Fundamental Problems in Quantum Field Theory

By

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In: Basic Quantum Field Theory

Foreword

The inherent purpose of a textbook is to teach the reader the basics of a topic, but ideally, also to inspire a learner to seek further knowledge. Memorizing the contents of textbooks, along with analytical thinking, is without argument an important component of higher education. For readers with some knowledge of physics, this book will make the reader think beyond the boundaries of current knowledge, about what physics is and may suggest a different view from what they have learnt from other textbooks, ”What are the differences?” ”Why?” Such serious contemplation is the first step in an intrinsic approach to physics.

To discover something truly new, it is important not only to enrich one’s own knowledge but keep an open mind and pursue answers to the Why’s, Why not’s, and discover the How’s. Sometimes such innovative thinkers may appear as if they are slow learners. They think and contemplate for a long time. It is, however, from among such people who engage in deep thought that new ideas in Science are born. Pierre Curie was a poetic physicist. A so-called dropout, refused by schools, he spent his childhood immersed in Nature yet he became a rare physicist who formulated principles on symmetry, piezo-electricity and magnetism - approaching the essence of Nature. Curie also discovered radium and found an application for radioactivity.

I believe that this is a book that will challenge the reader, and it is my hope, the reader, in being challenged, will be inspired to seek new answers in physics.

Hideaki Koizumi, Fellow and Corporate Officer, Hitachi, Ltd.

Preface

Quantum field theory has been a central subject in physics research for a long time. This is basically because the fundamental physics law is essentially written in terms of quantum field theory terminology. The Newton equation is the exception in this respect since it is not the field theory equation, but it is the equation for the coordinate of the particle object. The Newton equation can be derived from the Schrödinger equation in terms of the Ehrenfest theorem, and thus it cannot be a fundamental equation of motion. Therefore, the general relativity that aimed at generalizing the Newton equation to a relativistic equation is not a fundamental equation of motion, either.

In this respect, the physical world is described in terms of field theory terminology, and physically interesting objects must be always the field Ψ which depends on space and time. This presents a physical state of the corresponding object in nature, and the basic equation of motion can determine the behavior of the field.

This world is described by fields of photon A^μ, leptons Ψ_ℓ and quarks $\Psi_{f,c}$ which are all quantized. In addition, there are a gravitational field $\mathcal{G}$ and the weak vector bosons $W^\mu,\ Z^\mu$ which are well included into the Lagrangian density. The fields of photon, leptons, quarks and weak vector bosons should be quantized, but the quantization of the gravitational field is not yet clear from the experimental point of view since there is no discovery of the graviton until now. In this sense, it is most likely true that the gravitational field $\mathcal{G}$ should not be quantized. The Lagrangian density that governs the equation of motion for all the fields with four interactions (QED, QCD, weak and gravitational interactions) can be uniquely written, and at the present stage, there is no experiment which is in contradiction to theoretical predictions of the above fields.

The quantum field theory has an infinite number of freedom once its field is quantized, and therefore the theory cannot be solved exactly. This indicates that we should rely on the perturbation theory when we wish to calculate any physical observables. The evaluation of the perturbation theory is well established in terms of the S-matrix theory which is essentially the same as the non-static perturbation theory in non-relativistic quantum mechanics. In the course of the evaluations of the physical observables, some of the Feynman diagrams contain the infinity in the momentum integral. The treatment of the infinity is developed in terms of the renormalization scheme in QED. The basic strategy is that the infinity in the evaluation of the physical observables should be renormalized into the wave function since its infinity in the physical observables is just the same as that of the self-energy contribution. At present, physical observables calculated by the renormalization scheme are consistent with the experiment. However, any physical observables like the vertex corrections should be finite if the theoretical framework is sound, and in this sense, we still believe that they should not have any logarithmic divergences if we can treat them properly with correct propagators. Therefore, it is most probable that the renormalization scheme should meet a major modification in near future.

Here, we should notice that science is only to understand nature, in contrast to engineering which may be connected to the invention of human technology. Therefore, science is always faced to a difficulty and, in some sense, to a fear that some of the research areas should fail to keep highest activities after this research area is completely understood. In this respect, the field theory should survive at any time of research in science since it presents the fundamental technique to understand nature whatever one wishes to study.

In this textbook, we clarify the fundamental part of basic physics law which can be well understood by now. The most important of all is to understand physics in depth, which is very difficult indeed. To remember the text book knowledge is not as important as one would have thought at the beginning of his physics study. Once one can understand physics in depth, then one can apply the physics law to understanding many interesting phenomena in nature, which should be basically complicated many body problems.

In the last chapter, we discuss some problems which are not understood very well at the present stage of the renormalization scheme. Some of the open problems should be solved by experimental observations, and some are solved by modifying the theoretical considerations.

The motivation of writing this textbook is initiated by Asma Ahmed who repeatedly pushed one of the authors (TF) who was reluctant to preparing a new textbook which may well displease quite a few physicists with vested rights. As a result, we concentrated on writing this book from intensive discussions and hard works with our collaborators to achieve deeper but simpler understanding of the quantum field theory than ever.

We should be grateful to all of our collaborators, in particular, R. Abe, H. Kato, H. Kubo, Y. Munakata, S. Obata, S. Oshima, T. Sakamoto and T. Tsuda for their great contributions to this book.

Contents

Chapter 1

Maxwell and Dirac Equations

Abstract: This chapter discusses the basic equations in quantum field theory. First, we clarify some important properties of Maxwell equation so that the main part of the electromagnetisms can be easily understood. Then, we present some useful properties of the Dirac equation and its free wave solution. These two equations are the basic ingredients in understanding quantum field theory. We also give the exact energy spectrum of Dirac equation with Coulomb plus gravity potential in hydrogen-like atom

Keywords: Maxwell equation, Dirac equation, photon, oscillator of electromagnetic wave, free Dirac equation, energy eigenvalue in Coulomb and gravity

1.1 Introduction

Science is to study and understand nature, and it is always fascinating even though it is quite difficult. The physics research is intended to clarify the fundamental law of physical world. At the present stage, the dynamics of electrons and nuclei is well described by the Dirac and Maxwell equations. In addition to the electromagnetic interactions, we have now the gravitational and weak interactions which are included into the same Lagrangian density that describes the field equations of the Dirac and Maxwell fields. The Dirac equation now contains the gravitational potential in the mass term, and the weak decay processes can be just calculated in the same manner as the standard treatment of quantum field theory after the field quantization.

The success of the Dirac equation is explained in the field theory textbooks, and therefore there is no need to add anything further to the standard description. However, the real examination of the Dirac equation is only done basically for one body problem and free case, and as long as the limited range of applications of the Dirac equation are concerned, it is perfectly successful. This does not mean that the Dirac equation is all correct for everything in nature. This is clear since we cannot solve even two body problems for the Dirac equation in an exact fashion. It should be interesting to note that the full relativistic treatment of the positronium has its intrinsic difficulty and up to the present stage, there is no solid method to solve the spectrum of the positronium in a correct manner.

Nevertheless the property of the matter fields is determined by the Dirac equation if we can luckily solve the many body problems. The dynamics of fermions becomes very complicated since the motion of charged particles can generate electromagnetic fields which, in turn, should affect on the motion of fermions.

Here, we intend to clarify the basic physics law as clearly as possible, and the difficulty of the many body nature should be treated in future. The most important of all should be that the fundamental four interactions (electromagnetic, strong, weak and gravitational interactions) can be well described in terms of the Lagrangian density, and therefore all the physical law should be written by the Lagrange equations which are common to four fundamental interactions.

1.2 Maxwell Equation

The most fundamental equation in physics is the Maxwell equation. This equation is discovered by extracting physical law from experiments, and therefore the equation is basically related to describing nature itself. The Maxwell equation is written for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = e\rho, \qquad \text{(Gauss law)} \tag{1.1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \text{(No magnetic monopole)} \tag{1.1b}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0, \qquad \text{(Faraday law)} \tag{1.1c}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial\boldsymbol{E}}{\partial t} = e\boldsymbol{j}, \qquad \text{(Ampere} - \text{Maxwell law)} \tag{1.1d}$$

where ρ and $\boldsymbol{j}$ denote the charge and current densities, respectively, and we explicitly write the charge e. Here, the charge density means the density of fermions which should be later denoted as $\rho = \psi^\dagger\psi$ for one fermion state, and therefore it does not include the charge e. This is because the charge e denotes the strength of the electromagnetic interaction with fermions, and the charge of electron, for example, should be considered as a quantum number of electron state, which is -1. Therefore, if there exist n electrons in the small area V, then the charge Q of the area V becomes $Q = -en$, and the charge is measured in units of e.

The behavior of the charge density ρ and the current density $\boldsymbol{j}$ should be understood by solving the equations of motion for fermions. In this sense, it is important to realize that the Maxwell equation cannot tell us anything about the charge and current densities. In reality, the behavior of the charge and current density in the metal is very complicated, and it is mostly impossible to produce and understand the physics of the charge and current density in the metal in a proper manner. This is, of course, related to the fact that many body problems cannot be solved even for the non-relativistic equations of motion.

It may be important to note that the Maxwell equation does not contain any $\hbar$ even though it is a field theory equation of motion. However, if one considers the energy of photon, then one should introduce the $\hbar$ to express the photon energy like $\hbar\omega$. In this respect, one may say that the free photon is the result of the quantization of the vector field, and the classical field equation which is derived for the vector field in the absence of the matter fields does not prove the existence of photon. It only says that the wave equation for the vector field $\boldsymbol{A}$ indicates that it should behave like a free massless particle.

In this sense, the Maxwell equation itself does not know about the quantization of fields, and the basic theoretical reason why one should quantize the fields is one of the most important problems left for readers as a home work. There must be some fundamental principle to understand the field quantization in connection with the electromagnetic field. On the other hand, the quantization of the Dirac field should be originated from the negative energy states which should require the field quantization with the anti-commutation relation for the creation and annihilation operators within the theoretical framework.

1.2.1 Vector Potential

In order to describe the Maxwell equation in a different way, one normally introduces the vector potential $(A_0, \boldsymbol{A})$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1.2}$$

In this case, the Faraday law ($\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$) and no magnetic monopole ($\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$) can be automatically satisfied. In this case, the Maxwell equation can be written in terms of the vector potential $(A_0, \boldsymbol{A})$ as

$$\boldsymbol{\nabla}^2 A_0 = -e\rho, \quad \text{(Poisson equation)} \tag{1.3a}$$

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) \boldsymbol{A} + \frac{\partial}{\partial t}\boldsymbol{\nabla} A_0 = e\boldsymbol{j}, \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{1.3b}$$

In this expression, we take the Coulomb gauge fixing since this is simple and best.

1.2.2 Static Fields

If the field does not depend on time, then the electric field $\boldsymbol{E}$ can be written as $\boldsymbol{E} = -\boldsymbol{\nabla} A_0$ because $\frac{\partial \boldsymbol{A}}{\partial t} = 0$. By making use of the identity equation for the $\delta-$function,

$$\boldsymbol{\nabla}^{\mathbf{2}} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{1.4}$$

we can obtain the solution for the Poisson equation as

$$A_0(\boldsymbol{r}) = \frac{e}{4\pi} \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' \tag{1.5}$$

and thus we obtain the electric field by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{e}{4\pi} \int \frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d^3r'. \tag{1.6}$$

On the other hand, the Ampere law becomes

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -e\boldsymbol{j}$$

which can be easily solved, and we can obtain the solution of the above equation as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{e}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r'. \tag{1.7}$$

In this case, the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ can be expressed as

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{e}{4\pi} \int \frac{Jd\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \quad \text{with} \quad Jd\boldsymbol{r}' \equiv \boldsymbol{j}(\boldsymbol{r}')d^3r' \tag{1.8}$$

which is Biot-Savart law.

1.2.3 Free Vector Field and Its Quantization

When there exist neither charge nor current densities, that is, the vacuum state, then the Maxwell equation becomes

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) \boldsymbol{A}(t,\boldsymbol{r}) = 0 \tag{1.9}$$

which is the wave equation. However, it is clear that the vector field is a real field, and therefore there is no free field solution at the present condition for the vector field. More explicitly, the solution of the free field should be an eigenstate of the momentum operator $\boldsymbol{p} = -i\boldsymbol{\nabla}$. This means the solution of the vector field with its momentum $\boldsymbol{k}$ should have the following shape

$$\boldsymbol{A}(t,\boldsymbol{r}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}, \quad \text{or} \quad \frac{1}{\sqrt{V}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t}$$

which are, however, complex functions. Therefore, we should have another condition on the vector field if we wish to have a free field solution, corresponding to a photon state. This is indeed connected to the quantization of the vector field and we write

$$\hat{\boldsymbol{A}}(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \left[c^\dagger_{\boldsymbol{k},\lambda} e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{r}} + c_{\boldsymbol{k},\lambda} e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right] \tag{1.10}$$

where $c_{\boldsymbol{k},\lambda}$, $c^\dagger_{\boldsymbol{k},\lambda}$ denote the creation and annihilation operators, and $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. Here, $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ denotes the polarization vector which should satisfy the following condition from the Coulomb gauge fixing

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0 \tag{1.11}$$

which is the most reasonable gauge fixing condition, and up to now, it does not give rise to any problems concerning the evaluation of all the physical observables in quantum electrodynamics.

Commutation Relations

Since the gauge fields are bosons, the quantization procedure must be done in the commutation relations, instead of anti-commutation relations. Therefore, the quantization can be done by requiring that $c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k},\lambda}$ should satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} \tag{1.12}$$

and all other commutation relations vanish.

1.2.4 Photon

For this quantized vector field, we can define one-photon state with $(\boldsymbol{k}, \lambda)$, and it can be written as

$$\boldsymbol{A}_{\boldsymbol{k},\lambda}(x) = \langle \boldsymbol{k}, \lambda|\hat{\boldsymbol{A}}(x)|0\rangle = \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega_{\boldsymbol{k}}t} \tag{1.13}$$

which is indeed the eigenstate of the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$. Here, we see that photon is the result of the field quantization. In this respect, photon cannot survive in the classical field theory of the Maxwell equation even though the wave equation suggests that there must be some wave that can propagate like a free particle. Indeed, eq.(1.9) indicates that there should be a free wave solution. However, the vector potential $\boldsymbol{A}$ itself is a real field, and therefore it cannot behave like a free particle which should be a complex function ($e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$). In this respect, the existence of photon should be understood only after the vector field $\boldsymbol{A}$ is quantized. After the field quantization, the energy of photon is measured in units of $\hbar$, that is,

$$E_{\text{photon}} = \hbar\omega. \tag{1.14}$$

The fact that the Maxwell equation does not contain any $\hbar$ may be a good reason why it could not lead us to the concept of the first quantization even though it is indeed a field theory equation.

1.2.5 Field Energy of Photon

The energy of the gauge field can be calculated from the energy momentum tensor $\mathcal{T}^{\mu\nu}$ of the electromagnetic fields and it becomes

$$H_0 = \int \mathcal{T}^{00} d^3r = \frac{1}{2}\int\left[\left(\frac{\partial \boldsymbol{A}}{\partial t}\right)^2 + (\boldsymbol{\nabla}\times\boldsymbol{A})^2\right]d^3r = \sum_{\boldsymbol{k},\lambda}\omega_{\boldsymbol{k}}\left(c^\dagger_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right). \tag{1.15}$$

This represents the energy of photons, and it is written in terms of the field quantized expression.

1.2.6 Static Field Energy per Time

The energy increase per second can be written as

$$W_0 = e \int \boldsymbol{j} \cdot \boldsymbol{E} d^3 r. \tag{1.16}$$

This equation can be rewritten by making use of the Maxwell equation as

$$W_0 = -\frac{d}{dt} \int \left(\frac{1}{2} |\boldsymbol{B}|^2 + \frac{1}{2} |\boldsymbol{E}|^2 \right) d^3 r - \int \boldsymbol{\nabla} \cdot \boldsymbol{S} d^3 r \tag{1.17}$$

where the Poynting vector $\boldsymbol{S}$ is defined as

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}. \tag{1.18}$$

This first term in this equation corresponds to the normal field energy increase of the static fields $\boldsymbol{E}$ and $\boldsymbol{B}$. The second term is the energy flow from the Poynting vector, but we should note that the energy should flow into the inner part of the system and should be accumulated into the condenser thorough the Poynting vector. But it never flows out into the air. That means that the emission of photons has nothing to do with the Poynting vector. This is, of course, clear since the emission of photon should be only possible through electrons (fermions) as we see below.

1.2.7 Oscillator of Electromagnetic Wave

Photon can be emitted from the oscillator when the electromagnetic field is oscillating. A question is as to how it can emit photons. Now the electromagnetic interaction H_I with electrons can be written as

$$H_I = -e \int \boldsymbol{j} \cdot \boldsymbol{A} d^3 r \tag{1.19}$$

and thus we should start from this expression. The interaction energy increase per time can be written as

$$W \equiv \frac{dH_I}{dt} = -e \int \left[\frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} + \boldsymbol{j} \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right] d^3 r \tag{1.20}$$

where we consider the case without A_0 term, and thus the electric field can be written as

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}. \tag{1.21}$$

Thus, W becomes

$$W = -e \int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} d^3 r + e \int \boldsymbol{j} \cdot \boldsymbol{E} d^3 r. \tag{1.22}$$

¿From the above equation, we see that the second term is just W_0, and thus there is no need to discuss it further. Therefore, defining the first term by W_1, we obtain

$$W_1 = -e \int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} d^3 r = -\frac{e}{m} \int \left\{ \frac{\partial}{\partial t} (\psi^\dagger \hat{\boldsymbol{p}} \psi) \right\} \cdot \boldsymbol{A} d^3 r \tag{1.23}$$

where we take the non-relativistic current $\boldsymbol{j}$ as

$$\boldsymbol{j} = \frac{1}{m}\psi^\dagger \hat{\boldsymbol{p}}\psi, \quad \text{with} \quad \hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}. \tag{1.24}$$

Since the Zeeman Hamiltonian H_Z is written as

$$H_Z = -\frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0 \tag{1.25}$$

we can evaluate the current variation with respect to time as

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{1}{m}\left[\frac{\partial \psi^\dagger}{\partial t}\hat{\boldsymbol{p}}\psi + \psi^\dagger \hat{\boldsymbol{p}}\frac{\partial \psi}{\partial t}\right] = \frac{e}{2m^2}\boldsymbol{\nabla} B_0(\boldsymbol{r}) \tag{1.26}$$

where we assume that the $\boldsymbol{B}_0$ is in the $z-$direction $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. Thus, we find

$$W_1 = -\frac{e^2}{2m^2}\int (\boldsymbol{\nabla} B_0(\boldsymbol{r}))\cdot \boldsymbol{A} d^3r \tag{1.27}$$

where we note that $\boldsymbol{A}$ is associated with current electrons while $\boldsymbol{B}_0$ is an external magnetic field. This is the basic mechanism for the production of the electromagnetic waves (low energy photons) through the oscillators. This clearly shows that the electromagnetic wave can be produced only when there are, at least, two coils where one coil should produce the change of the magnetic field which can affect on electrons in another coil.

In most of the textbooks in electromagnetism, the description of the photon emission is insufficient, and we should be very careful for the photon emission processes.

1.3 Dirac Equations

The fundamental equation for fermions is the Dirac equation which can describe the energy spectrum of the hydrogen atom to a very high accuracy. The Dirac equation can naturally describe the spin part of the wave function and this is essentially connected to the relativistic wave equations. In addition to the spin degree of freedom, the Dirac equation contains the negative energy states which are quite new to the non-relativistic wave equations. The existence of the negative energy states requires the Pauli principle which enables us to build the vacuum state, and it should be defined as the state in which all the negative energy states are filled. In this case, this vacuum state becomes stable since no particle can be decayed into the vacuum state due to the Pauli principle.

It should be noted that the Pauli principle can be derived if we ask the quantization of the Dirac field in terms of the anti-commutation relations. In this respect, the quantization of the Dirac field is essential because of the Pauli principle, and the field quantization is basically necessary within the theoretical framework.

1.3.1 Free Field Solutions

The Dirac equation for free fermion with its mass m is written as

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta \right)\psi(\boldsymbol{r},t) = 0 \tag{1.28}$$

where ψ has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{1.29}$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix.

The derivation of the Dirac equation and its application to hydrogen atom can be found in the standard textbooks. One can learn from the procedure of deriving the Dirac equation that the number of components of the electron fields is important, and it is properly obtained in the Dirac equation. That is, among the four components of the field ψ, two degrees of freedom should correspond to the positive and negative energy solutions and another two degrees should correspond to the spin with $s = \frac{1}{2}$. It is also important to note that the factorization procedure indicates that the four component spinor is the minimum number of fields which can take into account the negative energy degree of freedom in a proper way.

Eq.(1.28) can be rewritten in terms of the wave function components by multiplying β from the left hand side

$$(i\partial_\mu\gamma^\mu - m)_{ij}\psi_j = 0 \quad \text{for} \quad i = 1,2,3,4 \tag{1.30}$$

where the repeated indices of j indicate the summation of $j = 1,2,3,4$. Here, gamma matrices

$$\gamma^\mu = (\gamma_0, \boldsymbol{\gamma}) \equiv (\beta, \beta\boldsymbol{\alpha})$$

are introduced, and the repeated indices of Greek letters μ indicate the summation of $\mu = 0,1,2,3$. The expression of eq.(1.30) is called *covariant* since its Lorentz invariance is manifest. It is indeed written in terms of the Lorentz scalars, but, of course there is no deep physical meaning in covariance.

Lagrangian Density for Free Dirac Fields

The Lagrangian density for free Dirac fermions can be constructed as

$$\mathcal{L} = \psi_i^\dagger[\gamma_0(i\partial_\mu\gamma^\mu - m)]_{ij}\psi_j = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi \tag{1.31}$$

where $\bar{\psi}$ is defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0.$$

This Lagrangian density is just constructed so as to reproduce the Dirac equation of (1.30) from the Lagrange equation. It should be important to realize that the Lagrangian density of eq.(1.31) is invariant under the Lorentz transformation since it is a Lorentz scalar. This is clear since the Lagrangian density should not depend on the system one chooses.

Lagrange Equation for Free Dirac Fields

The Lagrange equation for $\psi_i^\dagger$ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \equiv \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} \tag{1.32}$$

and one can easily calculate the following equations

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} = 0, \qquad \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} = [\gamma_0(i\partial_\mu \gamma^\mu - m)]_{ij} \psi_j$$

and thus, this leads to the following equation

$$[\gamma_0(i\partial_\mu \gamma^\mu - m)]_{ij} \psi_j = 0 \tag{1.33}$$

which is just the free Dirac equation. Here, it should be noted that the ψ_i and $\psi_i^\dagger$ are independent functional variables, and the functional derivative with respect to ψ_i or $\psi_i^\dagger$ gives the same equation of motion.

Plane Wave Solutions of Free Dirac Equation

The free Dirac equation of eq.(1.33) can be solved exactly, and it has plane wave solutions. A simple way to solve eq.(1.33) can be shown as follows. First, one writes the wave function ψ in the following shape

$$\psi_s(\boldsymbol{r}, t) = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{1.34}$$

where φ and ϕ are two component spinors

$$\varphi = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \qquad \phi = \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}. \tag{1.35}$$

In this case, eq.(1.33) becomes

$$\begin{pmatrix} m-E & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m-E \end{pmatrix}\begin{pmatrix}\varphi \\ \phi\end{pmatrix} = 0 \tag{1.36}$$

which leads to

$$E^2 = m^2 + \boldsymbol{p}^2. \tag{1.37}$$

This equation has the following two solutions.

Positive Energy Solution ($E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(+)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} u_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{1.38a}$$

$$u_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{p}}+m}{2E_{\boldsymbol{p}}}}\begin{pmatrix}\chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_{\boldsymbol{p}}+m}\chi_s\end{pmatrix}, \quad \text{with} \quad s = \pm\frac{1}{2} \tag{1.38b}$$

where χ_s denotes the spin wave function and is written as

$$\chi_{\frac{1}{2}} = \begin{pmatrix}1 \\ 0\end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix}0 \\ 1\end{pmatrix}.$$

Negative Energy Solution ($E_{\boldsymbol{p}} = -\sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(-)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} v_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{1.39a}$$

$$v_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{|E_{\boldsymbol{p}}|+m}{2|E_{\boldsymbol{p}}|}}\begin{pmatrix}-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E_{\boldsymbol{p}}|+m}\chi_s \\ \chi_s\end{pmatrix}. \tag{1.39b}$$

Some Properties of Spinor

The spinor wave function $u_{\boldsymbol{p}}^{(s)}$ and $v_{\boldsymbol{p}}^{(s)}$ are normalized according to

$$u_{\boldsymbol{p}}^{(s)\dagger} u_{\boldsymbol{p}}^{(s)} = 1 \tag{1.40a}$$

$$v_{\boldsymbol{p}}^{(s)\dagger} v_{\boldsymbol{p}}^{(s)} = 1. \tag{1.40b}$$

Further, they satisfy the following equations when the spin is summed over

$$\sum_{s=1}^{2} u_{\boldsymbol{p}}^{(s)} \bar{u}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}} \tag{1.41a}$$

$$\sum_{s=1}^{2} v_{\boldsymbol{p}}^{(s)} \bar{v}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}. \tag{1.41b}$$

1.3.2 Quantization of Dirac Fields

Here, we discuss the quantization of free Dirac fields and write the free Dirac field as

$$\psi(\boldsymbol{r},t) = \sum_{\boldsymbol{n},s} \frac{1}{\sqrt{L^3}} \left(a_{\boldsymbol{n}}^{(s)} u_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p}_{\boldsymbol{n}}\cdot\boldsymbol{r} - iE_{\boldsymbol{n}}t} + b_{\boldsymbol{n}}^{(s)} v_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p}_{\boldsymbol{n}}\cdot\boldsymbol{r} + iE_{\boldsymbol{n}}t} \right), \tag{1.42}$$

where $u_{\boldsymbol{n}}^{(s)}$ and $v_{\boldsymbol{n}}^{(s)}$ denote the spinor part of the plane wave solutions as given in eqs.(1.38). Here, the basic method to quantize the fields is to require that the annihilation and creation operators $a_{\boldsymbol{n}}^{(s)}$ and $a^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for positive energy states and $b_{\boldsymbol{n}}^{(s)}$ and $b^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for negative energy states become operators which should satisfy the anti-commutation relations.

Anti-commutation Relations

The creation and annihilation operators for positive and negative energy states should satisfy the following anti-commutation relations,

$$\left\{ a_{\boldsymbol{n}}^{(s)}, a^{\dagger}{}_{\boldsymbol{n}'}^{(s')} \right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}, \quad \left\{ b_{\boldsymbol{n}}^{(s)}, b^{\dagger}{}_{\boldsymbol{n}'}^{(s')} \right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}. \tag{1.43a}$$

All the other cases of the anti-commutations vanish, for examples,

$$\left\{ a_{\boldsymbol{n}}^{(s)}, a_{\boldsymbol{n}'}^{(s')} \right\} = 0, \quad \left\{ b_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')} \right\} = 0, \quad \left\{ a_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')} \right\} = 0. \tag{1.43b}$$

1.3.3 Quantization in Box with Periodic Boundary Conditions

In field theory, one often puts the theory into the box with its volume $V = L^3$ and requires that the wave function should satisfy the periodic boundary conditions (PBC). This is mainly because the free field solutions are taken as the basis states, and in this case, one can only calculate physical observables if one works in the box. It is clear that the free field can be defined well only if it is confined in the box. Since the wave function $\psi_s(\boldsymbol{r},t)$ for a free particle in the box should be proportional to

$$\psi_s(\boldsymbol{r},t) \simeq \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}}$$

the PBC equations become

$$e^{ip_x x} = e^{ip_x(x+L)}, \quad e^{ip_y y} = e^{ip_y(y+L)}, \quad e^{ip_z z} = e^{ip_z(z+L)}.$$

Therefore, one obtains the constraints on the momentum p_k as

$$p_x = \frac{2\pi}{L} n_x, \quad p_y = \frac{2\pi}{L} n_y, \quad p_z = \frac{2\pi}{L} n_z, \qquad n_k = 0, \pm 1, \pm 2, \cdots . \tag{1.44}$$

In this case, the number of states N in the large L limit becomes

$$N = \sum_{n_x,n_y,n_z} \sum_{s} = 2\frac{L^3}{(2\pi)^3} \int d^3p \tag{1.45}$$

where a factor of two comes from the spin degree of freedom.

At this point, we should make a comment on the validity of the periodic boundary conditions. After we solve the Schrödinger equation, we should impose some boundary conditions on the wave function. Normally, one puts the condition that the wave function should vanish at infinity when solving the bound state problems, and this can determine the energy eigenvalues of the Hamiltonian. On the other hand, the plane wave solutions as given in eq.(1.42) cannot satisfy this type of boundary condition that the wave function should be zero at infinity. Nevertheless, we want to confine the waves within the box, and the only possible boundary condition is the periodic boundary conditions. Up to now, there are no physical observables which are in contradiction with this condition of PBC. The important requirement is that any physical observables should not depend on the box length L if it is sufficiently large, which is called the thermodynamic limit.

1.3.4 Hamiltonian Density for Free Dirac Fermion

The Hamiltonian density for free fermion can be constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \partial^\nu \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \partial^\nu \psi_i^\dagger \right) - \mathcal{L} g^{\mu\nu}. \tag{1.46}$$

Hamiltonian Density from Energy Momentum Tensor

Now, one defines the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i)} \partial_0 \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} \partial_0 \psi_i^\dagger \right) - \mathcal{L}. \tag{1.47}$$

Since the Lagrangian density of free fermion is given in eq.(1.31) and is rewritten as

$$\mathcal{L} = i\psi_i^\dagger \partial_0 \psi_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j. \tag{1.48}$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = \mathcal{T}^{00} = \bar{\psi}_i \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \tag{1.49}$$

Hamiltonian for Free Dirac Fermion

The Hamiltonian for free fermion fields is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H} d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi d^3r. \tag{1.50}$$

As we discussed in the Schrödinger field, the Hamiltonian itself cannot give us many information on the dynamics. One can learn some properties of the system described by the Hamiltonian, but one cannot obtain any dynamical information of the system from the

Hamiltonian. In order to calculate the dynamics of the system in the classical field theory model, one has to solve the equation of motions which are obtained from the Lagrange equations for fields.

When one wishes to consider the quantum effects of the fields or, in other words, creations of particles and anti-particles, then one should quantize the fields. In this case, the Hamiltonian becomes an operator. Therefore, one has to prepare the Fock states on which the Hamiltonian can operate. Most of the difficulties of the field theory models should be to find the correct vacuum of the interacting system. In four dimensional field theory models, only the free field theory can be solved exactly, and therefore we are all based on the perturbation theory to obtain physical observables.

1.3.5 Fermion Current and its Conservation Law

Dirac equation has a very important equation of current conservation. This is, in fact, related to the global gauge symmetry which should be always satisfied in Dirac as well as Schrödinger equations. If the Lagrangian density should have the following shape

$$\mathcal{L} = F(\psi^\dagger \psi)$$

then it is invariant under the global gauge transformation of

$$\psi' = e^{i\alpha}\psi.$$

In this case, if one defines the Noether current j^μ as

$$j^\mu \equiv -i\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\psi^\dagger\right] \tag{1.51}$$

then one has the conservation of current

$$\partial_\mu j^\mu = 0. \tag{1.52}$$

For Dirac fields, one can obtain as a conserved current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{1.53}$$

while the conserved current $j^\mu = (\rho, \boldsymbol{j})$ for the Schrödinger field is written as

$$\rho = \psi^\dagger\psi, \qquad \boldsymbol{j} = \frac{1}{2im}\left(\psi^\dagger \boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^\dagger)\psi\right) \tag{1.54}$$

1.3.6 Dirac Equation for Coulomb Potential

For a hydrogen-like atomic system, one can write the Dirac equation as

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta + \frac{Ze^2}{r}\right)\psi(\boldsymbol{r}, t) = 0 \tag{1.55}$$

where ψ has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{1.56}$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix. In this case, one can easily prove that the quantities that can commute with the Dirac Hamiltonian must be $\boldsymbol{J}$ and K as defined below

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{s}, \qquad K = \beta(2\boldsymbol{s} \cdot \boldsymbol{L} + 1) \tag{1.57}$$

where $\boldsymbol{L}$ and $\boldsymbol{s}$ are defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \hat{\boldsymbol{p}}, \qquad \boldsymbol{s} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}.$$

Therefore, the energy eigenvalue of the Dirac field can be specified by the quantum numbers of $J,\ J_z,\ K$.

Energy Eigenvalue with Coulomb in Hydrogen-like Atom

The energy eigenvalue of the Dirac equation can be obtained for the hydrogen-like atomic system. The Dirac equation can be written as

$$\left(-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + m_e\beta - \frac{Ze^2}{r}\right)\psi(\boldsymbol{r}, t) = E\psi(\boldsymbol{r}, t) \tag{1.58}$$

where m_e denotes the electron mass. This can be solved exactly, and the energy eigenvalue is given as

$$E_{n,j} = m_e \left[1 - \frac{(Z\alpha)^2}{n^2 + 2\left(n - (j + \frac{1}{2})\right)\left[\sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2} - (j + \frac{1}{2})\right]}\right]^{\frac{1}{2}} \tag{1.59}$$

where α denotes the fine structure constant with $\alpha = \frac{1}{137}$. The quantum number n runs as $n = 1, 2, \ldots$. The energy $E_{n,j}$ can be expanded up to the order α^4 as

$$E_{n,j} - m_e = -\frac{m_e(Z\alpha)^2}{2n^2} - \frac{m_e(Z\alpha)^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \mathcal{O}\left((Z\alpha)^6\right). \tag{1.60}$$

The first term in the energy eigenvalue is the familiar energy spectrum of the hydrogen-like atom in the non-relativistic quantum mechanics.

It should be noted that this result is mathematically exact, but the Dirac equation eq.(1.58) itself is simply obtained within one body problem, and it is, of course, an approximation. A question may arise as to how much the reduction of the one body problem can be justified. Namely, the hydrogen atom should be, at least, a two body problem since it involves electron and proton in the hydrogen atom. In fact, the relativistic two body Dirac equation cannot be solved or cannot be reduced to one body problem in a proper manner. The Dirac equation of eq.(1.58) is indeed one body equation, but the mass m should be replaced by the reduced mass, and this is indeed a very artificial procedure.

In reality, it may well be even more complicated than the tow body problems, and once the fields are quantized, then the hydrogen atom should become many body problems. This means that one electron state could be mixed up by the two electron-one positron states in the electron wave function after the field quantization. At present, however, there is no reliable calculation with this additional configuration, and therefore we do not know how large these contributions to the energy should be for the hydrogen atom.

1.3.7 Dirac Equation for Coulomb and Gravity Potential

Even when one considers the hydrogen-like atom, there is a gravitational interaction between electron and proton. Here, we write a full Dirac equation in the hydrogen-like atom when the gravitational interaction is included

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m_e-\frac{Gm_eM_pZ}{r}\right)\beta-\frac{Ze^2}{r}\right]\Psi=E\Psi \tag{1.61}$$

where M_p and G denote the proton mass and the gravitational constant, respectively. The gravity is too weak to make any influence on the spectrum in the hydrogen-like atom, but theoretically it should be important that all the interactions in the hydrogen-like atom are now included in the Dirac equation.

Energy Eigenvalue with Coulomb and Gravity in Hydrogen-like Atom

The equation (1.61) can be solved exactly, and we obtain

$$\frac{E}{m_e}=\frac{-Z^2\alpha c'+(\gamma+n_r)\sqrt{(Z\alpha)^2-(Zc')^2+(\gamma+n_r)^2}}{(Z\alpha)^2+(\gamma+n_r)^2} \tag{1.62}$$

where

$$c'\equiv GM_pm_e,\quad \gamma\equiv\sqrt{\kappa^2-(Z\alpha)^2+(Zc')^2}$$

and

$$\kappa\equiv\mp\left(j+\frac{1}{2}\right)\qquad\text{for}\qquad \begin{array}{l} j=l+\frac{1}{2}\\ j=l-\frac{1}{2}.\end{array}$$

When we solve the equation, we see that the allowed region of Z is changed as

$$Z\ <\ \frac{1+\sqrt{1+2(GM_pm_e)^2}}{2\alpha}\ .$$

Therefore, the energy eigenvalue is rewritten as

$$E_{n_r,\,j} = m_e \left[\frac{-Z^2\alpha c' + (\gamma + n_r)\sqrt{(Z\alpha)^2 - (Zc')^2 + (\gamma + n_r)^2}}{(Z\alpha)^2 + (\gamma + n_r)^2} \right] \tag{1.63}$$

where

$$n_r = -\gamma + \frac{Z\alpha E + Zc' m_e}{\lambda}$$
$$\lambda = \sqrt{m_e^2 - E^2}$$

and

$$n_r = \begin{cases} 0, 1, 2, \cdots & \kappa < 0 \\ \quad 1, 2, 3, \cdots & \kappa > 0. \end{cases}$$

n_r is related to the principal quantum number n as

$$n_r + |\kappa| = n \quad (n = 1, 2, 3, \cdots).$$

If we expand it in terms of $Zc' (= ZGM_p m_e)$, then the energy eigenvalue becomes

$$E_{n_r,\,j} \simeq m_e \left[1 + \left(\frac{Z\alpha}{n_r + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}} \right)^2 \right]^{-\frac{1}{2}}$$
$$- \frac{Z^2\alpha \cdot GM_p m_e^2}{\left(n_r + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2} \right)^2 + (Z\alpha)^2} + \mathcal{O}((GM_p Z m_e)^2). \tag{1.64}$$

The frst term is just the well known Dirac's eigenvalue of hydrogen like atom. The second term corresponds to the correction of gravitational effects. As an example, we consider $1s_{\frac{1}{2}}$ state in hydrogen atom $(Z = 1)$, and the correction of gravitational effects becomes

$$E_{gr.} = -\frac{\alpha G M_p m_e^2}{(0 + \sqrt{1-\alpha^2})^2 + \alpha^2} = -\alpha G m_e^2 M_p \simeq -1.2 \times 10^{-38} \ \text{eV} \tag{1.65}$$

which is too small to observe, but it is finite. In addition, we obtain the eigenfunction [1]

$$\left.\begin{matrix} f \\ g \end{matrix}\right\} = \frac{\pm(2\lambda)^{\frac{3}{2}}}{\Gamma(2\gamma+1)} \left[\frac{(m_e \pm E)\Gamma(2\gamma + n_r + 1)}{4m_e \frac{(Z\alpha m_e + Zc'E)}{\lambda} \left(\frac{Z\alpha m_e + Zc'E}{\lambda} - \kappa \right) n_r!} \right]^{\frac{1}{2}} (2\lambda r)^{\gamma - 1} e^{-\lambda r}$$
$$\times \left\{ \left(\frac{Z\alpha m_e + Zc'E}{\lambda} - \kappa \right) {}_1F_1(-n_r, 2\gamma+1; 2\lambda r) \mp n_r {}_1F_1(1-n_r, 2\gamma+1; 2\lambda r) \right\} \tag{1.66}$$

where ${}_1F_1(\alpha, \beta; z)$ denotes the hypergeometric function. Also, f and g are radial wave functions, and therefore, the total wave function becomes

$$\Psi(r, \theta, \phi) = \begin{pmatrix} f(r)\Omega_{j,l,m}(\theta, \phi) \\ (-)^{\frac{1+l-l'}{2}} g(r)\Omega_{j,l',m}(\theta, \phi) \end{pmatrix} \tag{1.67}$$

where

$$l = j \pm \frac{1}{2}$$

$$l' = 2j - l$$

$$\Omega_{j,l',m}(\theta,\phi) = i^{l-l'}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}\right)\Omega_{j,l,m}(\theta,\phi)$$

and

$$\Omega_{l+\frac{1}{2},l,m}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}Y_l^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j-m}{2j}}Y_l^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix}$$

$$\Omega_{l-\frac{1}{2},l,m}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{j-m+1}{2j+2}}Y_l^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j+m+1}{2j+2}}Y_l^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix}.$$

Classical Limits

As we see in the later chapter, the gravitational force becomes important when we discuss the motion of the planets in the Newton equation. When we make the non-relativistic reduction of the Dirac Hamiltonian, then we find

$$H = \frac{\boldsymbol{p}^2}{2m_e} - \frac{e^2}{r} - \frac{Gm_eM_p}{r} + \frac{GM_p}{2m_er}\boldsymbol{p}^2. \tag{1.68}$$

Now, we make the classical limit of the Hamiltonian and obtain a new potential for the Newton equation with an additional gravitational potential

$$V(r) = -\frac{e^2}{r} - \frac{Gm_eM_p}{r} + \frac{1}{2m_ec^2}\left(\frac{Gm_eM_p}{r}\right)^2. \tag{1.69}$$

If the new potential is applied to the motion of the planets, then this additional gravitational potential turns out to be responsible for the description of the observed advance shifts of the Mercury perihelion, the GPS satellite motion and the earth rotation around the sun.

Conflict of Interest
The author(s) confirm that this article content has no conflicts interest.

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