
Symmetry and Its Breaking in Quantum Field Theory

By
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Preface

Physics is always difficult, though it is extremely interesting. Many times I thought I understood it sufficiently profoundly, but after some time, it turned out that my understanding of physics was far from satisfactory. In particular, field theory has special complexities which may not be common to other fields of research. The symmetry and its breaking are most exotic and sometimes almost mysterious to even those who can normally understand the basic physics in a clear manner.

In this textbook, I focused on presenting a simple and clear picture of the symmetry and its breaking in quantum field theory. For this purpose, I explained physics of elementary field theory of fermions interacting by gauge fields as well as by four body fermion fields. In this respect, the interpretation of the basic field theory is repeatedly done such that physicists including graduate students may understand the essential points of the symmetry breaking in this textbook.

Also, this book is intended for researchers who look for the basic problems in their investigations. In many fields of research, field theory is used as a computational tool. In this regard, I present some elaborate technical tools which are quite useful and sometimes incentive for new ideas in fundamental researches.

In physics, deeper understanding is more important than quicker understanding. In particular, graduate students should realize that, if someone else can understand the basic physics very quickly, then he is most likely a good interpreter of the textbook knowledge. Slow but deep understanding of physics is most important since it should definitely take much time to understand physics in depth. The shortest path of understanding physics is only one of many paths, and interesting physics may well be found in the paths which are far from the shortest one.

Physics must be simple once we understand it all. For example, I believe that QCD can surely describe the strong interaction physics. However, it may well be difficult to justify the perturbative calculation of the interactions between quarks, unless the gauge independence of the quark-quark interactions is guaranteed. In other words, when the unperturbed as well as interaction Hamiltonians are gauge dependent, we should make it sure that any physical quantities evaluated perturbatively are indeed gauge invariant, which seems to be very difficult.

In this textbook, there are quite a few issues which are still debating. I believe that the present understanding of the basic field theory in this textbook must be reasonably good,

and as far as physics of the symmetry and its breaking is concerned, it should be the best of all. The spontaneous symmetry breaking of the global symmetry is by now understood in this textbook in terms of a simple physics terminology, and there is nothing mysterious from the standard way of understanding physics. However, it is still not yet settled whether the local gauge symmetry can be broken in terms of Higgs mechanism or not. At least, the gauge fixing for the non-gauge field is physically not at all easy to understand. For this problem, we need a lot to think over in future what should be physical observables in the Higgs mechanism.

This textbook contains a brief description of the lattice field theory even though it is not directly connected to the symmetry breaking physics. Still it may be interesting for readers to understand the basic point of the lattice field theory. For example, the continuum field theory must be richer than the lattice version, and it is most likely true that the lattice field theory can give only limited information on the continuum field theory, particularly when the latter keeps some symmetry while the former does not.

In Appendix, I explain some elementary physics so that readers may grasp the essence of the symmetry breaking phenomena in fermion field theory with little advanced knowledge. In some sense, Appendix can be read in its own interests since it includes non-relativistic quantum mechanics, Dirac equation and Maxwell equation, in addition to the notations which are often used in field theory. At the same time, Appendix contains some new physics interpretation for bosons, Dirac fields and quantization procedure. In particular, I believe that the first quantization of $[x, p_x] = i\hbar$, etc. may well be the result of the Dirac equation in that the Dirac Lagrangian density can be derived from the gauge principle as well as the Maxwell equations without involving the first quantization procedure. In the final chapter of Appendix, I briefly explain the renormalization in QED which is the most successful theory in quantum field theory. The perturbation theory is not the main issue of this textbook, but nevertheless readers may learn the essence of the renormalization scheme in quantum field theory.

The motive force of writing this textbook is initiated by Frank Columbus who understands the importance of the new picture of spontaneous symmetry breaking physics prior to experts and has encouraged me to write it into a textbook form. Indeed, I started to write this book from intensive discussions and hard works with my collaborators on this subject to achieve deeper but simpler understanding of the symmetry and its breaking in quantum field theory.

I should be grateful to all of my collaborators, in particular, Tomoko Asaga, Makoto Hiramoto, Takashi Homma, Seiji Kanemaki, Sachiko Oshima and Hidenori Takahashi for their great contributions to this book. Quite a few physicists and students also helped me a great deal for their critical reading of this manuscript. However, it is trivial to note that any mistakes in this book are entirely due to my carelessness.

To the Second Edition

The revision of this textbook is made mainly because of the following two reasons. Firstly, the first edition contained the wrong description of the path integral formulation. Even though it is normally found in the field theory textbooks, the path integral description in the field theory textbooks is not a correct one, and therefore I had to rewrite it into a correct formulation which was originally presented by Feynman. Secondly, the revision is concerned with the quantum gravity, and fortunately, the Lagrangian density that includes the gravitational interactions with fermions is properly constructed. Therefore, I included quantum gravity in this textbook, and one can now understand the basic physics of quantum gravity with our standard knowledge of quantum field theory, without referring to the space deformation.

In this occasion, I would like to express my sincere gratitude to late Prof. Kazuhiko Nishijima for his many useful comments and encouragements. His continuous supports for our works encouraged me a great deal, and in particular, the discussions of quantum gravity helped me to improve the description of the graviton propagation.

Finally I should like to thank numerous students and physicists for their interesting comments and suggestions to the first edition as well as the draft of the second edition. In particular, I should be grateful to Atsushi Kusaka, Kazuhiro Tsuda, Naohiro Kanda, Hiroshi Kato, Hiroaki Kubo and Yasunori Munakata for their careful reading of the manuscript.

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Contents

1	Classical Field Theory of Fermions	1
1.1	Non-relativistic Fields	1
1.1.1	Schrödinger Equation	2
1.1.2	Lagrangian Density for Schrödinger Fields	3
1.1.3	Lagrange Equation for Schrödinger Fields	4
1.1.4	Hamiltonian Density for Schrödinger Fields	5
1.1.5	Hamiltonian for Schrödinger Fields	6
1.1.6	Conservation of Vector Current	7
1.2	Dirac Fields	7
1.2.1	Dirac Equation for Free Fermion	8
1.2.2	Lagrangian Density for Free Dirac Fields	8
1.2.3	Lagrange Equation for Free Dirac Fields	9
1.2.4	Plane Wave Solutions of Free Dirac Equation	9
1.2.5	Quantization in Box with Periodic Boundary Conditions	11
1.2.6	Hamiltonian Density for Free Dirac Fermion	12
1.2.7	Hamiltonian for Free Dirac Fermion	12
1.2.8	Conservation of Vector Current	13
1.3	Electron and Electromagnetic Fields	13
1.3.1	Lagrangian Density	13
1.3.2	Gauge Invariance	14
1.3.3	Lagrange Equation for Dirac Field	15
1.3.4	Lagrange Equation for Gauge Field	15
1.3.5	Hamiltonian Density for Fermions with Electromagnetic Field	16
1.3.6	Hamiltonian for Fermions with Electromagnetic Field	17
1.4	Self-interacting Fermion Fields	18
1.4.1	Lagrangian and Hamiltonian Densities of NJL Model	18
1.4.2	Lagrangian Density of Thirring Model	19
1.4.3	Hamiltonian Density for Thirring Model	19
1.5	Quarks with Electromagnetic and Chromomagnetic Interactions	20
1.5.1	Lagrangian Density	20
1.5.2	EDM Interactions	21

2	Symmetry and Conservation Law	23
2.1	Introduction to Transformation Property	23
2.2	Lorentz Invariance	24
2.2.1	Lorentz Covariance	25
2.3	Time Reversal Invariance	26
2.3.1	T -invariance in Quantum Mechanics	26
2.3.2	T -invariance in Field Theory	27
2.3.3	T -violating Interactions (Imaginary Mass Term)	27
2.3.4	T and P -violating Interactions (EDM)	28
2.4	Parity Transformation	28
2.5	Charge Conjugation	29
2.5.1	Charge Conjugation in Maxwell Equation	29
2.5.2	Charge Conjugation in Dirac Field	30
2.5.3	Charge Conjugation in Quantum Chromodynamics	31
2.6	Translational Invariance	31
2.6.1	Energy Momentum Tensor	32
2.6.2	Hamiltonian Density from Energy Momentum Tensor	33
2.7	Global Gauge Symmetry	33
2.8	Chiral Symmetry	34
2.8.1	Expression of Chiral Transformation in Two Dimensions	34
2.8.2	Mass Term	35
2.8.3	Chiral Anomaly	36
2.8.4	Chiral Symmetry Breaking in Massless Thirring Model	37
2.9	$SU(3)$ Symmetry	37
2.9.1	Dimension of Representation $[\lambda, \mu]$	38
2.9.2	Useful Reduction Formula	39
3	Quantization of Fields	41
3.1	Quantization of Free Fermion Field	42
3.1.1	Creation and Annihilation Operators	42
3.1.2	Equal Time Quantization of Field	43
3.1.3	Quantized Hamiltonian of Free Dirac Field	44
3.1.4	Vacuum of Free Field Theory	45
3.2	Quantization of Thirring Model	46
3.2.1	Vacuum of Thirring Model	47
3.3	Quantization of Gauge Fields in QED	48
3.4	Quantization of Schrödinger Field	49
3.4.1	Creation and Annihilation Operators	50
3.4.2	Fermi Gas Model	50
3.5	Quantized Hamiltonian of QED and Eigenstates	51
3.5.1	Quantized Hamiltonian	51
3.5.2	Eigenvalue Equation	52

3.5.3	Vacuum State $ \Omega\rangle$	52
4	Goldstone Theorem and Spontaneous Symmetry Breaking	53
4.1	Symmetry and Its Breaking in Vacuum	54
4.1.1	Symmetry in Quantum Many Body Theory	55
4.1.2	Symmetry in Field Theory	56
4.2	Goldstone Theorem	57
4.2.1	Conservation of Chiral Charge	57
4.2.2	Symmetry of Vacuum	57
4.2.3	Commutation Relation	58
4.2.4	Momentum Zero State	59
4.2.5	Pole in S -matrix	60
4.3	New Interpretation of Goldstone Theorem	60
4.3.1	Eigenstate of Hamiltonian and \hat{Q}_5	60
4.3.2	Index of Symmetry Breaking	61
4.4	Chiral Symmetry in Quantized Thirring Model	61
4.4.1	Lagrangian Density	62
4.4.2	Quantized Hamiltonian	62
4.4.3	Chiral Transformation for Operators	62
4.4.4	Unitary Operator with Chiral Charge \hat{Q}_5	63
4.4.5	Symmetric and Symmetry Broken Vacuum	63
4.5	Spontaneous Chiral Symmetry Breaking	63
4.5.1	Exact Vacuum of Thirring Model	64
4.5.2	Condensate Operator	64
4.6	Symmetry Breaking in Two Dimensions	65
4.6.1	Fermion Field Theory in Two Dimensions	65
4.6.2	Boson Field Theory in Two Dimensions	65
4.7	Symmetry Breaking in Boson Fields	65
4.7.1	Double Well Potential	65
4.7.2	Change of Field Variables	66
4.7.3	Current Density of Fields	67
4.8	Breaking of Local Gauge Symmetry?	67
4.8.1	Higgs Mechanism	67
4.8.2	Gauge Fixing	68
4.8.3	What Is Physics Behind Higgs Mechanism?	69
5	Quantum Electrodynamics	71
5.1	General Properties of QED	72
5.1.1	QED Lagrangian Density	72
5.1.2	Local Gauge Invariance	72
5.1.3	Equation of Motion	73
5.1.4	Noether Current and Conservation Law	73
5.1.5	Gauge Invariance of Interaction Lagrangian	74

5.1.6	Gauge Fixing	74
5.1.7	Gauge Choices	75
5.1.8	Gauge Dependence without $\partial_\mu j^\mu = 0$	77
5.2	S -matrix in QED	78
5.2.1	Definition of S -matrix	78
5.2.2	Fock Space of Free Fields	80
5.2.3	Electron-Electron Interactions	81
5.2.4	Feynman Rules for QED	83
5.3	Schwinger Model (Massless QED ₂)	84
5.3.1	QED with Massless Fermions in Two Dimensions	84
5.3.2	Gauge Fixing	85
5.3.3	Quantized Hamiltonian of Schwinger Model	85
5.3.4	Bosonization of Schwinger Model	86
5.3.5	Chiral Anomaly	87
5.3.6	Regularization of Vacuum Energy	89
5.3.7	Bosonized Hamiltonian of Schwinger Model	90
5.4	Quantized QED ₂ Hamiltonian in Trivial Vacuum	91
5.4.1	Hamiltonian and Gauge Fixing	91
5.4.2	Field Quantization in Anti-particle Representation	91
5.4.3	Dirac Representation of γ -matrices	92
5.4.4	Quantized Hamiltonian of QED ₂	92
5.4.5	Boson Fock States	94
5.4.6	Boson Wave Function	94
5.4.7	Boson Mass	94
5.5	Bogoliubov Transformation in QED ₂	96
5.5.1	Bogoliubov Transformation	96
5.5.2	Boson Mass in Bogoliubov Vacuum	99
5.5.3	Chiral Condensate	100
5.6	QED ₂ in Light Cone	100
5.6.1	Light Cone Quantization	101
6	Quantum Chromodynamics	105
6.1	Properties of QCD with $SU(N_c)$ Colors	106
6.1.1	Lagrangian Density of QCD	106
6.1.2	Infinitesimal Local Gauge Transformation	107
6.1.3	Local Gauge Invariance	107
6.1.4	Noether Current in QCD	108
6.1.5	Conserved Charge of Color Octet State	108
6.1.6	Gauge Non-invariance of Interaction Lagrangian	109
6.1.7	Equations of Motion	109
6.1.8	Hamiltonian Density of QCD	110
6.1.9	Hamiltonian of QCD	111

6.2	Hamiltonian of QCD in Two Dimensions	111
6.2.1	Gauge Fixing	112
6.2.2	Quantization of Fields	113
6.2.3	Quantized Hamiltonian of QCD ₂ with $SU(N_c)$	113
6.2.4	Bogoliubov Transformed Hamiltonian	114
6.2.5	Determination of Bogoliubov Angle	115
6.2.6	Fermion Condensate	115
6.2.7	Boson Mass	115
6.2.8	Condensate and Boson Mass in $SU(N_c)$	116
6.3	't Hooft Model	117
6.3.1	$1/N_c$ Expansion	118
6.3.2	Examination of 't Hooft Model	119
6.4	Spontaneous Symmetry Breaking in QCD ₂	119
6.5	Explicit Expression of H'	121
7	Thirring Model	123
7.1	Bethe Ansatz Method for Massive Thirring Model	124
7.1.1	Free Fermion System	124
7.1.2	Bethe Ansatz State in Two Particle System	125
7.1.3	Bethe Ansatz State in N Particle System	126
7.2	Bethe Ansatz Method for Field Theory	128
7.2.1	Vacuum State of Massive Thirring Model	128
7.2.2	Excited States	129
7.2.3	Lowest Excited State (Boson)	130
7.2.4	Higher Excited States	130
7.2.5	Continuum States	130
7.3	Bethe Ansatz Method for Massless Thirring Model	131
7.3.1	Vacuum State of Massless Thirring Model	132
7.3.2	Symmetric Vacuum State	132
7.3.3	True Vacuum (Symmetry Broken) State	132
7.3.4	$1p - 1h$ State	134
7.3.5	Momentum Distribution of Negative Energy States	135
7.4	Bosonization of Thirring Model	135
7.4.1	Massless Thirring Model	137
7.4.2	Massive Thirring Model	138
7.4.3	Physics of Zero Mode	139
7.5	Massive Thirring vs Sine-Gordon Models	140
7.5.1	Sine-Gordon Field Theory Model	140
7.5.2	Correlation Functions	141
7.5.3	Correspondence	142
7.6	Bogoliubov Method for Thirring Model	143
7.6.1	Massless Thirring Model	143

7.6.2	Bogoliubov Transformation	143
7.6.3	Bogoliubov Transformed Hamiltonian	144
7.6.4	Eigenvalue Equation for Boson	145
7.6.5	Solution of Separable Interactions	145
7.6.6	Boson Spectrum	146
7.6.7	Axial Vector Current Conservation	147
7.6.8	Fermion Condensate	147
7.6.9	Massive Thirring Model	147
7.6.10	NJL Model	148
8	Lattice Field Theory	151
8.1	General Remark on Discretization of Space	151
8.1.1	Equal Spacing	152
8.1.2	Continuum Limit	152
8.2	Bethe Ansatz Method in Heisenberg Model	153
8.2.1	Exchange Operator $P_{i,j}$	154
8.2.2	Heisenberg XXZ for One Magnon State	154
8.2.3	Heisenberg XXZ for Two Magnon States	156
8.2.4	Heisenberg XXZ for m Magnon States	157
8.3	Equivalence between Heisenberg XYZ and Massive Thirring Models	158
8.3.1	Jordan-Wigner Transformation	158
8.3.2	Continuum Limit	159
8.3.3	Heisenberg XXZ and Massless Thirring Models	161
8.4	Gauge Fields on Lattice	162
8.4.1	Discretization of Space	162
8.4.2	Wilson's Action	162
8.4.3	Wilson Loop	164
8.4.4	Critical Review on Wilson's Results	165
8.4.5	Problems in Wilson's Action	166
8.4.6	Confinement of Quarks	168
9	Quantum Gravity	169
9.1	Problems of General Relativity	169
9.1.1	Field Equation of Gravity	170
9.1.2	Principle of Equivalence	170
9.1.3	General Relativity	171
9.2	Lagrangian Density for Gravity	172
9.2.1	Lagrangian Density for QED	172
9.2.2	Lagrangian Density for QED plus Gravity	173
9.2.3	Dirac Equation with Gravitational Interactions	173
9.2.4	Total Hamiltonian for QED plus Gravity	173
9.3	Static-dominance Ansatz for Gravity	174
9.4	Quantization of Gravitational Field	175

9.4.1	No Quantization of Gravitational Field	175
9.4.2	Quantization Procedure	175
9.4.3	Graviton	176
9.5	Interaction of Photon with Gravity	176
9.6	Renormalization Scheme for Gravity	179
9.6.1	Self-Energy of Graviton	179
9.6.2	Fermion Self-Energy from Gravity	180
9.6.3	Vertex Correction from Gravity	180
9.6.4	Renormalization Procedure	181
9.7	Gravitational Interaction of Photon with Matter	181
9.7.1	Photon-Gravity Scattering Process	182
9.8	Cosmology	182
9.8.1	Cosmic Fireball Formation	182
9.8.2	Relics of Preceding Universe	183
9.8.3	Remarks	183
9.9	Time Shifts of Mercury and Earth Motions	184
9.9.1	Non-relativistic Gravitational Potential	184
9.9.2	Time Shifts of Mercury, GPS Satellite and Earth	185
9.9.3	Mercury Perihelion Shift	186
9.9.4	GPS Satellite Advance Shift	186
9.9.5	Time Shift of Earth Rotation – Leap Second	187
9.9.6	Observables from General Relativity	187
9.9.7	Prediction from General Relativity	188
9.9.8	Summary of Comparisons between Calculations and Data	188
9.9.9	Intuitive Picture of Time Shifts	189
9.9.10	Leap Second Dating	190
A	Introduction to Field Theory	191
A.1	Natural Units	192
A.2	Hermite Conjugate and Complex Conjugate	193
A.3	Scalar and Vector Products (Three Dimensions) :	194
A.4	Scalar Product (Four Dimensions)	194
A.4.1	Metric Tensor	195
A.5	Four Dimensional Derivatives ∂_μ	195
A.5.1	\hat{p}^μ and Differential Operator	195
A.5.2	Laplacian and d'Alembertian Operators	196
A.6	γ -Matrices	196
A.6.1	Pauli Matrices	196
A.6.2	Representation of γ -matrices	197
A.6.3	Useful Relations of γ -Matrices	197
A.7	Transformation of State and Operator	198
A.8	Fermion Current	198

A.9	Trace in Physics	199
A.9.1	Definition	199
A.9.2	Trace in Quantum Mechanics	199
A.9.3	Trace in $SU(N)$	199
A.9.4	Trace of γ -Matrices and \not{p}	200
A.10	Lagrange Equation	200
A.10.1	Lagrange Equation in Classical Mechanics	201
A.10.2	Hamiltonian in Classical Mechanics	201
A.10.3	Lagrange Equation for Fields	202
A.11	Noether Current	202
A.11.1	Global Gauge Symmetry	202
A.11.2	Chiral Symmetry	204
A.12	Hamiltonian Density	204
A.12.1	Hamiltonian Density from Energy Momentum Tensor	204
A.12.2	Hamiltonian Density from Conjugate Fields	205
A.12.3	Hamiltonian Density for Free Dirac Fields	206
A.12.4	Hamiltonian for Free Dirac Fields	206
A.12.5	Role of Hamiltonian	206
A.13	Variational Principle in Hamiltonian	208
A.13.1	Schrödinger Field	208
A.13.2	Dirac Field	209
B	Non-relativistic Quantum Mechanics	211
B.1	Procedure of First Quantization	211
B.2	Mystery of Quantization or Hermiticity Problem?	212
B.2.1	Free Particle in Box	212
B.2.2	Hermiticity Problem	213
B.3	Schrödinger Fields	214
B.3.1	Currents of Bound State	214
B.3.2	Free Fields (Static)	214
B.3.3	Degree of Freedom of Schrödinger Field	215
B.4	Hydrogen-like Atoms	216
B.5	Harmonic Oscillator Potential	217
B.5.1	Creation and Annihilation Operators	218
C	Relativistic Quantum Mechanics of Bosons	221
C.1	Klein–Gordon Equation	221
C.2	Scalar Field	222
C.2.1	Physical Scalar Field	222
C.2.2	Current Density	223
C.2.3	Complex Scalar Field	225
C.2.4	Composite Bosons	226
C.2.5	Gauge Field	227

C.3	Degree of Freedom of Boson Fields	227
D	Relativistic Quantum Mechanics of Fermions	229
D.1	Derivation of Dirac Equation	229
D.2	Negative Energy States	230
D.3	Hydrogen Atom	230
D.3.1	Conserved Quantities	231
D.3.2	Energy Spectrum	232
D.3.3	Ground State Wave Function ($1s_{\frac{1}{2}}$ – state)	232
D.4	Lamb Shifts	233
D.4.1	Quantized Vector Field	233
D.4.2	Non-relativistic Hamiltonian	234
D.4.3	Second Order Perturbation Energy	234
D.4.4	Mass Renormalization and New Hamiltonian	234
D.4.5	Lamb Shift Energy	235
D.4.6	Lamb Shift in Muonium	236
D.4.7	Lamb Shift in Anti-hydrogen Atom	237
D.4.8	Physical Meaning of Cutoff Λ	237
E	Maxwell Equation and Gauge Transformation	239
E.1	Gauge Invariance	239
E.2	Derivation of Lorenz Force in Classical Mechanics	240
E.3	Number of Independent Functional Variables	241
E.3.1	Electric and Magnetic fields \mathbf{E} and \mathbf{B}	241
E.3.2	Vector Field A_μ and Gauge Freedom	242
E.4	Lagrangian Density of Electromagnetic Fields	243
E.5	Boundary Condition for Photon	244
F	Regularizations and Renormalizations	247
F.1	Euler’s Regularization	247
F.1.1	Abelian Summation	247
F.1.2	Regularized Abelian Summation	247
F.2	Chiral Anomaly	248
F.2.1	Charge and Chiral Charge of Vacuum	248
F.2.2	Large Gauge Transformation	249
F.2.3	Regularized Charge	249
F.2.4	Anomaly Equation	250
F.3	Index of Renormalizability	250
F.3.1	Renormalizable	250
F.3.2	Unrenormalizable	251
F.3.3	Summary of Renormalizability	251
F.4	Infinity in Physics	252

G	Path Integral Formulation	253
G.1	Path Integral in Quantum Mechanics	253
G.1.1	Path Integral Expression	254
G.1.2	Physical Mmeaning of Path Integral	255
G.1.3	Advantage of Path Integral	257
G.1.4	Harmonic Oscillator Case	257
G.2	Path Integral in Field Theory	258
G.2.1	Field Quantization	258
G.2.2	Field Quantization in Path Integral (Feynman's Ansatz)	259
G.2.3	Electrons Interacting through Gauge Fields	260
G.3	Problems in Field Theory Path Integral	261
G.3.1	Real Scalar Field as Example	261
G.3.2	Lattice Field Theory	262
G.3.3	Physics of Field Quantization	263
G.3.4	No Connection between Fields and Classical Mechanics	263
G.4	Path Integral Function Z in Field Theory	264
G.4.1	Path Integral Function in QCD	264
G.4.2	Fock Space	265
H	New Concept of Quantization	267
H.1	Derivation of Lagrangian Density of Dirac Field from Gauge Invariance and Maxwell Equation	267
H.1.1	Lagrangian Density for Maxwell Equation	267
H.1.2	Four Component Spinor	268
H.2	Shape of Lagrangian Density	269
H.2.1	Mass Term	269
H.2.2	First Quantization	269
H.3	Two Component Spinor	270
H.4	Klein–Gordon Equation	270
H.5	Incorrect Quantization in Polar Coordinates	271
H.6	Interaction with Gravity	272
I	Renormalization in QED	273
I.1	Hilbert Space of Unperturbed Hamiltonian	273
I.2	Necessity of Renormalization	274
I.2.1	Intuitive Picture of Fermion Self-energy	274
I.2.2	Intuitive Picture of Photon Self-energy	274
I.3	Fermion Self-energy	275
I.4	Vertex Corrections	276
I.5	New Aspects of Renormalization in QED	277
I.5.1	Renormalization Group Equation in QED	277
I.6	Renormalization in QCD	278
I.6.1	Fock Space of Free Fields	278

I.6.2	Renormalization Group Equation in QCD	279
I.6.3	Serious Problems in QCD	279
I.7	Renormalization of Massive Vector Fields	279
I.7.1	Renormalizability	280
J	Photon Self-energy Contribution in QED	281
J.1	Momentum Integral with Cutoff Λ	282
J.1.1	Photon Self-energy Contribution	282
J.1.2	Finite Term in Photon Self-energy Diagram	282
J.2	Dimensional Regularization	283
J.2.1	Photon Self-energy Diagram with $D = 4 - \epsilon$	283
J.2.2	Mathematical Formula of Integral	283
J.2.3	Reconsideration of Photon Self-energy Diagram	284
J.3	Propagator Correction of Photon Self-energy	284
J.3.1	Lamb Shift Energy	284
J.3.2	Magnetic Hyperfine Interaction	285
J.3.3	QED Corrections for Hyperfine Splitting	286
J.3.4	Finite Size Corrections for Hyperfine Splitting	287
J.3.5	Finite Propagator Correction from Photon Self-energy	287
J.3.6	Magnetic Moment of Electron	288
J.4	Spurious Gauge Conditions	289
J.4.1	Gauge Condition of $\Pi^{\mu\nu}(k)$	290
J.4.2	Physical Processes Involving Vacuum Polarizations	291
J.5	Renormalization Scheme	291
J.5.1	Wave Function Renormalization—Fermion Field	292
J.5.2	Wave Function Renormalization—Vector Field	292
J.5.3	Mass Renormalization—Fermion Self-energy	293
J.5.4	Mass Renormalization—Photon Self-energy	293
	Bibliography	295
	Index	300

Chapter 1

Classical Field Theory of Fermions

The world of elementary particles is basically composed of fermions. Quarks, electrons and neutrinos are all fermions. On the other hand, elementary bosons are all gauge bosons, except Higgs particles though unknown at present. Therefore, if one wishes to understand field theory, then it should be the best to first study fermion field theory models.

In this chapter, we discuss the classical field theory in which “classical field” means that the field is not an operator but a c -number function. First, we treat the Schrödinger field and its equation in terms of the non-relativistic field theory model. In this case, the first quantization of $[x_i, p_j] = i\hbar\delta_{ij}$ is already done since we start from the Lagrangian density. In fact, the Lagrange equation leads to the Schrödinger equation or in other words, the Lagrangian density is constructed such that the Schrödinger equation can be derived from the Lagrange equation. The Dirac field is then discussed in terms of the Lagrangian density and the Lagrange equation. We also discuss the electromagnetic fields which interact with the Dirac field. The gauge invariance will be repeatedly discussed in this textbook, and the first introduction is given here. Finally, the field theory models with self-interacting fields are introduced and their Lagrangian density as well as Hamiltonian are described.

In this textbook, the basic parts of elementary physics can be found in Appendix, and in fact, Appendix is prepared such that it can be read in its own interests independently from the main part of the textbook.

Throughout this book, we employ the natural units

$$c = 1, \quad \hbar = 1.$$

This is, of course, due to its simplicity, and one can easily recover the right dimension of any physical quantities by making use of

$$\hbar c = 197 \text{ MeV} \cdot \text{fm}.$$

1.1 Non-relativistic Fields

If one treats a classical field $\psi(\mathbf{r})$, it does not matter whether it is a relativistic field or non-relativistic one. The kinematics becomes important when one solves the equation of

motion which is relativistic or non-relativistic. If the kinematics is non-relativistic, then the equation of motion that governs the field $\psi(\mathbf{r})$ is the Schrödinger equation. Therefore, we should first study the Schrödinger field from the point of view of the classical field theory.

1.1.1 Schrödinger Equation

Electron in classical mechanics is treated as a point particle whose equation of motion is governed by the Newton equation. When electrons are trapped by atoms, then their motions should be described by quantum mechanics. As long as electrons move much slowly in comparison with the velocity of light c , the equation of their motion is governed by the Schrödinger equation. The Schrödinger equation for electron with its mass m in the external field $U(\mathbf{r})$ can be written as [102]

$$\left(i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 - U(\mathbf{r}) \right) \psi(\mathbf{r}, t) = 0, \quad (1.1)$$

where $U(\mathbf{r})$ is taken to be a real potential. $\psi(\mathbf{r}, t)$ corresponds to the electron field in atoms, and $|\psi(\mathbf{r}, t)|^2$ can be interpreted as a probability density of finding the electron at (\mathbf{r}, t) .

Field $\psi(\mathbf{r}, t)$ is Complex

The Schrödinger field $\psi(\mathbf{r}, t)$ should be a complex function, and the complex field just corresponds to one particle state in the classical field theory. This is a well known fact, but below we will see what may happen when we assume *a priori* that the Schrödinger field $\psi(\mathbf{r}, t)$ should be a real function.

Real Field Condition is Unphysical

If one imposes the condition that the field $\psi(\mathbf{r}, t)$ should be real

$$\psi(\mathbf{r}, t) = \psi^\dagger(\mathbf{r}, t)$$

then, one sees immediately that the field $\psi(\mathbf{r}, t)$ becomes time-independent since eq.(1.1) and its complex conjugate equation give the following constraint for a real field $\psi(\mathbf{r}, t)$

$$\frac{\partial \psi(\mathbf{r}, t)}{\partial t} = 0.$$

Also, the field $\psi(\mathbf{r})$ should satisfy the following equation

$$\left(-\frac{1}{2m} \nabla^2 + U(\mathbf{r}) \right) \psi(\mathbf{r}) = 0.$$

Since the general solution of eq.(1.1) can be written as

$$\psi(\mathbf{r}, t) = e^{-iEt} \phi(\mathbf{r})$$

the field $\psi(\mathbf{r}, t)$ may become a real function only if the energy E of the system vanishes. That is, the energy eigenvalue of E is

$$E = 0.$$

Therefore, the real field cannot propagate and should be unphysical. This means that the real field condition of $\psi(\mathbf{r}, t)$ is physically too strong as a constraint.

1.1.2 Lagrangian Density for Schrödinger Fields

The Lagrangian density which can produce eq.(1.1) is easily found as

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m} \frac{\partial \psi^\dagger}{\partial x_k} \frac{\partial \psi}{\partial x_k} - \psi^\dagger U \psi, \quad (1.2)$$

where the repeated indices of k mean the summation of $k = 1, 2, 3$ and, in this text, this notation as well as the vector representation are employed depending on the situations. The repeated indices notation is mostly better for the calculation, but for memorizing the expressions or equations, the vector notation has some advantage.

The Lagrangian density of eq.(1.2) is constructed such that the Lagrange equation can reproduce the Schrödinger equation of eq.(1.1). It may also be important to note that the Lagrangian density of eq.(1.2) has a $U(1)$ symmetry, that is, it is invariant under the change of the field ψ as

$$\psi'(x) = e^{i\theta} \psi(x) \longrightarrow \mathcal{L}' = \mathcal{L},$$

where θ is a real constant. This invariance is clearly satisfied, and it is related to the conservation of vector current in terms of Noether's theorem which will be treated in the later chapters and in Appendix A.

Non-hermiticity of Lagrangian Density

At this point, we should discuss the non-hermiticity of the Lagrangian density. As one notices, the Lagrangian density of eq.(1.2) is not hermitian, and therefore some symmetry will be lost. One can build the Lagrangian density which is hermitian by replacing the first term by

$$i\psi^\dagger \frac{\partial \psi}{\partial t} \longrightarrow \left(\frac{i}{2} \psi^\dagger \frac{\partial \psi}{\partial t} - \frac{i}{2} \frac{\partial \psi^\dagger}{\partial t} \psi \right).$$

However, it is a difficult question whether the Lagrangian density must be hermitian or not since it is not an observable. In addition, when one introduces the conjugate fields

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}}, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}$$

in accordance with the fields ψ and ψ^\dagger , then the symmetry between them is lost. However, the conjugate fields themselves are again not observables, and therefore there is no reason

that one should keep this symmetry. In any case, one can, of course, work with the symmetric and hermitian Lagrangian density, but physical observables are just the same as eq.(1.2). In this textbook, we employ eq.(1.2) since it is simpler.

1.1.3 Lagrange Equation for Schrödinger Fields

The Lagrange equation for field theory can be obtained by the variational principle of the action S

$$S = \int \mathcal{L} dt d^3r$$

and the Lagrange equation is derived in Appendix A. Since the field ψ is a complex field, ψ and ψ^\dagger are treated as independent functional variables. The Lagrange equation for the field ψ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \equiv \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi}, \quad (1.3a)$$

where the four dimensional derivative

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

is introduced for convenience. Now, the following equations can be easily evaluated

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= i \frac{\partial \psi^\dagger}{\partial t}, \\ \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} &= -\frac{1}{2m} \frac{\partial}{\partial x_k} \frac{\partial \psi^\dagger}{\partial x_k}, \\ \frac{\partial \mathcal{L}}{\partial \psi} &= -\psi^\dagger U \end{aligned}$$

and therefore one obtains

$$\left(-i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 - U(r) \right) \psi^\dagger(\mathbf{r}, t) = 0$$

which is just the Schrödinger equation for ψ^\dagger in eq.(1.1).

It should be interesting to calculate the Lagrange equation for the field ψ^\dagger ,

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi^\dagger}. \quad (1.3b)$$

In this case, one finds

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0,$$

$$\begin{aligned}\frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial (\frac{\partial \psi^\dagger}{\partial x_k})} &= -\frac{1}{2m} \frac{\partial}{\partial x_k} \frac{\partial \psi}{\partial x_k}, \\ \frac{\partial \mathcal{L}}{\partial \psi^\dagger} &= i \frac{\partial \psi}{\partial t} - U \psi\end{aligned}$$

and therefore one obtains

$$\left(i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 - U(r) \right) \psi(\mathbf{r}, t) = 0$$

which is just the same equation as eq.(1.1).

Here, we note that the Lagrangian density is not a physical observable and therefore it does not necessarily have to be determined uniquely. It is by now clear that the Lagrangian density eq.(1.2) reproduces a desired Schrödinger equation and thus can be taken as the right Lagrangian density for Schrödinger fields.

1.1.4 Hamiltonian Density for Schrödinger Fields

From the Lagrangian density, one can build the Hamiltonian density \mathcal{H} which is the energy density of the field $\psi(\mathbf{r}, t)$. The Hamiltonian density \mathcal{H} is best constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\dagger)} \partial^\nu \psi^\dagger - \mathcal{L} g^{\mu\nu}$$

which will be derived in eq.(2.32) in Chapter 2. The energy momentum tensor $\mathcal{T}^{\mu\nu}$ satisfies the following equation of conservation law

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0$$

due to the invariance of the Lagrangian density under the translation. Therefore, the conserved charge associated with the $\mathcal{T}^{0\nu}$

$$\mathcal{Q}^\nu = \int \mathcal{T}^{0\nu} d^3r$$

should be a conserved quantity. Thus, it is natural that one defines the Hamiltonian in terms of the \mathcal{Q}^0 .

Hamiltonian Density from Energy Momentum Tensor

The Hamiltonian density \mathcal{H} is defined as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} \dot{\psi}^\dagger - \mathcal{L}. \quad (1.4a)$$

Therefore, introducing the conjugate fields Π_ψ and Π_{ψ^\dagger} by

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0$$

one can write the Hamiltonian density as

$$\mathcal{H} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger} \dot{\psi}^\dagger - \mathcal{L} = \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi + \psi^\dagger U \psi. \quad (1.4b)$$

1.1.5 Hamiltonian for Schrödinger Fields

The Hamiltonian for the Schrödinger field is obtained by integrating the Hamiltonian density over all space

$$H \equiv \int \mathcal{H} d^3r = \int \left[\frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi + \psi^\dagger U \psi \right] d^3r. \quad (1.4c)$$

By employing the Gauss theorem

$$\int_V \nabla \cdot (\psi^\dagger \nabla \psi) d^3r = \int_S (\psi^\dagger \nabla_n \psi) dS_n$$

one can rewrite eq.(1.4c)

$$H = \int \left[-\frac{1}{2m} \psi^\dagger \nabla^2 \psi + \psi^\dagger U \psi \right] d^3r, \quad (1.4d)$$

where the following identity is employed

$$\nabla \cdot (\psi^\dagger \nabla \psi) = \nabla \psi^\dagger \cdot \nabla \psi + \psi^\dagger \nabla^2 \psi.$$

In addition, the surface integral term is neglected since it should vanish at the surface of sphere at infinity.

Now, it may be interesting to note that the Hamiltonian in eq.(1.4d) by itself does not give us much information on the dynamics. As long as we stay in the classical field theory, then the dynamics can be obtained from the equation of motion, that is, the Schrödinger equation. The static Schrödinger equation can be derived from the variational principle of the Hamiltonian with respect to ψ , and this treatment is given in Appendix A.

The Hamiltonian of eq.(1.4c) becomes important when the field ψ is quantized, that is, the field ψ is assumed to be written in terms of the annihilation operator a_k as discussed in Chapter 3. In this case, the Schrödinger field becomes an operator and therefore the Hamiltonian as well. This means that one has to prepare the Fock state on which the Hamiltonian can operate, and if one solves the eigenvalue equation for the Hamiltonian, then one can obtain the energy eigenvalue of the Hamiltonian corresponding to the Fock state.

However, the quantization of the Schrödinger field is not needed in the normal circumstances. The field quantization is necessary for the relativistic fields which contain negative energy solutions, and it becomes important when one wishes to treat the quantum fluctuation of the fields which corresponds to the creation and annihilation of particles.

1.1.6 Conservation of Vector Current

From the Schrödinger equation, one can derive the current conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

where ρ and \mathbf{j} are defined as

$$\rho = \psi^\dagger \psi, \quad \mathbf{j} = \frac{i}{2m} \left[(\nabla \psi^\dagger) \psi - \psi^\dagger \nabla \psi \right].$$

This continuity equation of the vector current can also be derived as Noether's theorem from the Lagrangian density of eq.(1.2) which is invariant under the global gauge transformation

$$\psi' = e^{i\alpha} \psi.$$

As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right], \quad \text{with } j^\mu = (\rho, \mathbf{j})$$

which just gives the above current density ρ and \mathbf{j} when one employs the Lagrangian density of eq.(1.2).

It may be interesting to observe that the Lagrange equation, energy momentum tensor and the current conservation are all written in a relativistically covariant fashion when the properties of the Schrödinger field are derived. That is, apart from the shape of the Lagrangian density of the Schrödinger field, all the treatments are just the same as the relativistic description.

1.2 Dirac Fields

Electron in hydrogen atom moves much slowly compared with the velocity of light c . However, if one considers a hydrogen-like ${}_{83}^{209}\text{Bi}$ atom where $Z = 83$, for example, then the motion of electron becomes relativistic since its velocity v can be given as

$$\frac{v}{c} \sim (Z\alpha)^2 \sim \left(\frac{83}{137} \right)^2 \sim 0.37$$

which is already comparable with c .

In this case, one should employ the relativistic kinematics, and therefore the Schrödinger equation should be replaced by the Dirac equation which is obtained by a natural extension of the relativistic kinematics. However, the Dirac equation contains new properties which are essentially different from the Schrödinger equation, apart from the kinematics. They have negative energy solutions and spin degrees of freedom. Both properties are very important in physics and will be repeatedly discussed in this textbook.

1.2.1 Dirac Equation for Free Fermion

The Dirac equation for free fermion with its mass m is written as [25, 26]

$$\left(i \frac{\partial}{\partial t} + i \nabla \cdot \boldsymbol{\alpha} - m \beta \right) \psi(\mathbf{r}, t) = 0, \quad (1.5)$$

where ψ has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix.

The derivation of the Dirac equation and its application to hydrogen atom are given in Appendix D. One can learn from the procedure of deriving the Dirac equation that the number of components of the electron fields is important, and it is properly obtained in the Dirac equation. That is, among the four components of the field ψ , two degrees of freedom should correspond to the positive and negative energy solutions and another two degrees should correspond to the spin with $s = \frac{1}{2}$. It is also important to note that the factorization procedure indicates that the four component spinor is the minimum number of fields which can take into account the negative energy degree of freedom in a proper way.

Eq.(1.5) can be rewritten in terms of the wave function components by multiplying β from the left hand side

$$(i\partial_\mu \gamma^\mu - m)_{ij} \psi_j = 0 \quad \text{for } i = 1, 2, 3, 4, \quad (1.6)$$

where the repeated indices of j indicate the summation of $j = 1, 2, 3, 4$. Here, gamma matrices

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) \equiv (\beta, \beta \boldsymbol{\alpha})$$

are introduced, and the repeated indices of Greek letters μ indicate the summation of $\mu = 0, 1, 2, 3$ as defined in Appendix A. The expression of eq.(1.6) is called *covariant* since the Lorentz invariance of eq.(1.6) is manifest. It is indeed written in terms of the Lorentz scalars, but, of course there is no deep physical meaning in covariance.

1.2.2 Lagrangian Density for Free Dirac Fields

The Lagrangian density for free Dirac fermions can be constructed as

$$\mathcal{L} = \psi_i^\dagger [\gamma_0 (i\partial_\mu \gamma^\mu - m)]_{ij} \psi_j = \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi, \quad (1.7)$$

where $\bar{\psi}$ is defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0.$$

This Lagrangian density is just constructed so as to reproduce the Dirac equation of (1.6) from the Lagrange equation. It should be important to realize that the Lagrangian density of eq.(1.7) is invariant under the Lorentz transformation since it is a Lorentz scalar. This is clear since the Lagrangian density should not depend on the system one chooses.

Non-hermiticity of Lagrangian Density

This Lagrangian density is not hermitian, and it is easy to construct a hermitian Lagrangian density. However, as we discussed in the context of Schrödinger field, there is no strong reason that one should take the hermitian Lagrangian density since proper physical equations can be obtained from eq.(1.7).

1.2.3 Lagrange Equation for Free Dirac Fields

The Lagrange equation for ψ_i^\dagger is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \equiv \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} \quad (1.8)$$

and one can easily calculate the following equations

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} &= 0, \\ \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} &= 0, \\ \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} &= [\gamma_0(i\partial_\mu \gamma^\mu - m)]_{ij} \psi_j \end{aligned}$$

and thus, this leads to the following equation

$$[\gamma_0(i\partial_\mu \gamma^\mu - m)]_{ij} \psi_j = 0$$

which is just eq.(1.6). Here, it should be noted that the ψ_i and ψ_i^\dagger are independent functional variables, and the functional derivative with respect to ψ_i or ψ_i^\dagger gives the same equation of motion.

1.2.4 Plane Wave Solutions of Free Dirac Equation

The free Dirac equation of eq.(1.5) can be solved exactly, and it has plane wave solutions. A simple way to solve eq.(1.5) can be shown as follows. First, one writes the wave function

ψ in the following shape

$$\psi_s(\mathbf{r}, t) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iEt+i\mathbf{p}\cdot\mathbf{r}}, \quad (1.9)$$

where ζ_1 and ζ_2 are two component spinors

$$\zeta_1 = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}.$$

In this case, eq.(1.5) becomes

$$\begin{pmatrix} -m - E & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & m - E \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0 \quad (1.10)$$

which leads to

$$E^2 = m^2 + \mathbf{p}^2.$$

This equation has the following two solutions.

Positive Energy Solution ($E_p = \sqrt{\mathbf{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(+)}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} u_{\mathbf{p}}^{(s)} e^{-iE_p t + i\mathbf{p}\cdot\mathbf{r}}, \quad (1.11a)$$

$$u_{\mathbf{p}}^{(s)} = \sqrt{\frac{E_p + m}{2E_p}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_s \end{pmatrix}, \quad \text{with } s = \pm \frac{1}{2}, \quad (1.11b)$$

where χ_s denotes the spin wave function and is written as

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Negative Energy Solution ($E_p = -\sqrt{\mathbf{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(-)}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} v_{\mathbf{p}}^{(s)} e^{-iE_p t + i\mathbf{p}\cdot\mathbf{r}}, \quad (1.12a)$$

$$v_{\mathbf{p}}^{(s)} = \sqrt{\frac{|E_p| + m}{2|E_p|}} \begin{pmatrix} -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|E_p| + m} \chi_s \\ \chi_s \end{pmatrix}. \quad (1.12b)$$

Some Properties of Spinor

The spinor wave function $u_{\mathbf{p}}^{(s)}$ and $v_{\mathbf{p}}^{(s)}$ are normalized according to

$$u_{\mathbf{p}}^{(s)\dagger} u_{\mathbf{p}}^{(s)} = 1,$$

$$v_{\mathbf{p}}^{(s)\dagger} v_{\mathbf{p}}^{(s)} = 1.$$

Further, they satisfy the following equations when the spin is summed over

$$\sum_{s=1}^2 u_{\mathbf{p}}^{(s)} \bar{u}_{\mathbf{p}}^{(s)} = \frac{p_{\mu} \gamma^{\mu} + m}{2E_{\mathbf{p}}}, \quad (1.13a)$$

$$\sum_{s=1}^2 v_{\mathbf{p}}^{(s)} \bar{v}_{\mathbf{p}}^{(s)} = \frac{p_{\mu} \gamma^{\mu} + m}{2E_{\mathbf{p}}}. \quad (1.13b)$$

1.2.5 Quantization in Box with Periodic Boundary Conditions

In field theory, one often puts the theory into the box with its volume $V = L^3$ and requires that the wave function should satisfy the periodic boundary conditions (PBC). This is mainly because the free field solutions are taken as the basis states, and in this case, one can only calculate physical observables if one works in the box. It is clear that the free field can be defined well only if it is confined in the box.

Since the wave function $\psi_s(\mathbf{r}, t)$ for a free particle in the box should be proportional to

$$\psi_s(\mathbf{r}, t) \simeq \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iEt + i\mathbf{p}\cdot\mathbf{r}}$$

the PBC equations become

$$e^{ip_x x} = e^{ip_x(x+L)}, \quad e^{ip_y y} = e^{ip_y(y+L)}, \quad e^{ip_z z} = e^{ip_z(z+L)}. \quad (1.14a)$$

Therefore, one obtains the constraints on the momentum p_k as

$$p_x = \frac{2\pi}{L} n_x, \quad p_y = \frac{2\pi}{L} n_y, \quad p_z = \frac{2\pi}{L} n_z, \quad n_k = 0, \pm 1, \pm 2, \dots \quad (1.14b)$$

In this case, the number of states N in the large L limit becomes

$$N = \sum_{n_x, n_y, n_z} \sum_s = 2 \frac{L^3}{(2\pi)^3} \int d^3 p, \quad (1.15)$$

where a factor of two comes from the spin degree of freedom.

1.2.6 Hamiltonian Density for Free Dirac Fermion

The Hamiltonian density for free fermion can be constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \partial^\nu \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \partial^\nu \psi_i^\dagger \right) - \mathcal{L} g^{\mu\nu}$$

which will be treated in eq.(A.12.3) of Appendix A.

Hamiltonian Density from Energy Momentum Tensor

Now, one defines the Hamiltonian density \mathcal{H} as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L}. \quad (1.16)$$

Since the Lagrangian density of free fermion is given in eq.(1.7) and is rewritten as

$$\mathcal{L} = i\psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger [i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0]_{ij} \psi_j$$

one can introduce the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$, and calculate them

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0. \quad (1.17)$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L} = \bar{\psi}_i [-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m]_{ij} \psi_j = \bar{\psi} [-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m] \psi. \quad (1.18)$$

1.2.7 Hamiltonian for Free Dirac Fermion

The Hamiltonian for free fermion fields is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H} d^3r = \int \bar{\psi} [-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m] \psi d^3r. \quad (1.19)$$

As we discussed in the Schrödinger field, the Hamiltonian itself cannot give us much information on the dynamics. One can learn some properties of the system described by the Hamiltonian, but one cannot obtain any dynamical information of the system from the Hamiltonian. In order to calculate the dynamics of the system in the classical field theory model, one has to solve the equation of motions which are obtained from the Lagrange equations for fields.

When one wishes to consider the fluctuations of the fields or, in other words, creations of particles and anti-particles, then one should quantize the fields. In this case, the Hamiltonian becomes an operator. Therefore, one has to prepare the Fock states on which the Hamiltonian can operate. Most of the difficulties of the field theory models should be to find the vacuum of the system.

1.2.8 Conservation of Vector Current

The Lagrangian density of the Dirac field has a global gauge invariance,

$$\psi' = e^{i\alpha}\psi \longrightarrow \mathcal{L}' = \mathcal{L}$$

and therefore there is a Noether current associated with the symmetry. As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right]$$

and therefore the vector current j_μ becomes

$$j^\mu = \bar{\psi} \gamma^\mu \psi.$$

Due to the global gauge invariance of the Lagrangian density, the vector current j_μ satisfies the continuity equation

$$\partial_\mu j^\mu = 0.$$

1.3 Electron and Electromagnetic Fields

The main part of the physical world is governed by the interaction between electrons and electromagnetic fields. Therefore, the Dirac equation, the Maxwell equation and their interactions are most important to understand the basic physics in many fundamental researches.

1.3.1 Lagrangian Density

When electron interacts with electromagnetic fields, the Lagrangian density becomes

$$\mathcal{L} = \bar{\psi} (i\partial_\mu \gamma^\mu - gA_\mu \gamma^\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (1.20)$$

where $F_{\mu\nu}$ denotes the field strength and is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

A^μ denotes the gauge field with

$$A^\mu = (A_0, \mathbf{A}),$$

where A_0 and \mathbf{A} are the scalar and vector potentials, respectively. g denotes the gauge coupling constant, and in the classical electromagnetism, it corresponds to the electric charge e .

In the four dimensional field theory of QED, the coupling constant g is dimensionless, and therefore it is renormalizable in the perturbation calculation. In the two dimensional case, the coupling constant g has a mass dimension, and thus it is called *super-renormalizable*. In this case, there appear no infinities from the momentum integral in the perturbative calculations, and therefore one does not have to renormalize the coupling constant.

1.3.2 Gauge Invariance

The Lagrangian density of eq.(1.20) has an interesting feature. The free fermion Lagrangian density part

$$\bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi$$

is just the same as free Dirac Lagrangian density, and the last term in eq.(1.20)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

corresponds to the field energy term of the electromagnetic fields. The important point is that the shape of the interaction term

$$-g\bar{\psi}A_\mu\gamma^\mu\psi$$

can be determined by the requirement of the invariance under the local gauge transformation.

Local Gauge Transformation

We consider the following local gauge transformation

$$\psi' = e^{-ig\chi}\psi, \quad A'_\mu = A_\mu + \partial_\mu\chi, \quad (1.21)$$

where χ is an arbitrary real function of space and time, that is, $\chi(\mathbf{r}, t)$ which is therefore called *local*. It is easy to prove that the shape of the field energy term of the electromagnetic fields does not change under the local gauge transformation of eq.(1.21)

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\chi) - \partial_\nu(A_\mu + \partial_\mu\chi) = F_{\mu\nu}.$$

In addition, one can easily prove that the Lagrangian density of

$$\bar{\psi}(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi$$

does not change its shape under the local gauge transformation of eq.(1.21). That is,

$$\begin{aligned} & \bar{\psi}'(i\partial_\mu\gamma^\mu - gA'_\mu\gamma^\mu - m)\psi' \\ &= \bar{\psi}e^{-ig\chi}e^{ig\chi}(i\partial_\mu\gamma^\mu + g\partial_\mu\chi\gamma^\mu - gA_\mu\gamma^\mu - g\partial_\mu\chi\gamma^\mu - m)\psi \\ &= \bar{\psi}(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi. \end{aligned} \quad (1.22)$$

Therefore, a new Lagrangian density \mathcal{L}' becomes equal to the original one \mathcal{L}

$$\mathcal{L}' = \bar{\psi}'(i\partial_\mu\gamma^\mu - gA'_\mu\gamma^\mu - m)\psi' - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} = \mathcal{L}.$$

The invariance of the Lagrangian density under the local gauge transformation determines the shape of the interaction between electron and electromagnetic fields. This is surprising, but it is, in a sense, the same as the classical mechanics as discussed in Appendix E. In this respect, it is interesting to realize that the gauge invariance that arises from the redundancy of the vector potential in solving the Maxwell equations plays an important role for determining the shape of the fundamental interactions.

1.3.3 Lagrange Equation for Dirac Field

The Dirac equation with the electromagnetic interaction can be easily obtained from the Lagrange equation for ψ

$$(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi = 0. \quad (1.23)$$

This is the Dirac equation for the hydrogen atom when the potential is static, that is

$$\mathbf{A} = 0$$

and

$$gA_0 = -\frac{Ze^2}{r},$$

where we put $g = e$ with e the electric charge.

1.3.4 Lagrange Equation for Gauge Field

The Lagrange equation for the gauge field A_ν is written as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu}.$$

Since one can easily calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= -g\bar{\psi}\gamma^\nu\psi, \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{2} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \times 2 = -\partial_\mu F^{\mu\nu} \end{aligned}$$

one obtains

$$\partial_\mu F^{\mu\nu} = g\bar{\psi}\gamma^\nu\psi = gj^\nu, \quad (1.24)$$

where the current density j^ν is defined as

$$j^\nu = \bar{\psi}\gamma^\nu\psi = (\bar{\psi}\gamma^0\psi, \bar{\psi}\boldsymbol{\gamma}\psi). \quad (1.25)$$

Eq.(1.24) is the Maxwell equation, and more explicitly, one can evaluate eq.(1.24)

$$[\nu = 0] \longrightarrow \frac{\partial F^{k0}}{\partial x_k} = \frac{\partial E_k}{\partial x_k} = \boldsymbol{\nabla} \cdot \mathbf{E} = gj_0, \quad (1.26a)$$

$$[\nu = k] \longrightarrow \frac{\partial F^{0k}}{\partial t} + \frac{\partial F^{ik}}{\partial x_i} = -\dot{E}_k + \epsilon_{kij} \frac{\partial B_j}{\partial x_i} = -\dot{E}_k + (\boldsymbol{\nabla} \times \mathbf{B})_k = gj_k \quad (1.26b)$$

which are just the Maxwell equations. It is of course easy to see that no magnetic monopole

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

and Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

are automatically satisfied in terms of the vector potential A_μ since

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &\implies \nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} = \nabla \times \nabla \cdot \mathbf{A} = 0, \\ \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A_0 &\implies \nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla A_0 \right) = -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned}$$

1.3.5 Hamiltonian Density for Fermions with Electromagnetic Field

Now, one can construct the Hamiltonian density of fermion with electromagnetic field. The Hamiltonian density \mathcal{H} can be defined by the energy momentum tensor $\mathcal{T}^{\mu\nu}$ [eq.(A.12.3)] as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \left(\frac{\partial \mathcal{L}}{\partial \dot{A}_k} \dot{A}_k \right) - \mathcal{L}$$

since $\mathcal{T}^{0\nu}$ is a conserved quantity. By introducing the conjugate fields Π_{ψ_i} , $\Pi_{\psi_i^\dagger}$ and Π_{A_k} as

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}, \quad \Pi_{\psi_i^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger}, \quad \Pi_{A_k} = \frac{\partial \mathcal{L}}{\partial \dot{A}_k}$$

one can rewrite the Hamiltonian density as

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \Pi_{A_k} \dot{A}_k - \mathcal{L}. \quad (1.27)$$

The conjugate fields Π_{ψ_i} , $\Pi_{\psi_i^\dagger}$ and Π_{A_k} can be calculated by employing the Lagrangian density of eq.(1.20)

$$\Pi_{\psi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0, \quad \Pi_{A_k} = \dot{A}_k + \frac{\partial A_0}{\partial x_k} = -E_k.$$

It should be noted that there is no corresponding conjugate field for A_0 in the Hamiltonian density, and thus there is no kinetic energy term present for A_0 . Now, the Hamiltonian density can be calculated as

$$\begin{aligned} \mathcal{H} = \bar{\psi} \left[-i\gamma_k \frac{\partial}{\partial x_k} + m + gA_\mu \gamma^\mu \right] \psi \\ + \frac{1}{2} \left[\dot{A}_k^2 - \left(\frac{\partial A_0}{\partial x_k} \right)^2 + \left(\frac{\partial A_k}{\partial x_j} \frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j} \frac{\partial A_j}{\partial x_k} \right) \right]. \quad (1.28a) \end{aligned}$$

Eq.(1.28a) can be written in a familiar form

$$\mathcal{H} = \bar{\psi} (-i\boldsymbol{\gamma} \cdot \nabla + m) \psi - g\mathbf{j} \cdot \mathbf{A} + gj_0 A_0 + \frac{1}{2} \left[\dot{\mathbf{A}}^2 - (\nabla A_0)^2 + \mathbf{B}^2 \right]. \quad (1.28b)$$

1.3.6 Hamiltonian for Fermions with Electromagnetic Field

The Hamiltonian can be obtained by integrating the Hamiltonian density over all space

$$H = \int \left[\bar{\psi} (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m) \psi - g\mathbf{j} \cdot \mathbf{A} + gj_0 A_0 + \frac{1}{2} \left(\dot{\mathbf{A}}^2 - (\boldsymbol{\nabla} A_0)^2 + \mathbf{B}^2 \right) \right] d^3r. \quad (1.28c)$$

Now, one makes use of the equation of motion

$$\boldsymbol{\nabla} \cdot \mathbf{E} = gj_0$$

in order to rewrite the A_0 in terms of the fermion current density j_0 . Since there is a gauge freedom left and one should fix it to avoid the redundancy of the field variables, one may take a Coulomb gauge, for example

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \quad (1.29)$$

In this case, the equation of motion for the gauge field A_0 becomes

$$\nabla^2 A_0 = -gj_0 \quad (1.30)$$

which is just a constraint. This is not an equation of motion any more since it does not depend on time. This constraint can be easily solved, and one obtains

$$A_0(r) = \frac{g}{4\pi} \int \frac{j_0(\mathbf{r}') d^3r'}{|\mathbf{r}' - \mathbf{r}|}. \quad (1.31)$$

Now, one can make use of the following equation

$$\frac{1}{2} \int (\boldsymbol{\nabla} A_0)^2 d^3r = -\frac{1}{2} \int (\nabla^2 A_0) A_0 d^3r = \frac{g^2}{8\pi} \int \frac{j_0(\mathbf{r}') j_0(\mathbf{r}) d^3r d^3r'}{|\mathbf{r}' - \mathbf{r}|}, \quad (1.32)$$

where the surface integrals are set to zero. Also, \mathbf{E}_T is introduced which denotes the transverse electric field

$$\mathbf{E}_T = -\dot{\mathbf{A}}$$

and it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{E}_T = 0.$$

Therefore, the Hamiltonian of fermions with electromagnetic fields becomes

$$H = \int \left\{ \bar{\psi} (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m) \psi - g\mathbf{j} \cdot \mathbf{A} \right\} d^3r + \frac{g^2}{8\pi} \int \frac{j_0(\mathbf{r}') j_0(\mathbf{r}) d^3r d^3r'}{|\mathbf{r}' - \mathbf{r}|} + \frac{1}{2} \int (\mathbf{E}_T^2 + \mathbf{B}^2) d^3r \quad (1.33)$$

which is a desired form.

1.4 Self-interacting Fermion Fields

Interactions between fermions are mediated by the gauge fields and this is the basic principle for the description of the fundamental field theory models. The reason why the gauge field theory is employed in modern physics is partly because the electromagnetic interaction is described by the gauge field theory but also because the gauge field theory is a renormalizable field theory. This is important since the renormalizable field theory has a predictive power in the perturbative calculations.

On the other hand, the field theory model with current-current interactions is not renormalizable in four dimensions since the coupling constant has the dimension of mass inverse square. Nevertheless, the model proposed by Nambu and Jona-Lasinio has been discussed frequently since it demonstrates, for the first time, the spontaneous symmetry breaking in the vacuum state in fermion field theory models. Therefore, we briefly discuss the Lagrangian density of the Nambu-Jona-Lasinio (NJL) model [93]. In addition, we treat the Thirring model which is the current current interaction model in two dimensions [109]. This model becomes important for the discussion of the spontaneous symmetry breaking which will be discussed in detail in Chapter 4.

1.4.1 Lagrangian and Hamiltonian Densities of NJL Model

The Lagrangian density of the NJL model is given as

$$\mathcal{L} = i\bar{\psi}\gamma_{\mu}\partial^{\mu}\psi - m\bar{\psi}\psi + \frac{1}{2}G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]. \quad (1.34)$$

In this case, the Hamiltonian density of the NJL model can be written as

$$\mathcal{H} = -i\psi^{\dagger}\nabla\cdot\boldsymbol{\alpha}\psi + m\bar{\psi}\psi - \frac{1}{2}G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]. \quad (1.35)$$

The coupling constant in this model has a dimension of inverse mass square,

$$G \sim m^{-2}. \quad (1.36)$$

Therefore, the NJL model is not renormalizable in the perturbative sense. Some of physical observables calculated in terms of the first order perturbation theory should have divergences of Λ^2 . When the cut-off momentum Λ becomes very large, the physical quantity diverges very quickly, and there is no chance to renormalize this divergence into the coupling constant G .

The NJL model has been discussed often in the context of the spontaneous symmetry breaking physics [83, 84], and therefore we are bound to discuss it here since we will discuss the symmetry and its breaking in the later chapter of this book. Further, it should be fair to mention that, if one solves the field theory model exactly or non-perturbatively, then one may find that the theory has some predictive power. But this problem is too difficult to discuss further.

1.4.2 Lagrangian Density of Thirring Model

There is a popular field theory model in two dimensions with current current interactions. It is called Thirring model which has been extensively studied since it has an exact solution due to the Bethe ansatz technique. This will be treated in detail in the later chapter. Here, we should only introduce the model Lagrangian density. The Thirring model is described by the following Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m_0\bar{\psi}\psi - \frac{1}{2}gj^\mu j_\mu, \quad (1.37)$$

where the fermion current j_μ is given as

$$j_\mu = \bar{\psi}\gamma_\mu\psi. \quad (1.38)$$

The coupling constant g in two dimensional current current interaction model is a dimensionless constant. Therefore, it is renormalizable, and the model has a predictive power in the perturbation calculations.

1.4.3 Hamiltonian Density for Thirring Model

The Hamiltonian density of the Thirring model can be written as

$$\mathcal{H} = -i\bar{\psi}\gamma^1\partial_1\psi + m_0\bar{\psi}\psi + \frac{1}{2}gj^\mu j_\mu. \quad (1.39)$$

Here, the chiral representation for γ matrices in two dimensions is chosen

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 \equiv \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.40)$$

By introducing the state ψ as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \quad (1.41)$$

the Hamiltonian density can be written

$$\mathcal{H} = -i \left(\psi_a^\dagger \frac{\partial}{\partial x} \psi_a - \psi_b^\dagger \frac{\partial}{\partial x} \psi_b \right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b. \quad (1.42)$$

Therefore, the Hamiltonian of the Thirring model can be written as

$$H = \int dx \left[-i \left(\psi_a^\dagger \frac{\partial}{\partial x} \psi_a - \psi_b^\dagger \frac{\partial}{\partial x} \psi_b \right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b \right]. \quad (1.43)$$

In Chapter 7, we will discuss the diagonalization procedure of the Thirring model Hamiltonian in terms of the Bethe ansatz technique.

1.5 Quarks with Electromagnetic and Chromomagnetic Interactions

It should be worthwhile writing the total Lagrangian density which is composed of quarks interacting with electromagnetic fields as well as chromomagnetic fields. Normally, one considers either electromagnetic interactions or chromomagnetic interactions separately since they become important at the different physical stages. Here, we write them together since in reality there are always two different types of interactions (QED and QCD) for quarks present in nature. In addition, we include the interaction terms which violate the time reversal invariance as well as parity transformation just for academic interests.

1.5.1 Lagrangian Density

The Lagrangian density of quarks interacting with electromagnetic fields as well as chromomagnetic fields is given as

$$\begin{aligned} \mathcal{L} = \bar{\psi}_f \left[i (\partial_\mu + ig_s A_\mu^a T^a + ie_f A_\mu) \gamma^\mu - m_0 \right] \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} \\ - \frac{i}{2} \tilde{d}_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 T^a \psi_f G^{\mu\nu,a} - \frac{i}{2} d_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 \psi_f F^{\mu\nu}, \end{aligned} \quad (1.44)$$

where the summation of flavor runs $f = \text{up, down, strange, charm, bottom and top quarks}$. T^a denotes the generator of the $SU(3)$ color group. The last two terms represent the T - and P -violating interactions. $\sigma_{\mu\nu}$ and γ_5 are defined as

$$\sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \quad \gamma_5 \equiv i\gamma_0 \gamma_1 \gamma_2 \gamma_3.$$

Field Strength of Electromagnetic Field

$F_{\mu\nu}$ denotes the electromagnetic field strength and is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (1.45)$$

where A_μ is the gauge field as given in Section 1.3.

Field Strength of Chromomagnetic Field

$G_{\mu\nu}$ denotes the chromomagnetic field strength and is given as

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s C^{abc} A_\mu^b A_\nu^c, \quad (1.46)$$

where A_μ^a is the color gauge fields. C^{abc} denotes the structure constant in the $SU(3)$ group. The coupling constants g_s and e_f denote the gauge coupling constant of f -flavor quarks interacting with chromomagnetic field and electromagnetic field, respectively.

1.5.2 EDM Interactions

The last two terms in eq.(1.44) represent the interaction terms which violate the time reversal invariance as well as the space reflection at the same time. These terms are given just for references in order to understand the T -violating interactions in future in terms of EDM (Electric Dipole Moments). That is,

$$\begin{aligned}
 -\frac{i}{2} \tilde{d}_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 T^a \psi_f G^{\mu\nu,a} &: \text{EDM for chromomagnetic fields,} \\
 -\frac{i}{2} d_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 \psi_f F^{\mu\nu} &: \text{EDM for electromagnetic fields.}
 \end{aligned}$$

The coupling strengths \tilde{d}_f and d_f denote the strength of the time reversal and parity violating interactions of quark with the chromomagnetic fields and the electromagnetic fields, respectively. The \tilde{d}_f and d_f have the dimension of the mass inverse, and, in fact, they are related to the electric dipole moment.

The existence of the EDM interactions should be determined from experiments. If there is any finite EDM interaction observed in future experiment, it should indicate an existence of a new scale which is different from the quark masses. In this respect, the observation of the EDM interaction must be physically very interesting and important indeed.