's integrations over the parameter space in QED. In QCD, the Hamiltonian for gluons contains the fourth power of $q^a_{\boldsymbol{k},\lambda}$, and therefore one cannot carry out the Gaussian integrations over the parameters $q^a_{\boldsymbol{k},\lambda}$ in QCD. As Feynman stated repeatedly in his original papers, the path integral formulation is closely connected to the Gaussian integration where the kinetic energy term in non-relativistic quantum mechanics is always described in terms of the quadratic term $\frac{\boldsymbol{p}^2}{2m}$. This naturally leads to the conclusion that the path integral formulation cannot be properly constructed in QCD, and this is consistent with the picture that QCD does not have a free Fock space due to its gauge non-invariance.

G.4.2 Fock Space

In quantum field theory, one must prepare Fock space since the Hamiltonian becomes an operator. The second quantized formulation is based on the creation and annihilation operators which act on the Fock space. In Feynman's path integral formulation, he prepared states which determine the number of photons in terms of $|q_{\boldsymbol{k},\lambda}\rangle$. Therefore, he started from the second quantized expression of the path integral formulation. However, Faddeev and Popov simply employed the same formula of the Lagrangian density, but integrated over the function $\mathcal{D}A^a_\mu(x)$. This cannot specify any quantum numbers of the Fock space, and therefore the integration over the function $\mathcal{D}A^a_\mu(x)$ does not correspond to the field quantization.

Appendix H

New Concept of Quantization

In this textbook, the first quantization of $[x_i, p_j] = i\delta_{ij}$ is employed to derive the Dirac equation. Here, we present a new way of understanding the first quantization. From the local gauge invariance and the Maxwell equation, one can derive the Lagrangian density of the Dirac field without involving the first quantization. This leads to a new concept of the quantization itself.

H.1 Derivation of Lagrangian Density of Dirac Field from Gauge Invariance and Maxwell Equation

Dirac derived the Dirac equation by factorizing eq.(D.1.1) such that the field equation becomes the first order in time derivative. Now, one can derive the Lagrangian density of Dirac field in an alternative way by employing the local gauge invariance and the Maxwell equation as the most fundamental principle [50].

H.1.1 Lagrangian Density for Maxwell Equation

We start from the Lagrangian density that reproduces the Maxwell equation

$$\mathcal{L} = -gj_\mu A^\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \tag{H.1.1}$$

where A^μ is the gauge field, and $F_{\mu\nu}$ is the field strength and is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

j_μ denotes the current density of matter field which couples to the electromagnetic field. From the Lagrange equation, one obtains

$$\partial_\mu F^{\mu\nu} = gj^\nu \tag{H.1.2}$$

which is just the Maxwell equation as we also saw in Appendix E.

H.1.2 Four Component Spinor

Now, one can derive the kinetic energy term of the fermion Lagrangian density. First, one assumes that the Dirac fermion should have four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

This is based on the observation that electron has spin degree of freedom which is two. In addition, there must be positive and negative energy states since it is a relativistic field, and therefore the fermion field should have 4 components.

16 Independent Components

Now, the matrix elements

$$\psi^\dagger \hat{O} \psi$$

can be classified into 16 independent Lorentz invariant components as

$$\bar{\psi}\psi: \ \text{scalar}, \quad \bar{\psi}\gamma_5\psi: \ \text{pseudo} - \text{scalar}, \tag{$H.1.3a$}$$

$$\bar{\psi}\gamma_\mu\psi: \ \text{4 component vector}, \quad \bar{\psi}\gamma_\mu\gamma_5\psi: \ \text{4 component axial-vector}, \tag{$H.1.3b$}$$

$$\bar{\psi}\sigma_{\mu\nu}\psi: \ \text{6 component tensor}, \tag{$H.1.3c$}$$

where $\bar{\psi}$ is defined for convenience as

$$\bar{\psi} = \psi^\dagger \gamma_0.$$

These properties are determined by mathematics.

Shape of Vector Current

From the invariance consideration, one finds that the vector current j_μ must be written as

$$j_\mu = C_0 \bar{\psi}\gamma_\mu\psi, \tag{$H.1.4$}$$

where C_0 is a constant. Since one can renormalize the constant C_0 into the coupling constant g, one can set without loss of generality

$$C_0 = 1. \tag{$H.1.5$}$$

H.2 Shape of Lagrangian Density

Now, one can make use of the local gauge invariance of the Lagrangian density, and one sees that the following shape of the Lagrangian density can keep the local gauge invariance

$$\mathcal{L} = C_1 \bar{\psi} \partial_\mu \gamma^\mu \psi - g \bar{\psi} \gamma_\mu \psi A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{H.2.1}$$

where C_1 is a constant. At this point, one requires that the Lagrangian density should be invariant under the local gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \chi, \tag{H.2.2a}$$

$$\psi \longrightarrow e^{-ig\chi} \psi. \tag{H.2.2b}$$

In this case, it is easy to find that the constant C_1 must be

$$C_1 = i. \tag{H.2.3}$$

Here, the constant $\hbar$ should be included implicitly into the constant C_1. The determination of $\hbar$ can be done only when one compares calculated results with experiment such as the spectrum of hydrogen atom.

H.2.1 Mass Term

The Lagrangian density of eq.(H.2.1) still lacks the mass term. Since the mass term must be a Lorentz scalar, it should be described as

$$C_2 \bar{\psi} \psi \tag{H.2.4}$$

which is, of course, gauge invariant as well. This constant C_2 should be determined again by comparing the calculated results of hydrogen atom, for example, with experiment. By denoting C_2 as $(-m)$, one arrives at the Lagrangian density of a relativistic fermion which couples with the electromagnetic fields A^μ

$$\mathcal{L} = i \bar{\psi} \partial_\mu \gamma^\mu \psi - g \bar{\psi} \gamma_\mu \psi A^\mu - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{H.2.5}$$

which is just the Lagrangian density for the Dirac field interacting with electromagnetic fields.

H.2.2 First Quantization

It is important to note that, in the procedure of deriving the Lagrangian density of eq.(H.2.5), one has not made use of the quantization condition of

$$E \to i \frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i \boldsymbol{\nabla}. \tag{H.2.6}$$

Instead, the first quantization is automatically done by the gauge condition since the Maxwell equation knows the first quantization in advance. This indicates that there may be some chance to understand the first quantization procedure in depth since this method gives an alternative way of the quantization condition of the energy and momentum.

H.3 Two Component Spinor

The derivation of the Dirac equation in terms of the local gauge invariance shows that the current density that can couple to the gauge field A^μ must be rather limited. Here, we discuss a possibility of finding field equation for the two component spinor. When the field has only two components,

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

then one can prove that one cannot make the current j_μ that couples with the gauge field A_μ. This can be easily seen since the matrix elements

$$\phi^\dagger \hat{O} \phi$$

can be classified into 4 independent variables as

$$\phi^\dagger\phi: \quad \text{scalar}, \quad \phi^\dagger\sigma_k\phi: \quad \text{3 component vector}. \tag{H.3.1}$$

Therefore, there is no chance to make four vector currents which may couple to the gauge field A_μ. This way of making the Lagrangian density indicates that it should be difficult to find a Lagrangian density of relativistic bosons.

H.4 Klein–Gordon Equation

In the new picture, the correspondence between $(E, \boldsymbol{p})$ and the differential operators $(i\frac{\partial}{\partial t}, -i\boldsymbol{\nabla})$ can be obtained as a consequence from the Dirac equation

$$\text{Dirac equation} \Longrightarrow E \to i\,\frac{\partial}{\partial t}\,, \quad \boldsymbol{p} \to -i\boldsymbol{\nabla}.$$

Therefore, it should not be considered as the fundamental principle any more. This equation can be applied to the non-relativistic motion since the Dirac equation can be reduced to the Schrödinger equation in the non-relativistic limit. However, one cannot apply the above relations to the relativistic kinematics since they are not the fundamental principle any more. In this respect, the derivation of the Schrödinger equation by the replacement of the energy and momentum by eq.(H.2.6)

$$E = \frac{\boldsymbol{p}^2}{2m} \Longrightarrow i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\,\boldsymbol{\nabla}^2\psi$$

can be justified, but the Klein–Gordon equation

$$E^2 = \boldsymbol{p}^2 + m^2 \Longrightarrow -\frac{\partial^2\psi}{\partial t^2} = \left(-\boldsymbol{\nabla}^2 + m^2\right)\psi$$

may not be justified any more. Therefore, this picture shows that the Klein–Gordon equation should not be taken as the fundamental equation for the elementary particles.

At the present stage, there is no way to derive the Klein–Gordon equation within the new picture. Therefore, it is most likely that there should not exist any elementary Klein–Gordon field of scalar bosons in nature, and in fact there is no elementary scalar boson observed up to now. However, the study of scalar field is mathematically an interesting subject, and one may learn some new aspects in boson fields in a future study.

In nature, there exist bosons, and they are all composite objects which have integer spins. For the composite boson which is composed of fermion and anti-fermion, a Klein–Gordon type equation should be obtained from the two particle Dirac equation. However, it may not be very easy to derive the equation for the center of mass motion for the two particle system in the Dirac equation since it involves the relativistic kinematics. But still theoretically it must be a doable calculation.

H.5 Incorrect Quantization in Polar Coordinates

In the standard quantum mechanical treatment, the particle Hamiltonian in polar coordinates in classical mechanics cannot be quantized in the canonical formalism. In order to see this situation more explicitly, one can write a free particle Hamiltonian with its mass m in polar coordinates

$$H = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\varphi}^2}{2mr^2 \sin^2\theta}, \tag{H.5.1}$$

where p_r, p_θ, p_φ are the generalized momenta in polar coordinates in classical mechanics. If one quantizes the Hamiltonian with the canonical quantization conditions

$$[r, p_r] = i, \quad [\theta, p_\theta] = i, \quad [\varphi, p_\varphi] = i \tag{H.5.2}$$

then one obtains

$$H = -\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{H.5.3}$$

which is not a correct Hamiltonian for a free particle in polar coordinates. The correct Hamiltonian for a free particle is, of course, given as

$$H = -\frac{1}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right) - \frac{1}{2mr^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{H.5.4}$$

which is obtained by transforming the Schrödinger equation in Cartesian coordinates into polar coordinates.

Now, we can explain the reason why the quantization condition is valid for the Cartesian coordinates, but not for the polar coordinates. In the new picture, the differential operators first appear in the Dirac equation and then the momentum is identified as the corresponding differential operator. Therefore, the Schrödinger equation is obtained from the Dirac equation, but not from the Hamiltonian of classical mechanics in the canonical formalism by replacing the generalized momenta by the corresponding differential operators.

H.6 Interaction with Gravity

The chain of the fundamental equations from classical mechanics to Dirac equation is now reversed

$$\text{Dirac equation} \Longrightarrow \text{Schrödinger equation} \Longrightarrow \text{Newton equation.}$$

At this point, however, one realizes that there is no way to include the gravitational force into the new system. This may be a serious defect of this new concept of the first quantization. However, if the gravity is included in this formalism, then it means that the quantum theory of gravity can be constructed.

Appendix I

Renormalization in QED

This textbook treats mainly the non-perturbative aspects of quantum field theory of fermions and therefore we have only briefly discussed the perturbation theory in QED in Chapter 5. In this Appendix, we present some of the basic ingredients in the renormalization scheme in the perturbative evaluation in QED. However, we only describe the basic concept of the renormalization scheme since calculations in detail are found in the textbook of Bjorken-Drell [14]. The essential difference between the perturbative evaluation and the non-perturbative treatment should be found in the Fock space which can be prepared in advance in the perturbative treatment while one does not know the eigenstates of the Hamiltonian in the non-perturbative calculation where one has to solve the eigenvalue equation of eq.(3.1).

I.1 Hilbert Space of Unperturbed Hamiltonian

First, we start from the QED Lagrangian density $\mathcal{L}$ which is composed of the unperturbed Lagrangian density $\mathcal{L}_0$ and the interaction term $\mathcal{L}_I$

$$\mathcal{L}_0 = \bar{\psi}(\not{p} - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \tag{I.1.1}$$

$$\mathcal{L}_I = -eA^\mu \bar{\psi}\gamma_\mu \psi. \tag{I.1.2}$$

In this case, the unperturbed Hamiltonian H_0 can be constructed from the Lagrangian density $\mathcal{L}_0$ as given in eq.(5.28a) in Chapter 5. The Hilbert space of the quantized Hamiltonian $\hat{H}_0$ can be well constructed since one finds the exact eigenvalues and eigenstates of eq.(3.1) for the $\hat{H}_0$. In this case, the Fock space can be specified by the box length L and the cutoff momentum Λ as well as by the energies and momenta of the free fermion and free gauge field states

$$(L,\Lambda): \quad \begin{pmatrix} E_p = \sqrt{\boldsymbol{p}_n^2 + m^2}, & \boldsymbol{p}_n = \dfrac{2\pi \boldsymbol{n}}{L} \\ \omega_k = |\boldsymbol{k}_n|, & \boldsymbol{k}_n = \dfrac{2\pi \boldsymbol{n}}{L} \end{pmatrix}, \tag{I.1.3}$$

$$n_i = 0, \pm 1, \ldots, \pm N \;\; \text{with} \;\; \Lambda = \frac{2\pi N}{L} \, ,$$

where the maximum number of freedom N is taken to be the same between the fermion and the gauge fields. The perturbative evaluation can be made within this Hilbert space and one can calculate physical quantities in terms of the S-matrix formulation.

In the evaluation of the second order perturbation energy, there are some diagrams which are divergent. However, the momentum integral has the cutoff due to the Λ, and therefore, the second order energies are described as the function of $\ln(\Lambda/m)$. Among the divergent diagrams, the photon self-energy contribution (vacuum polarization diagram) is special, and therefore we will teat it separately in Appendix J. Here, we discuss the fermion self-energy diagram and the vertex correction contribution.

I.2 Necessity of Renormalization

Before explaining the divergent contribution of the self-energy diagrams, we should understand reasons why we should make a renormalization procedure [59, 103, 107]. The fermion self-energy diagram itself is obviously unphysical since the fermion with its momentum p goes to the same fermion with the same momentum p after emitting a photon and absorbing the same photon. This process itself cannot be observed and examined experimentally. Why is it then necessary to consider the fermion self-energy diagram into the renormalization procedure?

I.2.1 Intuitive Picture of Fermion Self-energy

The answer to the above question is simple. We should consider the following situation in which the fermion is found in the different quantum state from the free state. For example, if we consider electron in hydrogen atom, then its quantum state should be very different from the free state. In this case, the fermion self-energy diagram gives a physical contribution which is observed as the Lamb shift energy. Basically, in the renormalization procedure, we consider a counter term which can exactly cancel the divergent contribution of the second order self-energy diagrams in the free state of fermions. However, in atom, this cancellation does not have to take place in an exact fashion, and this small effect of difference gives rise to the Lamb shift energy in $1s_{\frac{1}{2}}$ and $2s_{\frac{1}{2}}$ states in hydrogen atom.

I.2.2 Intuitive Picture of Photon Self-energy

On the other hand, the photon self-energy is quite different in that photon can never become a bound state. It is always found as a free state. Therefore, there is no physical process in which the contribution of the photon self-energy is detectable in some way or the other. In fact, there has been no physical processes which show the effect of the simple self-energy diagrams.

I.3 Fermion Self-energy

Now, the fermion self-energy $\Sigma(p)$ is obtained from the corresponding Feynman diagram as

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma_\mu \, \frac{1}{\not p - \not k - m}\, \gamma^\mu \, \frac{1}{k^2}\,. \tag{$I.3.1$}$$

This can be calculated to be

$$\Sigma(p) = \frac{e^2}{8\pi^2}\, \ln\left(\frac{\Lambda}{m}\right)(-\not p + 4m) + \text{finite terms}. \tag{$I.3.2$}$$

Therefore, the Lagrangian density of the free fermion part

$$\mathcal{L}_{\rm F} = \bar\psi(\not p - m)\psi \tag{$I.3.3$}$$

should be modified, up to one loop contributions, by the counter term $\delta\mathcal{L}_{\rm F}$

$$\delta\mathcal{L}_{\rm F} = \bar\psi\left[\frac{e^2}{8\pi^2}\ln\left(\frac{\Lambda}{m}\right)(-\not p + 4m)\right]\psi.$$

In this case, the total Lagrangian density of fermion becomes

$$\mathcal{L'}_{\rm F} = \mathcal{L}_{\rm F} + \delta\mathcal{L}_{\rm F} = \bar\psi\left[(1+B)\not p - (1+A)m\right]\psi + \text{finite terms}\,, \tag{$I.3.4$}$$

where

$$A = -\frac{e^2}{2\pi^2}\,\ln\left(\frac{\Lambda}{m}\right), \quad B = -\frac{e^2}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right). \tag{$I.3.5$}$$

Now, one defines Z_2 and δm as

$$Z_2 \equiv 1 + B = 1 - \frac{e^2}{8\pi^2}\ln\left(\frac{\Lambda}{m}\right), \tag{$I.3.6a$}$$

$$\delta m \equiv \frac{3e^2 m}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right). \tag{$I.3.6b$}$$

Here, one can introduce the wave function renormalization and the bare mass m_0

$$\psi_{\rm b} \equiv \sqrt{Z_2}\psi = \sqrt{1+B}\psi, \tag{$I.3.7a$}$$

$$\begin{aligned} m_0 &\equiv Z_2^{-1} m(1+A) \\ &= m\left(1 + \frac{e^2}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right)\right)\left(1 - \frac{e^2}{2\pi^2}\,\ln\left(\frac{\Lambda}{m}\right)\right) \simeq m - \delta m, \end{aligned} \tag{$I.3.7b$}$$

where one should always keep up to order of e^2. In this case, one can rewrite $\mathcal{L'}_{\rm F}$ as

$$\mathcal{L'}_{\rm F} = \bar\psi_{\rm b}(\not p - m_0)\psi_{\rm b} + \text{finite terms} \tag{$I.3.8$}$$

which has just the same shape as the original one, and thus it is renormalizable.

I.4 Vertex Corrections

The vertex corrections can be evaluated as

$$\Lambda^\mu(p',p) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\nu \frac{1}{\not p' - \not k - m}\,\gamma^\mu \frac{1}{\not p - \not k - m}\,\gamma_\nu \frac{1}{k^2}$$

$$= \frac{e^2}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right)\gamma^\mu + \text{finite terms}. \qquad (I.4.1)$$

Therefore, the counter term of the interaction Lagrangian density $\delta\mathcal{L}_I$ becomes

$$\delta\mathcal{L}_I = \frac{e^3}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right) A^\mu \bar\psi\gamma_\mu\psi.$$

In this case, the total interaction Lagrangian density can be written as

$$\mathcal{L'}_I = -eA^\mu\bar\psi\gamma_\mu\psi + \delta\mathcal{L}_I = -Z_1 eA^\mu\bar\psi\gamma_\mu\psi + \text{finite terms}, \qquad (I.4.2)$$

where Z_1 is defined as

$$Z_1 \equiv 1 - \frac{e^2}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right). \qquad (I.4.3)$$

From eqs.(I.3.6) and (I.4.3), one finds

$$Z_1 = Z_2. \qquad (I.4.4)$$

This can be also understood from the following identity

$$\frac{\partial}{\partial p_\mu}\left(\frac{1}{\not p - m}\right) = -\frac{1}{\not p - m}\,\gamma^\mu\,\frac{1}{\not p - m}\,. \qquad (I.4.5)$$

Therefore, one finds

$$\Lambda^\mu(p,p) = -\frac{\partial\Sigma(p)}{\partial p_\mu} \qquad (I.4.6)$$

which leads to eq.(I.4.4). The interaction Lagrangian density can be rewritten in terms of the bare quantities

$$\psi_{\rm b} \equiv \sqrt{Z_2}\psi \qquad (I.4.7)$$

as

$$\mathcal{L'}_I = -Z_1 eA^\mu\bar\psi\gamma_\mu\psi = -Z_1 e\frac{1}{Z_2}\,A^\mu\bar\psi_{\rm b}\gamma_\mu\psi_{\rm b} = -e_{\rm b}A^\mu\bar\psi_{\rm b}\gamma_\mu\psi_{\rm b} + \text{finite terms}, \quad (I.4.8)$$

where the bare charge $e_{\rm b}$ turns out to be

$$e_{\rm b} = e. \qquad (I.4.9)$$

Therefore, all the infinite quantities are renormalized into the physical constants as well as the wave function. It should be important that the charge e is not affected from the renormalization.

I.5 New Aspects of Renormalization in QED

The renormalization scheme in QED is best studied and most reliable. Nevertheless there is one important modification in the renormalization procedure, and it is connected to the treatment of the photon self-energy diagram. It turns out that the photon self-energy contribution should not be considered for the renormalization procedure since the contribution itself is unphysical. The detailed discussion of the photon self-energy in QED is treated in Appendix J. This gives rise to some important consequences. The most important of all is that the charge is not affected by the renormalization procedure. This means that there is no renormalization group equation. Here, we should first like to discuss the old version of the renormalization group equation in QED and then compare the new simple picture which has no renormalization group equation since the charge has no renormalization effect.

I.5.1 Renormalization Group Equation in QED

(1) Old Version

For the renormalization group equation, people start from the following equation for the renormalized coupling constant $e_{\rm b}$

$$e_{\rm b} = eZ_1\lambda^{\frac{\epsilon}{2}}\,\frac{1}{Z_2\sqrt{Z_3}} = e\lambda^{\frac{\epsilon}{2}}\left(1+\frac{e^2}{12\pi^2\epsilon}\right), \qquad (I.5.1)$$

where λ is a parameter which is introduced in the dimensional regularization and ϵ is an infinitesimally small number. Here, Z_3 is defined as

$$Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon}$$

which comes from the wave function renormalization of the photon self-energy contribution. Even though eq.(I.5.1) is obtained from renormalizing all of the self-energy diagrams together with the vertex corrections, this renormalization of the coupling constant e_b is basically originated from the photon self-energy contribution since the renormalization constant Z_1 of the vertex correction and the fermion wave function renormalization Z_2 cancel with each other because of $Z_1 = Z_2$. Since the $e_{\rm b}$ should not depend on the scale λ, one can obtain the renormalization group equation

$$\lambda\frac{\partial e}{\partial\lambda} = -\frac{e}{2}\left(1-\frac{e^2}{4\pi^2\epsilon}\right)\left(\epsilon+\frac{e^2}{12\pi^2}\right) = -\frac{\epsilon}{2}\,e + \frac{1}{12\pi^2}\,e^3 + O(e^5). \qquad (I.5.2)$$

When one makes $\epsilon \to 0$, then one finds

$$\lambda\frac{\partial e}{\partial\lambda} = \frac{1}{12\pi^2}\,e^3 \qquad (I.5.3)$$

which is the renormalization group equation for QED. This equation can be easily solved for e. The expression for the running coupling constant $\alpha(\lambda) \equiv \frac{e^2}{4\pi}$ is given as

$$\alpha(\lambda) = \frac{\alpha(\lambda_0)}{1 - \frac{2\alpha(\lambda_0)}{3\pi} \ln(\frac{\lambda}{\lambda_0})}\,, \qquad (I.5.4)$$

where λ_0 denotes the renormalization point for the coupling constant. This is the standard procedure to obtain the behavior of the coupling constant as the function of the λ.

However, the λ does not appear in the Hilbert space of the unperturbed Hamiltonian H_0 and this means that one cannot interpret the result of eq.(I.5.4) in terms of the physical observables which should be found in the unperturbed Fock space. Therefore $\alpha(\lambda)$ does not have any physical meaning at all.

(2) New Version

The fact that the photon self-energy contribution should not be considered for the renormalization procedure has many important consequences. The most important is that the charge is not influenced from the renormalization as one can see it from eq.(I.4.9) and, therefore, there is no renormalization group equation in QED. The idea of the renormalization group equation itself is gone.

I.6 Renormalization in QCD

When one wishes to carry out the perturbation calculation, then one should first construct the free Fock space since all of the perturbative evaluations aim at describing physical observables in terms of the free state physical quantities. However, QCD has intrinsic difficulties to construct the free Fock space.

I.6.1 Fock Space of Free Fields

The normal perturbation theory always starts from the free field Hilbert space which is made from the unperturbed Hamiltonian H_0. This free field Fock space must be constructed from the physical states as long as one stays in the perturbation theory. However, in QCD, there is no free quark state in nature due to the gauge non-invariance of the quark color charge. In fact, if one defines the unperturbed Lagrangian density $\mathcal{L}_0$ for QCD as

$$\mathcal{L}_0 = \bar{\psi}(\not{p} - m)\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a})$$

then, one finds that it is not invariant under the local gauge transformation of eqs.(6.7) and (6.8). Even for the free fermion part, it transforms under eq.(6.7) as

$$\bar{\psi}'(\not{p} - m)\psi' = \bar{\psi}(\not{p} - m)\psi - g(\partial^\mu \chi^a) j^a_\mu = \bar{\psi}(\not{p} - m)\psi - g\partial^\mu(\chi^a j^a_\mu) + g(\partial^\mu j^a_\mu)\chi^a$$

which is not invariant since the vector current is not conserved ($\partial^\mu j^a_\mu \neq 0$) for QCD, contrary to QED.

I.6.2 Renormalization Group Equation in QCD

Therefore, one cannot find any perturbation scheme in QCD, and there is no renormalization. In this respect, the gluon coupling constant is not affected from the renormalization. This is always true even if one could define some type of the perturbation theory by hand. Therefore, there is no renormalization group equation for QCD, and thus the concept of the asymptotic freedom has lost its physical meaning.

I.6.3 Serious Problems in QCD

However, OCD has much more serious problems than the lack of the QCD renormalization group equation. As we stress above, we cannot carry out the perturbative calculations since there is no free Fock state in QCD. This means that one cannot describe any physical quantities in terms of the free Fock state terminology since the free states of quarks and gluons do not exist as the physical states. In this respect, the evaluation of QCD cannot be based on the perturbation scheme and therefore one has to invent a new scheme to solve QCD. This is a very important problem which should be solved in some way or the other in future [35].

I.7 Renormalization of Massive Vector Fields

Since the self-energy of boson fields should not be considered for the renormalization procedure, there are many important consequences concerning the renormalizability of the model field theory. Here, we should briefly discuss the renormalization procedure of the massive vector fields which interact with fermion fields. The simplest Lagrangian density for two flavor leptons which couple to the SU(2) vector fields W_μ^a can be written as

$$\mathcal{L} = \bar{\Psi}(i\partial_\mu\gamma^\mu + m)\Psi - gJ_\mu^a W^{\mu,a} + \frac{1}{2}\,M^2 W_\mu^a W^{\mu,a} - \frac{1}{4}\,G_{\mu\nu}^a G^{\mu\nu,a}, \qquad (I.7.1)$$

where M denotes the mass of the vector boson. The fermion wave function Ψ has two components.

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \qquad (I.7.2)$$

Correspondingly, the mass matrix can be written as

$$m = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}. \qquad (I.7.3)$$

The fermion current J_μ^a and the field strength $G_{\mu\nu}^a$ are defined as

$$J_\mu^a = \bar{\Psi}\gamma_\mu\tau^a\Psi, \quad G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a. \qquad (I.7.4)$$

This Lagrangian density is almost the same as the standard model Lagrangian density apart from the Higgs fields, as far as the basic structure of the field theory is concerned.

I.7.1 Renormalizability

Now, one can easily examine the renormalizability of the model field theory, and one can prove that the fermion self-energy has the logarithmic divergence which is just the same as the QED case. Also, the vertex correction in this model field theory has the logarithmic divergence and is the same as the QED case as long as the renormalizability is concerned. Since one should not consider the self-energy of the massive vector fields, one can safely renormalize the divergent contributions of the fermion self-energy as well as the vertex correction.

In this sense, this is just similar to checking the renormalizability of the *final* Lagrangian density of the standard model, and it is indeed renormalizable. This is quite important since the final Lagrangian density of the Weinberg-Salam model is quite successful for describing the experiment even though the Higgs mechanism itself is not a correct procedure in terms of the symmetry breaking physics as discussed in Chapter 4.

Appendix J

Photon Self-energy Contribution in QED

In the renormalization procedure of QED, one considers the vacuum polarization which is the contribution of the self-energy diagram of photon

$$\Pi^{\mu\nu}(k) = ie^2 \int \frac{d^4p}{(2\pi)^4} \,\mathrm{Tr}\left[\gamma^\mu \frac{1}{\not{p} - m} \gamma^\nu \frac{1}{\not{p} - \not{k} - m}\right]. \qquad (J.0.1)$$

This integral obviously gives rise to the quadratic divergence (Λ^2 term). However, when one considers the counter term of the Lagrangian density which should cancel this quadratic divergence term, then the counter Lagrangian density violates the gauge invariance since it should correspond to the mass term in the gauge field Lagrangian density. Therefore, one has to normally erase it by hand, and in the cutoff procedure of the renormalization scheme, one subtracts the quadratic divergence term such that one can keep the gauge invariance of the Lagrangian density. Here, we should notice that the largest part of the vacuum polarization contributions is discarded, and this indicates that there must be something which is not fully understandable in the renormalization procedure. Physically, it should be acceptable to throw away the Λ^2 term since this infinite term should not be connected to any physical observables. Nevertheless we should think it over why the unphysical infinity appears in the self-energy diagram of photon.

On the other hand, the quadratic divergence term disappears in the treatment of the dimensional regularization scheme. Here, we clarify why the quadratic divergence term does not appear in the dimensional regularization treatment. That is, the treatment of the dimensional regularization employs the mathematical formula which is not valid for the evaluation of the momentum integral. Therefore, the fact that there is no quadratic divergence term in the dimensional regularization is simply because one makes a mistake by applying the invalid mathematical formula to the momentum integral. This is somewhat surprising, but now one sees that the quadratic divergence is still there in the dimensional regularization, and this strongly indicates that we should reexamine the effect of the photon self-energy diagram itself.

J.1 Momentum Integral with Cutoff Λ

This procedure of the photon self-energy is well explained in the textbook of Bjorken and Drell, and therefore we describe here the simplest way of calculating the momentum integral. The type of integral one has to calculate can be summarized as

$$\int d^4p \, \frac{1}{(p^2 - s + i\varepsilon)^n} = i\pi^2 \int_0^{\Lambda^2} w \, dw \, \frac{1}{(w - s + i\varepsilon)^n} \,, \quad \text{with} \;\; w = p^2, \qquad (J.1.1)$$

where i appears because the integral is rotated into the Euclidean space and this corresponds to $D = 4$ in the dimensional regularization as we will see it below.

J.1.1 Photon Self-energy Contribution

The photon self-energy contribution $\Pi^{\mu\nu}(k)$ in eq.(J.0.1) can be easily evaluated as

$$\Pi_{\mu\nu}(k) = \frac{4ie^2}{(2\pi)^4} \int_0^1 dz \int d^4p \left[\frac{2p_\mu p_\nu - g_{\mu\nu}p^2 + s g_{\mu\nu} - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu})}{(p^2 - s + i\varepsilon)^2} \right]$$

$$= \frac{\alpha}{2\pi} \int_0^1 dz \int_0^{\Lambda^2} dw \left[\frac{w(w-2s)g_{\mu\nu} + 4z(1-z)w(k_\mu k_\nu - k^2 g_{\mu\nu})}{(w - s + i\varepsilon)^2} \right], \qquad (J.1.2)$$

where s is defined as $s = m^2 - z(1-z)k^2$. This can be calculated to be

$$\Pi_{\mu\nu}(k) = \Pi^{(1)}_{\mu\nu}(k) + \Pi^{(2)}_{\mu\nu}(k),$$

where

$$\Pi^{(1)}_{\mu\nu}(k) = \frac{\alpha}{2\pi} \left(\Lambda^2 + m^2 - \frac{k^2}{6} \right) g_{\mu\nu} \qquad (J.1.3a)$$

$$\Pi^{(2)}_{\mu\nu}(k) = \frac{\alpha}{3\pi}(k_\mu k_\nu - k^2 g_{\mu\nu}) \left[\ln\left(\frac{\Lambda^2}{m^2 e} \right) - 6 \int_0^1 dz z(1-z) \ln\left(1 - \frac{k^2}{m^2} z(1-z) \right) \right]. \qquad (J.1.3b)$$

Here, the $\Pi^{(1)}_{\mu\nu}(k)$ term corresponds to the quadratic divergence term and this should be discarded since it violates the gauge invariance. The $\Pi^{(2)}_{\mu\nu}(k)$ term can keep the gauge invariance, and therefore one can renormalize it into the new Lagrangian density.

J.1.2 Finite Term in Photon Self-energy Diagram

After the renormalization, one finds a finite term which should affect the propagator change in the process involving the exchange of the transverse photon $\boldsymbol{A}$. The propagator $\frac{1}{q^2}$ should

be replaced by

$$\frac{1}{q^2} \Longrightarrow \frac{1}{q^2}\left[1+\frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\ln\left(1-\frac{q^2 z(1-z)}{m^2}\right)\right], \qquad (J.1.4)$$

where q^2 should become $q^2 \approx -\boldsymbol{q}^2$ for small $\boldsymbol{q}^2$. It should be important to note that the correction term arising from the finite contribution of the photon self-energy should affect only for the renormalization of the vector field $\boldsymbol{A}$. Since the Coulomb propagator is not affected by the renormalization procedure of the transverse photon (vector field $\boldsymbol{A}$), one should not calculate its effect on the Lamb shift.

J.2 Dimensional Regularization

In the evaluation of the momentum integral, one can employ the dimensional regularization [72, 73] where the integral is replaced as

$$\int \frac{d^4p}{(2\pi)^4} \longrightarrow \lambda^{4-D}\int \frac{d^Dp}{(2\pi)^D}\,, \qquad (J.2.1)$$

where λ is introduced as a parameter which has a mass dimension in order to compensate the unbalance of the momentum integral dimension. This is the integral in the Euclidean space, but D is taken to be $D = 4-\epsilon$ where ϵ is an infinitesimally small number.

J.2.1 Photon Self-energy Diagram with $D = 4-\epsilon$

In this case, the photon self-energy $\Pi_{\mu\nu}(k)$ can be calculated to be

$$\begin{aligned}\Pi_{\mu\nu}(k) &= i\lambda^{4-D}e^2\int\frac{d^Dp}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma_\mu\frac{1}{\not p - m}\,\gamma_\nu\,\frac{1}{\not p - \not k - m}\right]\\ &= \frac{\alpha}{3\pi}\,(k_\mu k_\nu - g_{\mu\nu}k^2)\left[\frac{2}{\epsilon} + \text{finite term}\right], \qquad (J.2.2)\end{aligned}$$

where the finite term is just the same as eq.(J.1.3). In eq.(J.2.2), one sees that the quadratic divergence term ($\Pi^{(1)}_{\mu\nu}(k)$) is missing. This is surprising since the quadratic divergence term is the leading order contribution in the momentum integral, and whatever one invents in the integral, there is no way to erase it unless one makes a mistake.

J.2.2 Mathematical Formula of Integral

Indeed, in the treatment of the dimensional regularization, people employ the mathematical formula which is invalid for the integral in eq.(J.2.2). That is, the integral formula for $D = 4-\epsilon$

$$\int d^Dp\frac{p_\mu p_\nu}{(p^2-s+i\varepsilon)^n} = i\pi^{\frac{D}{2}}(-1)^{n+1}\frac{\Gamma(n-\frac{1}{2}D-1)}{2\Gamma(n)}\,\frac{g_{\mu\nu}}{s^{n-\frac{1}{2}D-1}} \quad (\text{for } n\geq 3) \qquad (J.2.3)$$

is only valid for $n \geq 3$ in eq.(J.2.3). For $n = 3$, the integral should have the logarithmic divergence, and this is nicely avoided by the replacement of $D = 4 - \epsilon$. However, the $n = 2$ case must have the quadratic divergence and the mathematical formula of eq.(J.2.3) is meaningless. In fact, one should recover the result of the photon self-energy contribution $\Pi_{\mu\nu}(k)$ of eqs.(J.1.3) at the limit of $D = 4$, apart from the $(2/\epsilon)$ term which corresponds to the logarithmic divergence term.

J.2.3 Reconsideration of Photon Self-energy Diagram

In mathematics, one may define the gamma function in terms of the algebraic equations with complex variables. However, the integral in the renormalization procedure is defined only in real space integral, and the infinity of the integral is originated from the infinite degrees of freedom which appear in the free Fock space as discussed in Appendix I.

Therefore, one sees that the disappearance of the quadratic divergence term in the evaluation of $\Pi_{\mu\nu}(k)$ in the dimensional regularization is not due to the mathematical trick, but simply due to a simple-minded mistake. In this respect, it is just accidental that the $\Pi^{(1)}_{\mu\nu}(k)$ term in the dimensional regularization vanishes to zero. Indeed, one should obtain the same expression of the $\Pi^{(1)}_{\mu\nu}(k)$ term as eq.(J.1.3a) when one makes $\epsilon \to 0$ in the calculation of the dimensional regularization. This strongly suggests that we should reconsider the photon self-energy diagram itself in the renormalization procedure [45].

J.3 Propagator Correction of Photon Self-energy

In order to examine whether the inclusion of the photon self-energy contribution is necessary for the renormalization procedure or not, we should consider the effect of the finite contribution from the photon self-energy diagram. As we see, there is a finite contribution of the transverse photon propagator to physical observables after the renormalization of the photon self-energy. The best application of the propagator correction must be the magnetic hyperfine splitting of the ground state ($1s_{\frac{1}{2}}$ state) in the hydrogen atom since this interaction is originated from the vector field $\boldsymbol{A}$ which gives rise to the magnetic hyperfine interaction between electrons and nucleus.

J.3.1 Lamb Shift Energy

In some of the textbooks [14], the correction term arising from the finite contribution of the photon self-energy diagram is applied to the evaluation of the Lamb shift energy in hydrogen atom. However, this is not a proper application since only the renormalization of the vector field $\boldsymbol{A}$ should be considered. This is closely connected to the understanding of the field quantization itself. One sees that the second quantization of the electromagnetic field should be made only for the vector field $\boldsymbol{A}$, and this is required from the experimental observation that photon is created from the vacuum of the electromagnetic field in the atomic

transitions. Therefore, it is clear that the renormalization becomes necessary only for the vector field $\boldsymbol{A}$.

Classical Picture of Polarization

This classical picture of the fermion pair creations (Uehling potential) should come from the misunderstanding of the structure of the vacuum state [65, 66, 110]. In the medium of solid state physics, the polarization can take place when there is an electric field present. In this case, the electric field can indeed induce the electric dipole moments in the medium, and this corresponds to the change of the charge density. However, this is a physical process which can happen in the real space (configuration space). On the other hand, the fermion pair creation in the vacuum in field theory is completely different in that the negative energy states are all filled in momentum space, and the time independent field of A^0 which is only a function of coordinates cannot induce any changes on the vacuum state. Therefore, unless some time dependent field is present in the reaction process, the pair creation of fermions cannot take place in physical processes. Therefore, in contrast to the common belief, there is, unfortunately, no change of the charge distribution in QED vacuum, even at the presence of two charges, and this is basically because the Coulomb field is not time dependent.

J.3.2 Magnetic Hyperfine Interaction

The magnetic hyperfine interaction between electron and proton in hydrogen atom can be written with the static approximation in the classical field theory as

$$H' = -\int \boldsymbol{j}_e(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r})\, d^3r, \tag{J.3.1}$$

where $\boldsymbol{j}_e(\boldsymbol{r})$ denotes the current density of electron, and $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential generated by proton and is given as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{\boldsymbol{J}_p(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3r', \tag{J.3.2}$$

where $\boldsymbol{J}_p(\boldsymbol{r})$ denotes the current density of proton. The hyperfine splitting of the ground state in the hydrogen atom can be calculated as

$$\Delta E_{hfs} = \langle 1s_{\frac{1}{2}}, I:F|H'|1s_{\frac{1}{2}}, I:F\rangle, \tag{J.3.3}$$

where I and F denote the spins of proton and atomic system, respectively. This can be explicitly calculated as

$$\Delta E_{hfs} = (2F(F+1)-3)\,\frac{\alpha g_p}{3M_p}\int_0^\infty F^{(1s)}(r)G^{(1s)}(r)\, dr, \tag{J.3.4}$$

where g_p and M_p denote the g-factor and the mass of proton, respectively. $F^{(1s)}(r)$ and $G^{(1s)}(r)$ are the small and large components of the radial parts of the Dirac wave function of electron in the atom. In the nonrelativistic approximation, the integral can be expressed as

$$\int_0^\infty F^{(1s)}(r)G^{(1s)}(r)\,dr \simeq \frac{(m_r\alpha)^3}{m_e}\,, \qquad (J.3.5a)$$

where m_e is the mass of electron and m_r denotes the reduced mass defined as

$$m_r = \frac{m_e}{1+\frac{m_e}{M_p}}\,. \qquad (J.3.6)$$

It should be noted that, in eq.(J.3.5), m_e appears in the denominator because it is originated from the current density of electron. Therefore, the energy splitting between $F=1$ and $F=0$ atomic states in the nonrelativistic limit with a point nucleus can be calculated from Eq.(J.3.4) as

$$\Delta E^{(0)}_{hfs} = \frac{8\alpha^4 m_r^3}{3m_e M_p}\,. \qquad (J.3.7)$$

J.3.3 QED Corrections for Hyperfine Splitting

There are several corrections which arise from the various QED effects such as the anomalous magnetic moments of electron and proton, nuclear recoil effects and relativistic effects. We write the result

$$\Delta E^{(QED)}_{hfs} = \frac{4g_p\alpha^4 m_r^3}{3m_e M_p}\,(1+a_e)\left(1+\frac{3}{2}\,\alpha^2\right)(1+\delta_R), \qquad (J.3.8)$$

where a_e denotes the anomalous magnetic moment of electron. The term $(1+\frac{3}{2}\,\alpha^2)$ appears because of the relativistic correction of the electron wave function

$$\int_0^\infty F^{(1s)}(r)G^{(1s)}(r)\,dr = \frac{(m_r\alpha)^3}{m_e}\left(1+\frac{3}{2}\,\alpha^2+\cdots\right). \qquad (J.3.5b)$$

The term δ_R corresponds to the recoil corrections and can be written as [16]

$$\delta_R = \alpha^2\left(\ln 2-\frac{5}{2}\right) - \frac{8\alpha^3}{3\pi}\,\ln\alpha\left(\ln\alpha-\ln 4+\frac{281}{480}\right)+\frac{15.4\alpha^3}{\pi}\,. \qquad (J.3.9)$$

Now the observed value of $\Delta E^{(exp)}_{hfs}$ is found to be [30]

$$\Delta E^{(exp)}_{hfs} = 1420.405751767\ \text{MHz}.$$

Also, we can calculate $\Delta E^{(0)}_{hfs}$ and $\Delta E^{(QED)}_{hfs}$ numerically and their values become

$$\Delta E^{(0)}_{hfs} = 1418.83712\ \text{MHz}, \quad \Delta E^{(QED)}_{hfs} = 1420.448815\ \text{MHz}.$$

Therefore, we find the deviation from the experimental value as

$$\frac{\Delta E_{hfs}^{(exp)} - \Delta E_{hfs}^{(QED)}}{\Delta E_{hfs}^{(0)}} \simeq -30 \ \text{ppm}. \tag{J.3.10}$$

J.3.4 Finite Size Corrections for Hyperfine Splitting

In addition to the QED corrections, there is a finite size correction of proton and its effect can be written as

$$\Delta E_{hfs}^{(FS)} = \Delta E_{hfs}^{(0)}(1+\varepsilon), \tag{J.3.11}$$

where the ε term corresponds to the Bohr-Weisskopf effect [17, 37, 114]

$$\varepsilon \simeq -m_e \alpha R_p, \tag{J.3.12}$$

where R_p denotes the radius of proton. It should be noted that the perturbative treatment of the finite proton size effect on the hyperfine splitting overestimates the correction by a factor of two. Now, the calculated value of ε becomes

$$\varepsilon \simeq -17 \ \text{ppm}.$$

Therefore, the agreement between theory and experiment is quite good.

J.3.5 Finite Propagator Correction from Photon Self-energy

The hyperfine splitting of the $1s_{\frac{1}{2}}$ state energy including the propagator correction can be written in terms of the momentum representation in the nonrelativistic limit as

$$\begin{aligned}\Delta E_{hfs}^{(VP)} &= (2F(F+1)-3)\,\frac{16}{3\pi}\,\frac{\alpha^5 m_r^4}{m_e M_p}\int_0^\infty \frac{q^2 dq}{(q^2+4(m_r\alpha)^2)^2}\left(1+M^R(q)\right) \\ &\equiv \Delta E_{hfs}^{(0)}(1+\delta_{vp}).\end{aligned} \tag{J.3.13}$$

$M^R(q)$ denotes the propagator correction and can be written as

$$M^R(q) = \frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\ln\left(1+\frac{q^2}{m_e^2}\,z(1-z)\right). \tag{J.3.14}$$

We can carry out numerical calculations of the finite term of the renormalization in the photon self-energy diagram, and we find

$$\delta_{vp} \simeq 18 \ \text{ppm} \tag{J.3.15}$$

which tends to make a deviation larger between theory and experiment of hyperfine splitting in hydrogen atom. This suggests that the finite correction from the photon self-energy contribution should not be considered for the renormalization procedure.

J.3.6 Magnetic Moment of Electron

The finite terms of the vacuum polarization contribute to the magnetic moment of electron. The evaluation of the magnetic moment of electron which is often described as g factor can be carried out within the renormalization scheme of QED. Here, we only quote the calculated results of one and two loop diagrams which are up to the order of α^2.

One loop calculation

The calculation of the electron g factor is just the result of the vertex correction in the renormalization procedure, and it is given as

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \cdots\right). \qquad (J.3.16)$$

It should be important to note that the diagram that contains the vacuum polarization in the one loop calculation vanishes to zero due to the condition that the magnetic moment of electron is measured at the zero momentum transfer.

Two loop calculation

The g factor of electron is calculated up to the two loop order, and it is given as

$$g = 2\left(1 + \frac{\alpha}{2\pi} + W_0\left(\frac{\alpha}{\pi}\right)^2 + W_{vp}\left(\frac{\alpha}{\pi}\right)^2 + \cdots\right), \qquad (J.3.17)$$

where W_0 denotes the contributions of the two loop diagrams without the vacuum polarization term while W_{vp} is the contribution of the two loop order from the vacuum polarization term. They are calculated to be [3]

$$W_0 = \frac{3}{4}\,\zeta(3) - \frac{\pi^2}{2}\,\ln 2 + \frac{5}{12}\,\pi^2 - \frac{279}{144} = -0.34416639, \qquad (J.3.18a)$$

$$W_{vp} = -\frac{\pi^2}{3} + \frac{119}{36} = 0.015687, \qquad (J.3.18b)$$

where $\zeta(3)$ is given as $\zeta(3) = 1.202056903$. It should be noted that the contribution of the finite term of the vacuum polarization is a few percents of the whole two loop order calculations, and in this sense it cannot be a very important contribution to the electron magnetic moment. In addition, the vacuum polarization in two loop order should have a $(\ln\Lambda)^2$ term which should give rise to a conceptual difficulty in the renormalization procedure.

Now, the observed g factor of electron is given [111]

$$g_{\rm exp} = 2\,(1 + 0.001\,159\,652\ ). \qquad (J.3.19a)$$

The calculated g factor of electron without the vacuum polarization diagram becomes

$$g_{\rm no-vp} = 2\,(1 + 0.001\,159\,553\ ) \qquad (J.3.19b)$$

while the total value of the calculated g factor of electron becomes

$$g_{\rm total} = 2\ (1 + 0.001\ 159\ 637\ \ ) \qquad (J.3.19c)$$

where we take the value of the fine structure constant of α as $\alpha^{-1} = 137.036$.

Three loop calculation

The three loop calculations are found to be around [77]

$$g_{\rm threeloopl} = 2\ \left(1 + \cdots + c_3 \left(\frac{\alpha}{\pi}\right)^3\ \right) + \cdots \qquad (J.3.19d)$$

where $c_3 \simeq 1.18$. In this sense, one sees that the total value of the two loop calculations seems to reproduce the experimental number better than the one without the vacuum polarization diagram.

Ambiguity in Fine Structure Constant α

From the observed value of the Zeeman splitting energy in electron

$$\Delta E_{exp} = \frac{e\hbar g}{2m_e c} B = \frac{\sqrt{4\pi\alpha}\hbar g}{2m_e c} B \qquad (J.3.20)$$

one can extract the g factor value. In this case, we should examine the accuracy of the constants which appear in eq.(J.3.20). Here, B denotes the external magnetic field. The experimental accuracies of $\hbar, m_e, c, B$ are all reliable up to 8th digit. However, we should be careful for the value of the coupling constant α since the experimental value of g_{exp} is very sensitive to the value of α. In fact, the difference between the α values at the 8-th digit may well be responsible for the g factor value at the corresponding accuracy. In this respect, it is most important that the experimental determination of the α value should be done at the accuracy of the 8-th digit independently from the $g-2$ experiment.

J.4 Spurious Gauge Conditions

The evaluation of the vacuum polarization contribution gives rise to the quadratic divergence and, if this should exist, there is no way to renormalize it into the standard renormalization scheme [14]. In fact, Pauli and Villars proposed [101] that the quadratic divergence term should be evaded by the requirement that the calculated result should be gauge invariant when renormalizing it into the Lagrangian density. This requirement of the gauge invariance should be based on the following relation for the vacuum polarization tensor $\Pi^{\mu\nu}(k)$ as

$$k_\mu \Pi^{\mu\nu}(k) = 0. \qquad (J.4.1)$$

However, this equation does not hold and it is indeed a spurious equation even though it has been employed as the *gauge condition*. The proof of eq.(J.4.1) in most of the field

theory textbooks is simply a mathematical mistake basically due to the wrong replacement of the integration variable in the infinite integral case. This problem of mathematics was, of course, realized and stated in the text book of Bjorken and Drell [14], but they accepted this relation up to the last step of their calculation of the vacuum polarization contributions.

J.4.1 Gauge Condition of $\Pi^{\mu\nu}(k)$

If one carries out the self-energy diagram of photon, then one obtains the vacuum polarization tensor which is given in eqs.(J.1.3), and this is obviously inconsistent with eq.(J.4.1). For a long time, people believe that the $\Pi^{\mu\nu}(k)$ should satisfy the relation of eq.(J.4.1)

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

and this equation is called "gauge condition of $\Pi^{\mu\nu}(k)$". Now, we present the proof of the above relation as discussed in the text book of Bjorken and Drell [14], and show that the proof of eq.(J.4.1) is a simple mathematical mistake. Therefore, this gauge condition is spurious, and the relation has no physical foundation at all. The standard method of the proof starts by rewriting the $k_\mu \Pi^{\mu\nu}(k)$ as

$$k_\mu \Pi^{\mu\nu}(k) = ie^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\left(\frac{1}{\not{p} - \not{k} - m + i\varepsilon} - \frac{1}{\not{p} - m + i\varepsilon}\right)\gamma^\nu\right]. \qquad (J.4.2)$$

In the first term, the integration variable should be replaced as

$$q = p - k$$

and thus one can prove that

$$k_\mu \Pi^{\mu\nu}(k) = ie^2 \left\{ \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{\not{q} - m + i\varepsilon}\gamma^\nu\right] - \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{\not{p} - m + i\varepsilon}\gamma^\nu\right] \right\} = 0. \qquad (J.4.3)$$

At a glance, this proof looks plausible. However, one can easily notice that the replacement of the integration variable is only meaningful when the integral is finite.

In order to clarify the mathematical mistake in eq.(J.4.3), we present a typical example which shows that one cannot make a replacement of the integration variable when the integral is infinity. Let us now evaluate the following integral

$$Q = \int_{-\infty}^{\infty} \left((x-a)^2 - x^2\right) dx. \qquad (J.4.4)$$

If we replace the integration variable in the first term as $x' = x - a$, then we can rewrite eq.(J.4.4) as

$$Q = \int_{-\infty}^{\infty} \left(x'^2 dx' - x^2 dx\right) = 0. \qquad (J.4.5)$$

However, if we calculate it properly, then we find

$$Q = \int_{-\infty}^{\infty} \left((x-a)^2 - x^2 \right) dx = \int_{-\infty}^{\infty} \left(a^2 - 2ax \right) dx = a^2 \times \infty \qquad (J.4.6)$$

which disagrees with eq.(J.4.5). If one wishes to carefully calculate eq.(J.4.4) by replacing the integration variable, then one should do as follows

$$Q = \lim_{\Lambda \to \infty} \left[\int_{-\Lambda-a}^{\Lambda-a} x'^2 dx' - \int_{-\Lambda}^{\Lambda} x^2 dx \right] = \lim_{\Lambda \to \infty} 2a^2 \Lambda. \qquad (J.4.7)$$

It is clear by now that the replacement of the integration variable in the infinite integral should not be made, and this is just the mistake which has been accepted as the gauge condition of the $\Pi^{\mu\nu}(k)$ in terms of eq.(J.4.1). Therefore, one sees that the requirement of the gauge condition of the vacuum polarization is unphysical.

J.4.2 Physical Processes Involving Vacuum Polarizations

In nature, there are a number of Feynman diagrams which involve the vacuum polarization. The best known physical process must be the π^0 decay into two photons, $\pi^0 \rightarrow \gamma + \gamma$. This process of the Feynman diagrams can be well calculated in terms of the nucleon and anti-nucleon pair creation where these fermions couple to photons [94]. In this calculation, one knows that the loop integral gives a finite result since the apparent logarithmic divergence vanishes to zero due to the kinematical cancellation. Also, the physical process of photon-photon scattering involves the box diagrams where electrons and positrons are created from the vacuum state. As is well known, the apparent logarithmic divergence of this box diagrams vanishes again due to the kinematical cancellation, and the evaluation of the Feynman diagrams gives a finite number. This is clear since all of the perturbative calculations employ the free fermion basis states which always satisfy the current conservation of $\partial_\mu j^\mu = 0$. In these processes, one does not have any additional "gauge conditions" in the evaluation of the Feynman diagrams. In this respect, if the process is physical, then the corresponding Feynman diagram should become finite without any further constraints of the gauge invariance.

J.5 Renormalization Scheme

Here, it is shown that there occurs no wave function renormalization of photon in the exact Lippmann-Schwinger equation for the vector potential [87]. The Lippmann-Schwinger equation for the fermion field ψ becomes

$$\psi(x) = \psi_0(x) + g \int G_F(x, x') A_\mu(x') \gamma^\mu \psi(x') d^4x', \qquad (J.5.1)$$

where $G_F(x, x')$ denotes the Green function which satisfies the following equation

$$(i\partial_\mu \gamma^\mu - m) G_F(x, x') = \delta^4(x - x'). \qquad (J.5.2)$$

$\psi_0(x)$ is the free fermion field solution. The Green function $G_F(x,x')$ can be explicitly written as

$$G_F(x,x') = \int \frac{1}{p_\mu \gamma^\mu - m + i\epsilon} e^{ip(x-x')} \frac{d^4p}{(2\pi)^4}. \tag{J.5.3}$$

On the other hand, the Lippmann-Schwinger equation for the vector field $\boldsymbol{A}$ becomes

$$\boldsymbol{A} = \boldsymbol{A}_0 + g \int G_0(x,x') \boldsymbol{j}(x') d^4x', \tag{J.5.4}$$

where $\boldsymbol{A}_0$ denotes the free field solution of the vector field. Here, the Green function $G_0(x,x')$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \right) G_0(x,x') = \delta^4(x-x'). \tag{J.5.5}$$

This can be explicitly written as

$$G_0(x,x') = \int \frac{1}{-p_0^2 + \boldsymbol{p}^2 + i\epsilon} e^{ip(x-x')} \frac{d^4p}{(2\pi)^4}. \tag{J.5.6}$$

Since we employ the Coulomb gauge fixing, the equation of motion for the A_0 field becomes a constraint equation and thus can be solved exactly as

$$A_0(\boldsymbol{r}) = \frac{g}{4\pi} \int \frac{j_0(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r'. \tag{J.5.7}$$

J.5.1 Wave Function Renormalization−Fermion Field

When we carry out the perturbation expansion, we can obtain the integral equations in powers of the coupling constant g as

$$\psi(x) = \psi_0(x) + g \int G_F(x,x') A_\mu \gamma^\mu \psi_0(x') d^4x' +$$

$$g^2 \int G_F(x,x') A_\mu(x') \gamma^\mu G_F(x',x'') A_\nu(x'') \gamma^\nu \psi_0(x'') d^4x' d^4x'' + \cdots . \tag{J.5.8}$$

This equation clearly shows that the fermion field should be affected by the perturbation expansion, and if it diverges, then we have to renormalize the wave function so as to absorb the infinity. Indeed, the infinity is logarithmic divergence and can be well renormalized into the wave function ψ_0.

J.5.2 Wave Function Renormalization−Vector Field

The vector field $\boldsymbol{A}$ can be determined from eq.(J.5.4) only when the fermion numbers are conserved. The best example can be found when the annihilation of the fermion pair takes place. In this case, we can write eq. (J.5.4) as

$$\langle 0|\boldsymbol{A}|f\bar{f}\rangle = g \int G_0(x,x') \langle 0|\boldsymbol{j}(x')|f\bar{f}\rangle d^4x', \tag{J.5.9}$$

where $|f\bar{f}\rangle$ denotes the fermion and anti-fermion state. Now, we can consider the following physical process of $e^+e^- \to e^+e^-$ which can be described in terms of the T-matrix as

$$T = -g\langle e\bar{e}| \int \boldsymbol{j}\cdot\boldsymbol{A}|e\bar{e}\rangle = -g^2 \int \langle e\bar{e}|\boldsymbol{j}(x)|0\rangle G_0(x,x')\langle 0|\boldsymbol{j}(x')|e\bar{e}\rangle d^3x d^4x', \quad (J.5.10)$$

where one can see that there appears no self-energy of photon term whatever one evaluates any physical processes in the Lippmann-Schwinger equation. From this equation, one finds that the vector field $\boldsymbol{A}$ cannot be affected by the renormalization procedure, and it always stays as a free state of photon. Since this is the exact equation of motion, there is no other possibility for the vector field. In this respect, it is just simple that the gauge field $\boldsymbol{A}$ always behaves as a free photon state in the evaluation of any Feynman diagrams.

J.5.3 Mass Renormalization−Fermion Self-energy

The evaluation of the self-energy of fermions can be carried out in a straight forward way, and one can obtain the self-energy which has a logarithmic divergence of the momentum cut-off Λ. Since electron has a mass, one can renormalize this logarithmic divergence term into the new mass term. In this procedure, there is no conceptual difficulty and indeed one can relate this renormalized effect to the observed value of the Lamb shift in hydrogen atom, which is indeed a great success of the QED renormalization scheme.

J.5.4 Mass Renormalization−Photon Self-energy

The evaluation of the self-energy of photon gives rise to the energy which has a quadratic divergence. There is no way to renormalize it into the renormalization scheme of QED since photon has no mass term. This clearly indicates that one should not take the contributions of the photon self-energy diagrams since they violate the Lorentz invariance. In this respect, one can now realize that the quadratic divergence term should be discarded because it is not consistent with the Lorentz invariance, and it has nothing to do with the gauge invariance.

The energy of photon is calculated in the system where fermion is at rest. The energy of photon with its momentum $\boldsymbol{k}$ must be described as $E_k = |\boldsymbol{k}|$, and there is no other expression. Therefore, the Lagrangian density of the vector field A_μ should be always written as

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (J.5.11)$$

and there should not be any modifications possible.

It may be interesting to note that Tomonaga stated a half century ago that the self-energy of photon should vanish to zero, even though he did not present any concrete proof at that time [108]. This claim must come from the understanding that the vacuum polarization energy of electromagnetic fields calculated by Heisenberg and Euler [65, 66] has no chance to be renormalized into the original Lagrangian density of electromagnetic fields. This is indeed the essence of the renormalization scheme.

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[1] [Textbooks]:
There are many field theory textbooks available for graduate students. Among them, the following two books may be suitable for the first year of graduate students,

(i) J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics", (McGraw-Hill Book Company,1964).

(ii) F. Mandl and G. Shaw, "Quantum field theory", (John Wiley & Sons, 1993).

When one wishes to learn physics in a concrete fashion,

(iii) F. Gross, "Relativistic quantum mechanics and field theory", (John Wiley & Sons, 1993)
may be suitable for graduate students.

If one wishes to understand realistic calculations in some old type of field theory,

(iv) K. Nishijima, " Fields and Particles " , (W.A. Benjamin,INC)
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