Symmetry and Its Breaking in Quantum Field Theory

By

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Preface

Physics is always difficult, though it is extremely interesting. Many times I thought I understood it sufficiently profoundly, but after some time, it turned out that my understanding of physics was far from satisfactory. In particular, field theory has special complexities which may not be common to other fields of research. The symmetry and its breaking are most exotic and sometimes almost mysterious to even those who can normally understand the basic physics in a clear manner.

In this textbook, I focused on presenting a simple and clear picture of the symmetry and its breaking in quantum field theory. For this purpose, I explained physics of elementary field theory of fermions interacting by gauge fields as well as by four body fermion fields. In this respect, the interpretation of the basic field theory is repeatedly done such that physicists including graduate students may understand the essential points of the symmetry breaking in this textbook.

Also, this book is intended for researchers who look for the basic problems in their investigations. In many fields of research, field theory is used as a computational tool. In this regard, I present some elaborate technical tools which are quite useful and sometimes incentive for new ideas in fundamental researches.

In physics, deeper understanding is more important than quicker understanding. In particular, graduate students should realize that, if someone else can understand the basic physics very quickly, then he is most likely a good interpreter of the textbook knowledge. Slow but deep understanding of physics is most important since it should definitely take much time to understand physics in depth. The shortest path of understanding physics is only one of many paths, and interesting physics may well be found in the paths which are far from the shortest one.

Physics must be simple once we understand it all. For example, I believe that QCD can surely describe the strong interaction physics. However, it may well be difficult to justify the perturbative calculation of the interactions between quarks, unless the gauge independence of the quark-quark interactions is guaranteed. In other words, when the unperturbed as well as interaction Hamiltonians are gauge dependent, we should make it sure that any physical quantities evaluated perturbatively are indeed gauge invariant, which seems to be very difficult.

In this textbook, there are quite a few issues which are still debating. I believe that the present understanding of the basic field theory in this textbook must be reasonably good, and as far as physics of the symmetry and its breaking is concerned, it should be the best of all. The spontaneous symmetry breaking of the global symmetry is by now understood in this textbook in terms of a simple physics terminology, and there is nothing mysterious from the standard way of understanding physics. However, it is still not yet settled whether the local gauge symmetry can be broken in terms of Higgs mechanism or not. At least, the gauge fixing for the non-gauge field is physically not at all easy to understand. For this problem, we need a lot to think over in future what should be physical observables in the Higgs mechanism.

This textbook contains a brief description of the lattice field theory even though it is not directly connected to the symmetry breaking physics. Still it may be interesting for readers to understand the basic point of the lattice field theory. For example, the continuum field theory must be richer than the lattice version, and it is most likely true that the lattice field theory can give only limited information on the continuum field theory, particularly when the latter keeps some symmetry while the former does not.

In Appendix, I explain some elementary physics so that readers may grasp the essence of the symmetry breaking phenomena in fermion field theory with little advanced knowledge. In some sense, Appendix can be read in its own interests since it includes non-relativistic quantum mechanics, Dirac equation and Maxwell equation, in addition to the notations which are often used in field theory. At the same time, Appendix contains some new physics interpretation for bosons, Dirac fields and quantization procedure. In particular, I believe that the first quantization of $[x, p_x] = i\hbar$, etc. may well be the result of the Dirac equation in that the Dirac Lagrangian density can be derived from the gauge principle as well as the Maxwell equations without involving the first quantization procedure. In the final chapter of Appendix, I briefly explain the renormalization in QED which is the most successful theory in quantum field theory. The perturbation theory is not the main issue of this textbook, but nevertheless readers may learn the essence of the renormalization scheme in quantum field theory.

The motive force of writing this textbook is initiated by Frank Columbus who understands the importance of the new picture of spontaneous symmetry breaking physics prior to experts and has encouraged me to write it into a textbook form. Indeed, I started to write this book from intensive discussions and hard works with my collaborators on this subject to achieve deeper but simpler understanding of the symmetry and its breaking in quantum field theory.

I should be grateful to all of my collaborators, in particular, Tomoko Asaga, Makoto Hiramoto, Takashi Homma, Seiji Kanemaki, Sachiko Oshima and Hidenori Takahashi for their great contributions to this book. Quite a few physicists and students also helped me a great deal for their critical reading of this manuscript. However, it is trivial to note that any mistakes in this book are entirely due to my carelessness.

To the Second Edition

The revision of this textbook is made mainly because of the following two reasons. Firstly, the first edition contained the wrong description of the path integral formulation. Even though it is normally found in the field theory textbooks, the path integral description in the field theory textbooks is not a correct one, and therefore I had to rewrite it into a correct formulation which was originally presented by Feynman. Secondly, the revision is concerned with the quantum gravity, and fortunately, the Lagrangian density that includes the gravitational interactions with fermions is properly constructed. Therefore, I included quantum gravity in this textbook, and one can now understand the basic physics of quantum gravity with our standard knowledge of quantum field theory, without referring to the space deformation.

In this occasion, I would like to express my sincere gratitude to late Prof. Kazuhiko Nishijima for his many useful comments and encouragements. His continuous supports for our works encouraged me a great deal, and in particular, the discussions of quantum gravity helped me to improve the description of the graviton propagation.

Finally I should like to thank numerous students and physicists for their interesting comments and suggestions to the first edition as well as the draft of the second edition. In particular, I should be grateful to Atsushi Kusaka, Kazuhiro Tsuda, Naohiro Kanda, Hiroshi Kato, Hiroaki Kubo and Yasunori Munakata for their careful reading of the manuscript.

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Contents

Chapter 1

Classical Field Theory of Fermions

The world of elementary particles is basically composed of fermions. Quarks, electrons and neutrinos are all fermions. On the other hand, elementary bosons are all gauge bosons, except Higgs particles though unknown at present. Therefore, if one wishes to understand field theory, then it should be the best to first study fermion field theory models.

In this chapter, we discuss the classical field theory in which "classical field" means that the field is not an operator but a c-number function. First, we treat the Schrödinger field and its equation in terms of the non-relativistic field theory model. In this case, the first quantization of $[x_i, p_j] = i\hbar\delta_{ij}$ is already done since we start from the Lagrangian density. In fact, the Lagrange equation leads to the Schrödinger equation or in other words, the Lagrangian density is constructed such that the Schrödinger equation can be derived from the Lagrange equation. The Dirac field is then discussed in terms of the Lagrangian density and the Lagrange equation. We also discuss the electromagnetic fields which interact with the Dirac field. The gauge invariance will be repeatedly discussed in this textbook, and the first introduction is given here. Finally, the field theory models with self-interacting fields are introduced and their Lagrangian density as well as Hamiltonian are described.

In this textbook, the basic parts of elementary physics can be found in Appendix, and in fact, Appendix is prepared such that it can be read in its own interests independently from the main part of the textbook.

Throughout this book, we employ the natural units

$$c = 1, \quad \hbar = 1.$$

This is, of course, due to its simplicity, and one can easily recover the right dimension of any physical quantities by making use of

$$\hbar c = 197 \ \ \mathrm{MeV} \cdot \mathrm{fm}.$$

1.1 Non-relativistic Fields

If one treats a classical field $\psi(\boldsymbol{r})$, it does not matter whether it is a relativistic field or non-relativistic one. The kinematics becomes important when one solves the equation of motion which is relativistic or non-relativistic. If the kinematics is non-relativistic, then the equation of motion that governs the field $\psi(\boldsymbol{r})$ is the Schrödinger equation. Therefore, we should first study the Schrödinger field from the point of view of the classical field theory.

1.1.1 Schrödinger Equation

Electron in classical mechanics is treated as a point particle whose equation of motion is governed by the Newton equation. When electrons are trapped by atoms, then their motions should be described by quantum mechanics. As long as electrons move much slowly in comparison with the velocity of light c, the equation of their motion is governed by the Schrödinger equation. The Schrödinger equation for electron with its mass m in the external field $U(\boldsymbol{r})$ can be written as [102]

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\boldsymbol{\nabla}^2 - U(\boldsymbol{r}) \right) \psi(\boldsymbol{r}, t) = 0, \tag{1.1}$$

where $U(\boldsymbol{r})$ is taken to be a real potential. $\psi(\boldsymbol{r}, t)$ corresponds to the electron field in atoms, and $|\psi(\boldsymbol{r}, t)|^2$ can be interpreted as a probability density of finding the electron at $(\boldsymbol{r}, t)$.

Field $\psi(\boldsymbol{r}, t)$ is Complex

The Schrödinger field $\psi(\boldsymbol{r}, t)$ should be a complex function, and the complex field just corresponds to one particle state in the classical field theory. This is a well known fact, but below we will see what may happen when we assume *a priori* that the Schrödinger field $\psi(\boldsymbol{r}, t)$ should be a real function.

Real Field Condition is Unphysical

If one imposes the condition that the field $\psi(\boldsymbol{r}, t)$ should be real

$$\psi(\boldsymbol{r}, t) = \psi^\dagger(\boldsymbol{r}, t)$$

then, one sees immediately that the field $\psi(\boldsymbol{r}, t)$ becomes time-independent since eq.(1.1) and its complex conjugate equation give the following constraint for a real field $\psi(\boldsymbol{r}, t)$

$$\frac{\partial \psi(\boldsymbol{r}, t)}{\partial t} = 0.$$

Also, the field $\psi(\boldsymbol{r})$ should satisfy the following equation

$$\left(-\frac{1}{2m}\boldsymbol{\nabla}^2 + U(r) \right) \psi(\boldsymbol{r}) = 0.$$

Since the general solution of eq.(1.1) can be written as

$$\psi(\boldsymbol{r}, t) = e^{-iEt}\phi(\boldsymbol{r})$$

the field $\psi(\boldsymbol{r},t)$ may become a real function only if the energy E of the system vanishes. That is, the energy eigenvalue of E is

$$E = 0.$$

Therefore, the real field cannot propagate and should be unphysical. This means that the real field condition of $\psi(\boldsymbol{r},t)$ is physically too strong as a constraint.

1.1.2 Lagrangian Density for Schrödinger Fields

The Lagrangian density which can produce eq.(1.1) is easily found as

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m}\frac{\partial \psi^\dagger}{\partial x_k}\frac{\partial \psi}{\partial x_k} - \psi^\dagger U \psi, \tag{1.2}$$

where the repeated indices of k mean the summation of $k = 1, 2, 3$ and, in this text, this notation as well as the vector representation are employed depending on the situations. The repeated indices notation is mostly better for the calculation, but for memorizing the expressions or equations, the vector notation has some advantage.

The Lagrangian density of eq.(1.2) is constructed such that the Lagrange equation can reproduce the Schrödinger equation of eq.(1.1). It may also be important to note that the Lagrangian density of eq.(1.2) has a $U(1)$ symmetry, that is, it is invariant under the change of the field ψ as

$$\psi'(x) = e^{i\theta}\psi(x) \longrightarrow \mathcal{L}' = \mathcal{L},$$

where θ is a real constant. This invariance is clearly satisfied, and it is related to the conservation of vector current in terms of Noether's theorem which will be treated in the later chapters and in Appendix A.

Non-hermiticity of Lagrangian Density

At this point, we should discuss the non-hermiticity of the Lagrangian density. As one notices, the Lagrangian density of eq.(1.2) is not hermitian, and therefore some symmetry will be lost. One can build the Lagrangian density which is hermitian by replacing the first term by

$$i\psi^\dagger \frac{\partial \psi}{\partial t} \longrightarrow \left(\frac{i}{2}\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{i}{2}\frac{\partial \psi^\dagger}{\partial t}\psi \right).$$

However, it is a difficult question whether the Lagrangian density must be hermitian or not since it is not an observable. In addition, when one introduces the conjugate fields

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}}, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}$$

in accordance with the fields ψ and $\psi^\dagger$, then the symmetry between them is lost. However, the conjugate fields themselves are again not observables, and therefore there is no reason that one should keep this symmetry. In any case, one can, of course, work with the symmetric and hermitian Lagrangian density, but physical observables are just the same as eq.(1.2). In this textbook, we employ eq.(1.2) since it is simpler.

1.1.3 Lagrange Equation for Schrödinger Fields

The Lagrange equation for field theory can be obtained by the variational principle of the action S

$$S = \int \mathcal{L} \, dt \, d^3r$$

and the Lagrange equation is derived in Appendix A. Since the field ψ is a complex field, ψ and $\psi^\dagger$ are treated as independent functional variables. The Lagrange equation for the field ψ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \equiv \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi}\,, \qquad (1.3a)$$

where the four dimensional derivative

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

is introduced for convenience. Now, the following equations can be easily evaluated

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \frac{\partial \psi^\dagger}{\partial t}\,,$$

$$\frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = -\frac{1}{2m} \frac{\partial}{\partial x_k} \frac{\partial \psi^\dagger}{\partial x_k}\,,$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\psi^\dagger U$$

and therefore one obtains

$$\left(-i \frac{\partial}{\partial t} + \frac{1}{2m} \boldsymbol{\nabla}^2 - U(r) \right) \psi^\dagger(\boldsymbol{r}, t) = 0$$

which is just the Schrödinger equation for $\psi^\dagger$ in eq.(1.1).

It should be interesting to calculate the Lagrange equation for the field $\psi^\dagger$,

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi^\dagger}\,. \qquad (1.3b)$$

In this case, one finds

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0,$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^\dagger}{\partial x_k})} = -\frac{1}{2m}\frac{\partial}{\partial x_k}\frac{\partial \psi}{\partial x_k},$$
$$\frac{\partial \mathcal{L}}{\partial \psi^\dagger} = i\frac{\partial \psi}{\partial t} - U\psi$$

and therefore one obtains

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\boldsymbol{\nabla}^2 - U(r)\right)\psi(\boldsymbol{r},t) = 0$$

which is just the same equation as eq.(1.1).

Here, we note that the Lagrangian density is not a physical observable and therefore it does not necessarily have to be determined uniquely. It is by now clear that the Lagrangian density eq.(1.2) reproduces a desired Schrödinger equation and thus can be taken as the right Lagrangian density for Schrödinger fields.

1.1.4 Hamiltonian Density for Schrödinger Fields

From the Lagrangian density, one can build the Hamiltonian density $\mathcal{H}$ which is the energy density of the field $\psi(\boldsymbol{r},t)$. The Hamiltonian density $\mathcal{H}$ is best constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\partial^\nu \psi^\dagger - \mathcal{L} g^{\mu\nu}$$

which will be derived in eq.(2.32) in Chapter 2. The energy momentum tensor $\mathcal{T}^{\mu\nu}$ satisfies the following equation of conservation law

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0$$

due to the invariance of the Lagrangian density under the translation. Therefore, the conserved charge associated with the $\mathcal{T}^{0\nu}$

$$\mathcal{Q}^\nu = \int \mathcal{T}^{0\nu}\, d^3r$$

should be a conserved quantity. Thus, it is natural that one defines the Hamiltonian in terms of the $\mathcal{Q}^0$.

Hamiltonian Density from Energy Momentum Tensor

The Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\,\dot{\psi}^\dagger - \mathcal{L}. \tag{1.4a}$$

Therefore, introducing the conjugate fields Π_ψ and $\Pi_{\psi^\dagger}$ by

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0$$

one can write the Hamiltonian density as

$$\mathcal{H} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger} \dot{\psi}^\dagger - \mathcal{L} = \frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi. \qquad (1.4b)$$

1.1.5 Hamiltonian for Schrödinger Fields

The Hamiltonian for the Schrödinger field is obtained by integrating the Hamiltonian density over all space

$$H \equiv \int \mathcal{H}\, d^3r = \int \left[\frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi \right] d^3r. \qquad (1.4c)$$

By employing the Gauss theorem

$$\int_V \boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi)\, d^3r = \int_S (\psi^\dagger \boldsymbol{\nabla}_n \psi)\, dS_n$$

one can rewrite eq.(1.4c)

$$H = \int \left[-\frac{1}{2m} \psi^\dagger \boldsymbol{\nabla}^2 \psi + \psi^\dagger U \psi \right] d^3r, \qquad (1.4d)$$

where the following identity is employed

$$\boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger \boldsymbol{\nabla}^2 \psi.$$

In addition, the surface integral term is neglected since it should vanish at the surface of sphere at infinity.

Now, it may be interesting to note that the Hamiltonian in eq.(1.4d) by itself does not give us much information on the dynamics. As long as we stay in the classical field theory, then the dynamics can be obtained from the equation of motion, that is, the Schrödinger equation. The static Schrödinger equation can be derived from the variational principle of the Hamiltonian with respect to ψ, and this treatment is given in Appendix A.

The Hamiltonian of eq.(1.4c) becomes important when the field ψ is quantized, that is, the field ψ is assumed to be written in terms of the annihilation operator $a_{\boldsymbol{k}}$ as discussed in Chapter 3. In this case, the Schrödinger field becomes an operator and therefore the Hamiltonian as well. This means that one has to prepare the Fock state on which the Hamiltonian can operate, and if one solves the eigenvalue equation for the Hamiltonian, then one can obtain the energy eigenvalue of the Hamiltonian corresponding to the Fock state.

However, the quantization of the Schrödinger field is not needed in the normal circumstances. The field quantization is necessary for the relativistic fields which contain negative energy solutions, and it becomes important when one wishes to treat the quantum fluctuation of the fields which corresponds to the creation and annihilation of particles.

1.1.6 Conservation of Vector Current

From the Schrödinger equation, one can derive the current conservation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0,$$

where ρ and $\boldsymbol{j}$ are defined as

$$\rho = \psi^\dagger \psi, \quad \boldsymbol{j} = \frac{i}{2m} \left[(\boldsymbol{\nabla} \psi^\dagger)\psi - \psi^\dagger \boldsymbol{\nabla} \psi \right].$$

This continuity equation of the vector current can also be derived as Noether's theorem from the Lagrangian density of eq.(1.2) which is invariant under the global gauge transformation

$$\psi' = e^{i\alpha} \psi.$$

As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right], \quad \text{with} \quad j^\mu = (\rho, \boldsymbol{j})$$

which just gives the above current density ρ and $\boldsymbol{j}$ when one employs the Lagrangian density of eq.(1.2).

It may be interesting to observe that the Lagrange equation, energy momentum tensor and the current conservation are all written in a relativistically covariant fashion when the properties of the Schrödinger field are derived. That is, apart from the shape of the Lagrangian density of the Schrödinger field, all the treatments are just the same as the relativistic description.

1.2 Dirac Fields

Electron in hydrogen atom moves much slowly compared with the velocity of light c. However, if one considers a hydrogen-like ${}^{209}_{83}\mathrm{Bi}$ atom where $Z = 83$, for example, then the motion of electron becomes relativistic since its velocity v can be given as

$$\frac{v}{c} \sim (Z\alpha)^2 \sim \left(\frac{83}{137} \right)^2 \sim 0.37$$

which is already comparable with c.

In this case, one should employ the relativistic kinematics, and therefore the Schrödinger equation should be replaced by the Dirac equation which is obtained by a natural extension of the relativistic kinematics. However, the Dirac equation contains new properties which are essentially different from the Schrödinger equation, apart from the kinematics. They have negative energy solutions and spin degrees of freedom. Both properties are very important in physics and will be repeatedly discussed in this textbook.

1.2.1 Dirac Equation for Free Fermion

The Dirac equation for free fermion with its mass m is written as [25, 26]

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta\right)\psi(\boldsymbol{r},t) = 0, \tag{1.5}$$

where ψ has four components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}.$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & \mathbf{0}\end{pmatrix}, \quad \beta = \begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix},$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix.

The derivation of the Dirac equation and its application to hydrogen atom are given in Appendix D. One can learn from the procedure of deriving the Dirac equation that the number of components of the electron fields is important, and it is properly obtained in the Dirac equation. That is, among the four components of the field ψ, two degrees of freedom should correspond to the positive and negative energy solutions and another two degrees should correspond to the spin with $s = \frac{1}{2}$. It is also important to note that the factorization procedure indicates that the four component spinor is the minimum number of fields which can take into account the negative energy degree of freedom in a proper way.

Eq.(1.5) can be rewritten in terms of the wave function components by multiplying β from the left hand side

$$(i\partial_\mu\gamma^\mu - m)_{ij}\psi_j = 0 \;\text{ for }\; i = 1,2,3,4, \tag{1.6}$$

where the repeated indices of j indicate the summation of $j = 1,2,3,4$. Here, gamma matrices

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) \equiv (\beta, \beta\boldsymbol{\alpha})$$

are introduced, and the repeated indices of Greek letters μ indicate the summation of $\mu = 0,1,2,3$ as defined in Appendix A. The expression of eq.(1.6) is called $covariant$ since the Lorentz invariance of eq.(1.6) is manifest. It is indeed written in terms of the Lorentz scalars, but, of course there is no deep physical meaning in covariance.

1.2.2 Lagrangian Density for Free Dirac Fields

The Lagrangian density for free Dirac fermions can be constructed as

$$\mathcal{L} = \psi_i^\dagger \left[\gamma_0(i\partial_\mu\gamma^\mu - m)\right]_{ij}\psi_j = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi, \tag{1.7}$$

where $\bar{\psi}$ is defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0.$$

This Lagrangian density is just constructed so as to reproduce the Dirac equation of (1.6) from the Lagrange equation. It should be important to realize that the Lagrangian density of eq.(1.7) is invariant under the Lorentz transformation since it is a Lorentz scalar. This is clear since the Lagrangian density should not depend on the system one chooses.

Non-hermiticity of Lagrangian Density

This Lagrangian density is not hermitian, and it is easy to construct a hermitian Lagrangian density. However, as we discussed in the context of Schrödinger field, there is no strong reason that one should take the hermitian Lagrangian density since proper physical equations can be obtained from eq.(1.7).

1.2.3 Lagrange Equation for Free Dirac Fields

The Lagrange equation for $\psi_i^\dagger$ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_i^\dagger}{\partial x_k}\right)} = \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} \tag{1.8}$$

and one can easily calculate the following equations

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} = 0,$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} = \left[\gamma_0(i\partial_\mu \gamma^\mu - m)\right]_{ij} \psi_j$$

and thus, this leads to the following equation

$$\left[\gamma_0(i\partial_\mu \gamma^\mu - m)\right]_{ij} \psi_j = 0$$

which is just eq.(1.6). Here, it should be noted that the ψ_i and $\psi_i^\dagger$ are independent functional variables, and the functional derivative with respect to ψ_i or $\psi_i^\dagger$ gives the same equation of motion.

1.2.4 Plane Wave Solutions of Free Dirac Equation

The free Dirac equation of eq.(1.5) can be solved exactly, and it has plane wave solutions. A simple way to solve eq.(1.5) can be shown as follows. First, one writes the wave function

ψ in the following shape

$$\psi_s(\boldsymbol{r}, t) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\sqrt{V}} \, e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.9}$$

where ζ_1 and ζ_2 are two component spinors

$$\zeta_1 = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}.$$

In this case, eq.(1.5) becomes

$$\begin{pmatrix} -m - E & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m - E \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0 \tag{1.10}$$

which leads to

$$E^2 = m^2 + \boldsymbol{p}^2.$$

This equation has the following two solutions.

Positive Energy Solution ($E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(+)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} \, u_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.11a}$$

$$u_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_{\boldsymbol{p}} + m}\chi_s \end{pmatrix}, \quad \text{with} \quad s = \pm\frac{1}{2}, \tag{1.11b}$$

where χ_s denotes the spin wave function and is written as

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Negative Energy Solution ($E_{\boldsymbol{p}} = -\sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(-)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} \, v_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.12a}$$

$$v_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{|E_{\boldsymbol{p}}| + m}{2|E_{\boldsymbol{p}}|}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E_{\boldsymbol{p}}| + m}\chi_s \\ \chi_s \end{pmatrix}. \tag{1.12b}$$

Some Properties of Spinor

The spinor wave function $u_{\boldsymbol{p}}^{(s)}$ and $v_{\boldsymbol{p}}^{(s)}$ are normalized according to

$$u_{\boldsymbol{p}}^{(s)\dagger} u_{\boldsymbol{p}}^{(s)} = 1,$$

$$v_{\boldsymbol{p}}^{(s)\dagger} v_{\boldsymbol{p}}^{(s)} = 1.$$

Further, they satisfy the following equations when the spin is summed over

$$\sum_{s=1}^{2} u_{\boldsymbol{p}}^{(s)} \bar{u}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}, \tag{1.13a}$$

$$\sum_{s=1}^{2} v_{\boldsymbol{p}}^{(s)} \bar{v}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}. \tag{1.13b}$$

1.2.5 Quantization in Box with Periodic Boundary Conditions

In field theory, one often puts the theory into the box with its volume $V = L^3$ and requires that the wave function should satisfy the periodic boundary conditions (PBC). This is mainly because the free field solutions are taken as the basis states, and in this case, one can only calculate physical observables if one works in the box. It is clear that the free field can be defined well only if it is confined in the box.

Since the wave function $\psi_s(\boldsymbol{r}, t)$ for a free particle in the box should be proportional to

$$\psi_s(\boldsymbol{r}, t) \simeq \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\sqrt{V}} \, e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}}$$

the PBC equations become

$$e^{ip_x x} = e^{ip_x(x+L)}, \quad e^{ip_y y} = e^{ip_y(y+L)}, \quad e^{ip_z z} = e^{ip_z(z+L)}. \tag{1.14a}$$

Therefore, one obtains the constraints on the momentum p_k as

$$p_x = \frac{2\pi}{L} n_x, \quad p_y = \frac{2\pi}{L} n_y, \quad p_z = \frac{2\pi}{L} n_z, \;\; n_k = 0, \pm 1, \pm 2, \ldots \,. \tag{1.14b}$$

In this case, the number of states N in the large L limit becomes

$$N = \sum_{n_x, n_y, n_z} \sum_{s} = 2\, \frac{L^3}{(2\pi)^3} \int d^3 p, \tag{1.15}$$

where a factor of two comes from the spin degree of freedom.

1.2.6 Hamiltonian Density for Free Dirac Fermion

The Hamiltonian density for free fermion can be constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \partial^\nu \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \partial^\nu \psi_i^\dagger \right) - \mathcal{L} g^{\mu\nu}$$

which will be treated in eq.(A.12.3) of Appendix A.

Hamiltonian Density from Energy Momentum Tensor

Now, one defines the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L}. \tag{1.16}$$

Since the Lagrangian density of free fermion is given in eq.(1.7) and is rewritten as

$$\mathcal{L} = i\psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j$$

one can introduce the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$, and calculate them

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0. \tag{1.17}$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L} = \bar{\psi}_i \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \tag{1.18}$$

1.2.7 Hamiltonian for Free Dirac Fermion

The Hamiltonian for free fermion fields is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi\, d^3r. \tag{1.19}$$

As we discussed in the Schrödinger field, the Hamiltonian itself cannot give us much information on the dynamics. One can learn some properties of the system described by the Hamiltonian, but one cannot obtain any dynamical information of the system from the Hamiltonian. In order to calculate the dynamics of the system in the classical field theory model, one has to solve the equation of motions which are obtained from the Lagrange equations for fields.

When one wishes to consider the fluctuations of the fields or, in other words, creations of particles and anti-particles, then one should quantize the fields. In this case, the Hamiltonian becomes an operator. Therefore, one has to prepare the Fock states on which the Hamiltonian can operate. Most of the difficulties of the field theory models should be to find the vacuum of the system.

1.2.8 Conservation of Vector Current

The Lagrangian density of the Dirac field has a global gauge invariance,

$$\psi' = e^{i\alpha}\psi \longrightarrow \mathcal{L}' = \mathcal{L}$$

and therefore there is a Noether current associated with the symmetry. As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\psi^\dagger\right]$$

and therefore the vector current j_μ becomes

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

Due to the global gauge invariance of the Lagrangian density, the vector current j_μ satisfies the continuity equation

$$\partial_\mu j^\mu = 0.$$

1.3 Electron and Electromagnetic Fields

The main part of the physical world is governed by the interaction between electrons and electromagnetic fields. Therefore, the Dirac equation, the Maxwell equation and their interactions are most important to understand the basic physics in many fundamental researches.

1.3.1 Lagrangian Density

When electron interacts with electromagnetic fields, the Lagrangian density becomes

$$\mathcal{L} = \bar{\psi}\big(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m\big)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{1.20}$$

where $F_{\mu\nu}$ denotes the field strength and is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

A^μ denotes the gauge field with

$$A^\mu = (A_0, \boldsymbol{A}),$$

where A_0 and $\boldsymbol{A}$ are the scalar and vector potentials, respectively. g denotes the gauge coupling constant, and in the classical electromagnetism, it corresponds to the electric charge e.

In the four dimensional field theory of QED, the coupling constant g is dimensionless, and therefore it is renormalizable in the perturbation calculation. In the two dimensional case, the coupling constant g has a mass dimension, and thus it is called *super-renormalizable*. In this case, there appear no infinities from the momentum integral in the perturbative calculations, and therefore one does not have to renormalize the coupling constant.

1.3.2 Gauge Invariance

The Lagrangian density of eq.(1.20) has an interesting feature. The free fermion Lagrangian density part

$$\bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi$$

is just the same as free Dirac Lagrangian density, and the last term in eq.(1.20)

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

corresponds to the field energy term of the electromagnetic fields. The important point is that the shape of the interaction term

$$-g\bar{\psi} A_\mu \gamma^\mu \psi$$

can be determined by the requirement of the invariance under the local gauge transformation.

Local Gauge Transformation

We consider the following local gauge transformation

$$\psi' = e^{-ig\chi}\psi, \quad {A'}_\mu = A_\mu + \partial_\mu \chi, \tag{1.21}$$

where χ is an arbitrary real function of space and time, that is, $\chi(\boldsymbol{r}, t)$ which is therefore called *local*. It is easy to prove that the shape of the field energy term of the electromagnetic fields does not change under the local gauge transformation of eq.(1.21)

$${F'}_{\mu\nu} = \partial_\mu {A'}_\nu - \partial_\nu {A'}_\mu = \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi) = F_{\mu\nu}.$$

In addition, one can easily prove that the Lagrangian density of

$$\bar{\psi}(i\partial_\mu \gamma^\mu - gA_\mu \gamma^\mu - m)\psi$$

does not change its shape under the local gauge transformation of eq.(1.21). That is,

$$\begin{aligned} &\bar{\psi}'(i\partial_\mu \gamma^\mu - g{A'}_\mu \gamma^\mu - m)\psi' \\ &= \bar{\psi} e^{-ig\chi} e^{ig\chi} \big(i\partial_\mu \gamma^\mu + g\partial_\mu \chi \gamma^\mu - gA_\mu \gamma^\mu - g\partial_\mu \chi \gamma^\mu - m\big)\psi \\ &= \bar{\psi}\big(i\partial_\mu \gamma^\mu - gA_\mu \gamma^\mu - m\big)\psi. \end{aligned} \tag{1.22}$$

Therefore, a new Lagrangian density $\mathcal{L}'$ becomes equal to the original one $\mathcal{L}$

$$\mathcal{L}' = \bar{\psi}' \left(i\partial_\mu \gamma^\mu - g{A'}_\mu \gamma^\mu - m\right) \psi' - \frac{1}{4} {F'}_{\mu\nu} F'^{\mu\nu} = \mathcal{L}.$$

The invariance of the Lagrangian density under the local gauge transformation determines the shape of the interaction between electron and electromagnetic fields. This is surprising, but it is, in a sense, the same as the classical mechanics as discussed in AppendixE. In this respect, it is interesting to realize that the gauge invariance that arises from the redundancy of the vector potential in solving the Maxwell equations plays an important role for determining the shape of the fundamental interactions.

1.3.3 Lagrange Equation for Dirac Field

The Dirac equation with the electromagnetic interaction can be easily obtained from the Lagrange equation for ψ

$$(i\partial_\mu \gamma^\mu - gA_\mu \gamma^\mu - m)\psi = 0. \tag{1.23}$$

This is the Dirac equation for the hydrogen atom when the potential is static, that is

$$\boldsymbol{A} = 0$$

and

$$gA_0 = -\frac{Ze^2}{r} \, ,$$

where we put $g = e$ with e the electric charge.

1.3.4 Lagrange Equation for Gauge Field

The Lagrange equation for the gauge field A_ν is written as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu} \, .$$

Since one can easily calculate

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -g\bar{\psi}\gamma^\nu \psi,$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\, \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \times 2 = -\partial_\mu F^{\mu\nu}$$

one obtains

$$\partial_\mu F^{\mu\nu} = g\bar{\psi}\gamma^\nu \psi = gj^\nu, \tag{1.24}$$

where the current density j^ν is defined as

$$j^\nu = \bar{\psi}\gamma^\nu \psi = (\bar{\psi}\gamma^0 \psi, \bar{\psi}\boldsymbol{\gamma}\psi). \tag{1.25}$$

Eq.(1.24) is the Maxwell equation, and more explicitly, one can evaluate eq.(1.24)

$$[\nu = 0] \longrightarrow \frac{\partial F^{k0}}{\partial x_k} = \frac{\partial E_k}{\partial x_k} = \boldsymbol{\nabla} \cdot \boldsymbol{E} = gj_0, \tag{1.26a}$$

$$[\nu = k] \longrightarrow \frac{\partial F^{0k}}{\partial t} + \frac{\partial F^{ik}}{\partial x_i} = -\dot{E}_k + \epsilon_{kij} \frac{\partial B_j}{\partial x_i} = -\dot{E}_k + (\boldsymbol{\nabla} \times \boldsymbol{B})_k = gj_k \tag{1.26b}$$

which are just the Maxwell equations. It is of course easy to see that no magnetic monopole

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

and Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

are automatically satisfied in terms of the vector potential A_μ since

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \Longrightarrow \nabla \cdot \boldsymbol{B} = \nabla \cdot \nabla \times \boldsymbol{A} = \nabla \times \nabla \cdot \boldsymbol{A} = 0,$$
$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A_0 \Longrightarrow \nabla \times \boldsymbol{E} = \nabla \times \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A_0\right) = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

1.3.5 Hamiltonian Density for Fermions with Electromagnetic Field

Now, one can construct the Hamiltonian density of fermion with electromagnetic field. The Hamiltonian density $\mathcal{H}$ can be defined by the energy momentum tensor $\mathcal{T}^{\mu\nu}$ [eq.(A.12.3)] as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \left(\frac{\partial \mathcal{L}}{\partial \dot{A}_k} \dot{A}_k \right) - \mathcal{L}$$

since $\mathcal{T}^{0\nu}$ is a conserved quantity. By introducing the conjugate fields Π_{ψ_i}, $\Pi_{\psi_i^\dagger}$ and Π_{A_k} as

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}, \quad \Pi_{\psi_i^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger}, \quad \Pi_{A_k} = \frac{\partial \mathcal{L}}{\partial \dot{A}_k}$$

one can rewrite the Hamiltonian density as

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \Pi_{A_k} \dot{A}_k - \mathcal{L}. \tag{1.27}$$

The conjugate fields Π_{ψ_i}, $\Pi_{\psi_i^\dagger}$ and Π_{A_k} can be calculated by employing the Lagrangian density of eq.(1.20)

$$\Pi_{\psi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0, \quad \Pi_{A_k} = \dot{A}_k + \frac{\partial A_0}{\partial x_k} = -E_k.$$

It should be noted that there is no corresponding conjugate field for A_0 in the Hamiltonian density, and thus there is no kinetic energy term present for A_0. Now, the Hamiltonian density can be calculated as

$$\mathcal{H} = \bar{\psi} \left[-i\gamma_k \frac{\partial}{\partial x_k} + m + gA_\mu \gamma^\mu \right] \psi + \frac{1}{2} \left[\dot{A_k}^2 - \left(\frac{\partial A_0}{\partial x_k} \right)^2 + \left(\frac{\partial A_k}{\partial x_j} \frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j} \frac{\partial A_j}{\partial x_k} \right) \right]. \tag{1.28a}$$

Eq.(1.28a) can be written in a familiar form

$$\mathcal{H} = \bar{\psi} \left(-i\boldsymbol{\gamma} \cdot \nabla + m \right) \psi - g\boldsymbol{j} \cdot \boldsymbol{A} + gj_0 A_0 + \frac{1}{2} \left[\dot{\boldsymbol{A}}^2 - (\nabla A_0)^2 + \boldsymbol{B}^2 \right]. \tag{1.28b}$$

1.3.6 Hamiltonian for Fermions with Electromagnetic Field

The Hamiltonian can be obtained by integrating the Hamiltonian density over all space

$$H=\int\left[\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\psi-g\boldsymbol{j}\cdot\boldsymbol{A}+gj_0A_0+\frac{1}{2}\left(\dot{\boldsymbol{A}}^2-(\boldsymbol{\nabla}A_0)^2+\boldsymbol{B}^2\right)\right]d^3r. \quad (1.28c)$$

Now, one makes use of the equation of motion

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=gj_0$$

in order to rewrite the A_0 in terms of the fermion current density j_0. Since there is a gauge freedom left and one should fix it to avoid the redundancy of the field variables, one may take a Coulomb gauge, for example

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0. \quad (1.29)$$

In this case, the equation of motion for the gauge field A_0 becomes

$$\boldsymbol{\nabla}^2A_0=-gj_0 \quad (1.30)$$

which is just a constraint. This is not an equation of motion any more since it does not depend on time. This constraint can be easily solved, and one obtains

$$A_0(r)=\frac{g}{4\pi}\int\frac{j_0(\boldsymbol{r}')\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}\,. \quad (1.31)$$

Now, one can make use of the following equation

$$\frac{1}{2}\int(\boldsymbol{\nabla}A_0)^2\,d^3r=-\frac{1}{2}\int(\boldsymbol{\nabla}^2A_0)A_0\,d^3r=\frac{g^2}{8\pi}\int\frac{j_0(\boldsymbol{r}')j_0(\boldsymbol{r})\;d^3r\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}\,, \quad (1.32)$$

where the surface integrals are set to zero. Also, $\boldsymbol{E}_T$ is introduced which denotes the transverse electric field

$$\boldsymbol{E}_T=-\dot{\boldsymbol{A}}$$

and it satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_T=0.$$

Therefore, the Hamiltonian of fermions with electromagnetic fields becomes

$$H=\int\left\{\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\psi-g\boldsymbol{j}\cdot\boldsymbol{A}\right\}d^3r$$

$$+\frac{g^2}{8\pi}\int\frac{j_0(\boldsymbol{r}')j_0(\boldsymbol{r})\;d^3r\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}+\frac{1}{2}\int\left(\boldsymbol{E}_T^2+\boldsymbol{B}^2\right)d^3r \quad (1.33)$$

which is a desired form.

1.4 Self-interacting Fermion Fields

Interactions between fermions are mediated by the gauge fields and this is the basic principle for the description of the fundamental field theory models. The reason why the gauge field theory is employed in modern physics is partly because the electromagnetic interaction is described by the gauge field theory but also because the gauge field theory is a renormalizable field theory. This is important since the renormalizable field theory has a predictive power in the perturbative calculations.

On the other hand, the field theory model with current-current interactions is not renormalizable in four dimensions since the coupling constant has the dimension of mass inverse square. Nevertheless, the model proposed by Nambu and Jona-Lasinio has been discussed frequently since it demonstrates, for the first time, the spontaneous symmetry breaking in the vacuum state in fermion field theory models. Therefore, we briefly discuss the Lagrangian density of the Nambu-Jona-Lasinio (NJL) model [93]. In addition, we treat the Thirring model which is the current current interaction model in two dimensions [109]. This model becomes important for the discussion of the spontaneous symmetry breaking which will be discussed in detail in Chapter 4.

1.4.1 Lagrangian and Hamiltonian Densities of NJL Model

The Lagrangian density of the NJL model is given as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi + \frac{1}{2}\,G\big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\big]. \tag{1.34}$$

In this case, the Hamiltonian density of the NJL model can be written as

$$\mathcal{H} = -i\psi^\dagger\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}\psi + m\bar{\psi}\psi - \frac{1}{2}\,G\big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\big]. \tag{1.35}$$

The coupling constant in this model has a dimension of inverse mass square,

$$G \sim m^{-2}. \tag{1.36}$$

Therefore, the NJL model is not renormalizable in the perturbative sense. Some of physical observables calculated in terms of the first order perturbation theory should have divergences of Λ^2. When the cut-off momentum Λ becomes very large, the physical quantity diverges very quickly, and there is no chance to renormalize this divergence into the coupling constant G.

The NJL model has been discussed often in the context of the spontaneous symmetry breaking physics [83, 84], and therefore we are bound to discuss it here since we will discuss the symmetry and its breaking in the later chapter of this book. Further, it should be fair to mention that, if one solves the field theory model exactly or non-perturbatively, then one may find that the theory has some predictive power. But this problem is too difficult to discuss further.

1.4.2 Lagrangian Density of Thirring Model

There is a popular field theory model in two dimensions with current current interactions. It is called Thirring model which has been extensively studied since it has an exact solution due to the Bethe ansatz technique. This will be treated in detail in the later chapter. Here, we should only introduce the model Lagrangian density. The Thirring model is described by the following Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m_0\bar{\psi}\psi - \frac{1}{2}\, g j^\mu j_\mu, \tag{1.37}$$

where the fermion current j_μ is given as

$$j_\mu = \bar{\psi}\gamma_\mu\psi. \tag{1.38}$$

The coupling constant g in two dimensional current current interaction model is a dimensionless constant. Therefore, it is renormalizable, and the model has a predictive power in the perturbation calculations.

1.4.3 Hamiltonian Density for Thirring Model

The Hamiltonian density of the Thirring model can be written as

$$\mathcal{H} = -i\bar{\psi}\gamma^1\partial_1\psi + m_0\bar{\psi}\psi + \frac{1}{2}\, g j^\mu j_\mu. \tag{1.39}$$

Here, the chiral representation for γ matrices in two dimensions is chosen

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 \equiv \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.40}$$

By introducing the state ψ as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{1.41}$$

the Hamiltonian density can be written

$$\mathcal{H} = -i\left(\psi_a^\dagger \frac{\partial}{\partial x}\,\psi_a - \psi_b^\dagger \frac{\partial}{\partial x}\,\psi_b\right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b. \tag{1.42}$$

Therefore, the Hamiltonian of the Thirring model can be written as

$$H = \int dx \left[-i\left(\psi_a^\dagger \frac{\partial}{\partial x}\,\psi_a - \psi_b^\dagger \frac{\partial}{\partial x}\,\psi_b\right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b\right]. \tag{1.43}$$

In Chapter 7, we will discuss the diagonalization procedure of the Thirring model Hamiltonian in terms of the Bethe ansatz technique.

1.5 Quarks with Electromagnetic and Chromomagnetic Interactions

It should be worthwhile writing the total Lagrangian density which is composed of quarks interacting with electromagnetic fields as well as chromomagnetic fields. Normally, one considers either electromagnetic interactions or chromomagnetic interactions separately since they become important at the different physical stages. Here, we write them together since in reality there are always two different types of interactions (QED and QCD) for quarks present in nature. In addition, we include the interaction terms which violate the time reversal invariance as well as parity transformation just for academic interests.

1.5.1 Lagrangian Density

The Lagrangian density of quarks interacting with electromagnetic fields as well as chromomagnetic fields is given as

$$\mathcal{L} = \bar{\psi}_f \Big[i \left(\partial_\mu + i g_s A_\mu^a T^a + i e_f A_\mu \right) \gamma^\mu - m_0 \Big] \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$
$$- \frac{i}{2} \tilde{d}_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 T^a \psi_f G^{\mu\nu,a} - \frac{i}{2} d_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 \psi_f F^{\mu\nu}, \tag{1.44}$$

where the summation of flavor runs $f =$ up, down, strange, charm, bottom and top quarks. T^a denotes the generator of the $SU(3)$ color group. The last two terms represent the T- and P-violating interactions. $\sigma_{\mu\nu}$ and γ_5 are defined as

$$\sigma_{\mu\nu} = \frac{i}{2} \left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \right), \quad \gamma_5 \equiv i \gamma_0 \gamma_1 \gamma_2 \gamma_3 .$$

Field Strength of Electromagnetic Field

$F_{\mu\nu}$ denotes the electromagnetic field strength and is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{1.45}$$

where A_μ is the gauge field as given in Section 1.3.

Field Strength of Chromomagnetic Field

$G_{\mu\nu}$ denotes the chromomagnetic field strength and is given as

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s C^{abc} A_\mu^b A_\nu^c , \tag{1.46}$$

where A_μ^a is the color gauge fields. C^{abc} denotes the structure constant in the $SU(3)$ group. The coupling constants g_s and e_f denote the gauge coupling constant of f-flavor quarks interacting with chromomagnetic field and electromagnetic field, respectively.

1.5.2 EDM Interactions

The last two terms in eq.(1.44) represent the interaction terms which violate the time reversal invariance as well as the space reflection at the same time. These terms are given just for references in order to understand the T-violating interactions in future in terms of EDM (Electric Dipole Moments). That is,

$$-\frac{i}{2}\,\tilde{d}_f\bar{\psi}_f\sigma_{\mu\nu}\gamma_5 T^a\psi_f G^{\mu\nu,a}: \quad \text{EDM for chromomagnetic fields,}$$
$$-\frac{i}{2}\,d_f\bar{\psi}_f\sigma_{\mu\nu}\gamma_5\psi_f F^{\mu\nu}: \quad \text{EDM for electromagnetic fields.}$$

The coupling strengths $\tilde{d}_f$ and d_f denote the strength of the time reversal and parity violating interactions of quark with the chromomagnetic fields and the electromagnetic fields, respectively. The $\tilde{d}_f$ and d_f have the dimension of the mass inverse, and, in fact, they are related to the electric dipole moment.

The existence of the EDM interactions should be determined from experiments. If there is any finite EDM interaction observed in future experiment, it should indicate an existence of a new scale which is different from the quark masses. In this respect, the observation of the EDM interaction must be physically very interesting and important indeed.