Chapter 2

S-Matrix Theory

Abstract: In this chapter, we discuss the S-matrix theory in quantum field theory. Here, we first treat the non-relativistic scattering theory and its relation to the T-matrix. In particular, we discuss the scattering problem in terms of the Lippmann-Schwinger equation. Then we discuss the S-matrix theory in quantum field theory. This is based on the perturbation theory and we present the example of the S-matrix evaluation. In particular, we discuss some basic problems in the Feynman propagator of photon and show a possible physical difference between Feynman and correct propagators of photon.

Keywords: S-matrix, T-matrix, scattering amplitude, Lippmann-Schwinger equation, time-dependent perturbation theory, Born approximation, cross section, Feynman and correct propagator of photon

2.1 Introduction

All of the evaluation in quantum field theory in four dimensions should be based on the perturbation theory. This is simply due to the fact that there is no case in which the quantum field theory models in four dimensions can be solved exactly, except a free field theory. In this sense, we should understand how we can obtain physical information from experiments. In most of the cases, the experimental information can be obtained by scattering processes in which the incident particles collide with targets. In this reaction process, people measure outgoing particles at fixed solid angles. Mostly they extract the differential cross section as the important information on the structure of the targets or on the interaction between incident particles and target matters. Here, we describe the basic theoretical framework in the S-matrix theory. The treatment should be first based on the non-relativistic quantum mechanics, and then we discuss the S-matrix expansion in the quantum field theory. It should be noted here that the difference of the theoretical frameworks between non-relativistic quantum mechanics and relativistic field theory is not very large as will be seen below. In

particular, the time development of the system is most important, and this property is just the same between relativistic and non-relativistic wave equations. The only but basic difference is the kinematics in the scattering process, and this is, in general, not very significant.

2.2 Time Dependent Perturbation Theory and T-matrix

In four dimensional field theory models, we should rely on the perturbation theory. This is, of course, due to the fact that there is no model field theory which can be solved exactly, except the free field theory. In this case, the strategy of the theoretical calculation is always based on the perturbation theory in which all the physical observables can be described in terms of the free field terminology, that is, electron and photon in the case of quantum electrodynamics.

2.2.1 Non-static Perturbation Expansion

Now we consider the system in which the total Hamiltonian can be written as the sum of H_0 and H_I

$$H = H_0 + H_I \tag{2.1}$$

where it is assumed that all the solutions of the Hamiltonian H_0 are known and also H_I is relatively small. In this case, the eigenfunctions and eigenvalues of the Hamiltonian H_0 can be written as

$$H_0 u_n(\boldsymbol{r}) = E_n u_n(\boldsymbol{r}), \quad n = 1, 2, \cdots . \tag{2.2}$$

The eigenfunction $u_n(\boldsymbol{r})$ should satisfy the following orthogonalty and completeness conditions

$$\langle u_n | u_m \rangle = \delta_{nm}, \quad \sum_n |u_n\rangle\langle u_n| = 1. \tag{2.3}$$

The state vector $\Psi(t, \boldsymbol{r})$ can be expanded in terms of the wave function $u_n(\boldsymbol{r})$ as

$$\Psi(t, \boldsymbol{r}) = \sum_n a_n(t) e^{-iE_n t} u_n(\boldsymbol{r}). \tag{2.4}$$

Now we insert this state vector into the Schrödinger equation

$$i\frac{\partial \Psi(t, \boldsymbol{r})}{\partial t} = (H_0 + H_I)\Psi(t, \boldsymbol{r}) \tag{2.5}$$

and we obtain

$$\sum_n \left[i\frac{da_n(t)}{dt} + E_n a_n(t) \right] e^{-iE_n t} u_n(\boldsymbol{r}) = \sum_n (E_n + H_I) a_n(t) e^{-iE_n t} u_n(\boldsymbol{r}). \tag{2.6}$$

By operating $u_k^\dagger(\boldsymbol{r})$ from the left and integrating over the three dimensional space, we obtain

$$\frac{da_k(t)}{dt} = -i\sum_n e^{i(E_k-E_n)t}\langle u_k|H_I|u_n\rangle a_n(t). \tag{2.7}$$

This integral equation can be formally solved, and we obtain

$$a_k(t) = a_k(0) - i\sum_n \int_0^t dt' e^{i(E_k-E_n)t'}\langle u_k|H_I|u_n\rangle a_n(t') \tag{2.8}$$

which is just the equation for the S-matrix evaluation. This is, of course, the same as the relativistic treatment which has the same time dependence as the Schrödinger equation.

2.2.2 T-matrix in Non-relativistic Potential Scattering

Here we consider the scattering process in the potential model where the Hamiltonian is given as

$$H = H_0 + V, \qquad \text{with} \qquad H_0 = \frac{\hat{\boldsymbol{p}}^2}{2m} = -\frac{1}{2m}\boldsymbol{\nabla}^2. \tag{2.9}$$

The main aim is to calculate the scattering T-matrix in the potential scattering process. For the scattering problem, it is easier if we start from the Lippmann-Schwinger equation

$$\psi = \varphi + \frac{1}{E - H_0 + i\varepsilon}V\psi \tag{2.10}$$

where $+i\varepsilon$ is introduced because of the boundary condition that we take out only the outgoing wave. This is very important, and the Lippmann-Schwinger equation has already the proper boundary condition in itself. Here, φ denotes the state vector which can satisfy the following free wave equation

$$(E - H_0)\varphi = 0 \tag{2.11}$$

and we normally take the plane wave solution with the incident energy of $E = \frac{\boldsymbol{k}^2}{2m}$

$$\varphi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{2.12}$$

where the normalization constant $\frac{1}{\sqrt{V}}$ is set to unity. Here, it may be worthwhile noting and making a comment on this normalization constant. Since we set the normalization constant to unity, the wave function does not have a proper dimension, and thus the T-matrix as well. However, this is all right since all the physical observables in scattering processes should be given as the ratio between outgoing flux divided by the incident flux.

Now the Lippmann-Schwinger equation can be rewritten more explicitly

$$\psi_k(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \int d^3r' G(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi_k(\boldsymbol{r}') \tag{2.13}$$

where the Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ is defined as

$$G(\boldsymbol{r}, \boldsymbol{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \frac{1}{\frac{k^2}{2m} - \frac{p^2}{2m} + i\varepsilon} = -\frac{m}{2\pi} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{2.14}$$

At large r compared to the potential range, we find

$$G(\boldsymbol{r}, \boldsymbol{r}') \simeq -\frac{m}{2\pi r} e^{ikr} e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'} \quad \text{with} \quad \boldsymbol{k}' \equiv k\hat{\boldsymbol{r}}. \tag{2.15}$$

Since the scattering amplitude $f(\boldsymbol{k}', \boldsymbol{k})$ is defined by the following equation

$$\psi_k(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\boldsymbol{k}', \boldsymbol{k}) \frac{e^{ikr}}{r} \tag{2.16}$$

the $f(\boldsymbol{k}', \boldsymbol{k})$ can be given as

$$f(\boldsymbol{k}', \boldsymbol{k}) = -\frac{m}{2\pi} \int d^3r' e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'} V(\boldsymbol{r}') \psi_k(\boldsymbol{r}'). \tag{2.17}$$

In this case, the probability P of finding particle at the area of $r^2 d\Omega$ should be

$$P d\Omega = \frac{j_r r^2 d\Omega}{j_{in}}. \tag{2.18}$$

Since the current density in the non-relativistic case is given in eq.(1.54), we find

$$j_{in} = \frac{k}{m}, \qquad j_r = \frac{k|f(\boldsymbol{k}', \boldsymbol{k})|^2}{mr^2}. \tag{2.19}$$

Therefore, we obtain the differential cross section $\frac{d\sigma}{d\Omega} = P$ as

$$\frac{d\sigma}{d\Omega} = |f(\boldsymbol{k}', \boldsymbol{k})|^2. \tag{2.20}$$

Here, we define the scattering T-matrix by

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle \equiv \langle \boldsymbol{k}'|V|\psi_k\rangle \tag{2.21a}$$

which is related to the scattering amplitude as

$$f(\boldsymbol{k}', \boldsymbol{k}) = -\frac{m}{2\pi} \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle. \tag{2.21b}$$

This T-matrix can, of course, satisfy the Lippmann-Schwinger type equation which can be written as

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle + \int \frac{d^3k''}{(2\pi)^3} \langle \boldsymbol{k}'|V|\boldsymbol{k}''\rangle \langle \boldsymbol{k}''| \frac{1}{E - H_0 + i\varepsilon} |\boldsymbol{k}''\rangle \langle \boldsymbol{k}''|T|\boldsymbol{k}\rangle. \tag{2.22}$$

It may also be important to note that the scattering amplitude as well as the T-matrix are not yet determined completely, and if one can solve the above equation exactly, then one can determine all the information on the scattering T-matrix.

2.2.3 Born Approximation

In the normal case of the potential scattering, one cannot easily solve the Lippmann-Schwinger equation, and therefore it is important to make a reasonably good approximation. The best known approximation method is the Born approximation in which we obtain the T-matrix by replacing the exact state vector $\psi_k(\boldsymbol{r}')$ by the plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}'}$. In this case, the scattering amplitude $f(\boldsymbol{k}', \boldsymbol{k})$ becomes

$$f_B(\boldsymbol{k}', \boldsymbol{k}) = -\frac{m}{2\pi}\int d^3r' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}'}V(\boldsymbol{r}'). \tag{2.23}$$

Correspondingly, we can obtain the T-matrix with the Born approximation as

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle_B = \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle. \tag{2.24}$$

Rutherford Scattering

If the potential is $V(r) = \frac{\alpha}{r}$ (Coulomb scattering), then the scattering amplitude can be easily calculated in the Born approximation, and one finds

$$f_B(\boldsymbol{k}', \boldsymbol{k}) = -\frac{m}{2\pi}\int d^3r' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}'}\frac{\alpha}{r'} = -\frac{2m\alpha}{|\boldsymbol{k}-\boldsymbol{k}'|^2}. \tag{2.25}$$

Therefore the differential cross section of the Rutherford scattering becomes

$$\frac{d\sigma}{d\Omega} = |f_B(\boldsymbol{k}', \boldsymbol{k})|^2 = \frac{\alpha^2}{4m^2v^4\sin^4\frac{\theta}{2}}. \tag{2.26}$$

Even though we make the approximation, it is well-known that the Rutherford cross section obtained by the Born approximation is found to be almost the same as the exact result by solving the Lippmann-Schwinger equation in a proper way.

This result can be compared with the Mott scattering formula which is obtained by calculating the relativistic T-matrix

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4(m^2+k^2)v^4\sin^4\frac{\theta}{2}}\left(1 - v^2\sin^2\frac{\theta}{2}\right) \tag{2.27}$$

where v is given as $v = \frac{k}{\sqrt{m^2+k^2}} \simeq \frac{k}{m}$ at the non-relativistic limit.

2.2.4 Separable Interaction

The Lippmann-Schwinger equation for the T-matrix cannot normally be solved analytically. However, there is a special potential which can be solved analytically, and this is the separable interaction. The separable interaction is assumed to have the following shape

$$\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle = \lambda g(\boldsymbol{k}')g(\boldsymbol{k}) \tag{2.28}$$

where $g(\boldsymbol{k})$ and λ denote some function and the interaction strength, respectively. Here, this interaction is a highly non-local, and therefore, this is a toy model. In this case, the T-matrix equation becomes

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \lambda g(\boldsymbol{k}')g(\boldsymbol{k}) + \lambda g(\boldsymbol{k}')\int \frac{d^3k''}{(2\pi)^3} g(\boldsymbol{k}'')G_E(\boldsymbol{k}'')\langle \boldsymbol{k}''|T|\boldsymbol{k}\rangle \tag{2.29}$$

where

$$G_E(\boldsymbol{k}'') = \langle \boldsymbol{k}''|\frac{1}{E - H_0 + i\varepsilon}|\boldsymbol{k}''\rangle. \tag{2.30}$$

This equation can be easily solved by assuming the following shape for the T-matrix

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \beta(E)g(\boldsymbol{k}')g(\boldsymbol{k}). \tag{2.31}$$

In this case, the equation for the T-matrix can be written as

$$\beta(E)g(\boldsymbol{k}')g(\boldsymbol{k}) = \lambda g(\boldsymbol{k}')g(\boldsymbol{k}) + \lambda\beta(E)g(\boldsymbol{k}')g(\boldsymbol{k})\int \frac{d^3k''}{(2\pi)^3}[g(\boldsymbol{k}'')]^2 G_E(\boldsymbol{k}'')$$

which can be solved for $\beta(E)$ and we find

$$\beta(E) = \frac{\lambda}{1 - \lambda F(E)} \tag{2.32}$$

where $F(E)$ is defined as

$$F(E) \equiv \int \frac{d^3k''}{(2\pi)^3}[g(\boldsymbol{k}'')]^2 G_E(\boldsymbol{k}''). \tag{2.33}$$

Therefore, the T-matrix is completely determined as

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \frac{\lambda}{1 - \lambda F(E)} g(\boldsymbol{k}')g(\boldsymbol{k}). \tag{2.34}$$

In this way, one sees that the T-matrix equation for the separable interaction can be solved exactly. This is quite important since we can understand the basic structure of the T-matrix, even though the separable interaction is not realistic.

2.3 Interaction Picture and Definition of S-matrix

In the non-static perturbation theory, we obtain the integral equation for the amplitude $a_k(t)$ as

$$a_k(t) = a_k(0) - i\sum_n \int_0^t dt' e^{i(E_k - E_n)t'}\langle u_k|H_I|u_n\rangle a_n(t').$$

Here, we want to derive the same type of the equation as above in terms of the interaction picture. The main difference should be that, in the interaction picture approach, we do not make any expectation values of the interaction Hamiltonian with the state vectors, and therefore the integration over the space is not yet done. In addition, the Hamiltonian should be taken as operators by introducing the Fock space.

2.3.1 Interaction Picture

Now, we introduce Ψ_I as

$$\Psi_I = e^{iH_0 t}\Psi. \tag{2.35}$$

It should be noted that the new state vector Ψ_I is different from the original one only because of the time rotation. In this case, the Schrödinger equation becomes

$$i\frac{\partial \Psi_I(t,\boldsymbol{r})}{\partial t} = H_I(t)\Psi_I(t,\boldsymbol{r}) \tag{2.36}$$

where $H_I(t)$ is defined as

$$H_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}. \tag{2.37}$$

2.3.2 S-matrix

Now we make the same procedure as above by operating $u_k^\dagger(\boldsymbol{r})$ from the left and integrating over the space. If we define $a_k(t)$ by

$$a_k(t) \equiv \langle u_k|\Psi_I\rangle$$

then we obtain exactly the same equation for $a_k(t)$ as eq.(2.7)

$$\frac{da_k(t)}{dt} = -i\sum_n e^{i(E_k-E_n)t}\langle u_k|H_I|u_n\rangle a_n(t).$$

Here, we take a slightly different approach by still keeping the Hamiltonian as operators. Instead, we introduce the matrix $U(t,t_0)$ as

$$\Psi_I(t) = U(t,t_0)\Psi_I(t_0) \tag{2.38}$$

and thus we obtain the following equation

$$i\frac{\partial U(t,t_0)}{\partial t} = H_I(t)U(t,t_0). \tag{2.39}$$

This equation can be easily solved in terms of iteration method, and we find

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \cdots. \tag{2.40}$$

However, this is not a very convenient shape when we define the S-matrix, and thus we introduce the T-product as

$$T\{H_I(t_1)H_I(t_2)\} = \begin{cases} H_I(t_1)H_I(t_2) & (t_1 > t_2) \\ \\ H_I(t_2)H_I(t_1) & (t_1 < t_2). \end{cases} \tag{2.41}$$

Therefore, we can rewrite the matrix $U(t,t_0)$ as

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1) + \frac{1}{2}(-i)^2 \int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 T\{H_I(t_1)H_I(t_2)\} + \cdots .$$

Now we can define the S-matrix as

$$S \equiv 1 - i\int_{-\infty}^{\infty} dt_1 H_I(t_1) + \frac{1}{2}(-i)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 T\{H_I(t_1)H_I(t_2)\} + \cdots \quad (2.42)$$

which can be symbolically written as

$$S = T\left\{\exp\left(-i\int d^4x \mathcal{H}_I\right)\right\} \qquad \text{with} \qquad H_I(t) = \int d^3r \mathcal{H}_I.$$

We note that the S-matrix is still an operator since the Hamiltonian should be an operator after the field quantization.

2.4 Photon Propagator

When one calculates the S-matrix elements in the process of the electromagnetic interaction $H' = e\int j_\mu A^\mu d^3x$ in the second order perturbation theory, then one has to evaluate the propagator of photon. This is written as

$$\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle \quad (2.43)$$

where $A^\mu(x)$ is given as

$$A^\mu(x) = \sum_{\boldsymbol{k}}\sum_{\lambda} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \epsilon^\mu_{\boldsymbol{k},\lambda}\left[c^\dagger_{\boldsymbol{k},\lambda} e^{-i\omega_{\boldsymbol{k}}t + i\boldsymbol{k}\cdot\boldsymbol{r}} + c_{\boldsymbol{k},\lambda} e^{i\omega_{\boldsymbol{k}}t - i\boldsymbol{k}\cdot\boldsymbol{r}}\right]. \quad (2.44)$$

This should be a solution of the following equation of motion for the gauge field

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \quad (2.45)$$

In this case, a question may arise as to how we can calculate the propagator of photon since the photon field $\boldsymbol{A}$ has one redundant degree of freedom. As we discuss below, there is some problem for determining the propagator of photon.

2.4.1 Free Wave of Photon

Now, the S-matrix evaluation should start from the free wave equation of motion. This means that we should not put the $\delta-$function in the right hand side of the equation of motion for photon. Indeed, if we start from the free wave equation of motion, then we should insert

the solution of $A^\mu(x)$ in eq.(2.44) into eq.(2.45) and obtain the following equation for the polarization vector $\epsilon^\mu_{k,\lambda}$ as

$$k^2\epsilon^\mu - (k_\nu\epsilon^\nu)k^\mu = 0. \tag{2.46}$$

This equation can be written in terms of the matrix equation for the polarization vector ϵ^μ as

$$\sum_{\nu=0}^{3}\{k^2 g^{\mu\nu} - k^\mu k^\nu\}\epsilon_\nu = 0$$

where we write the summation explicitly. In order that the ϵ^μ should have a non-zero solution, the determinant of the matrix should vanish, namely,

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{2.47}$$

Thus one finds the solution of this equation

$$k^2 = 0$$

which is a proper dispersion relation of photon. Then, we insert it into eq.(2.46) and obtain

$$k_\mu\epsilon^\mu = 0 \tag{2.48}$$

and this is the solution for the polarization vector. This is the same equation as the Lorentz gauge fixing and thus the Lorentz gauge is not a proper gauge fixing even though people often use it. At this point, we should require the gauge fixing condition, and for example, we choose the Coulomb gauge fixing of $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. In this case, we obtain a condition for the polarization vector as

$$\boldsymbol{k}\cdot\boldsymbol{\epsilon} = 0, \qquad \epsilon_0 = 0.$$

This gauge fixing is most natural, and we should take this condition which can guarantee the number of freedom of photon, which is two.

2.4.2 Feynman Propagator of Photon

Before going to the evaluation of the photon propagator in detail, we should make a comment on the Feynman propagator of photon [2]. The propagator of photon which is known as the Feynman propagator can be written as

$$D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 - i\varepsilon}. \tag{2.49}$$

This is a standard photon propagator which can be found in most of the field theory textbooks. However, it is also well-known that this propagator cannot satisfy the condition of the polarization summation in a correct way. This is clear since it cannot satisfy the following equation

$$k_\mu D_F^{\mu\nu}(k) = -\frac{k^\nu}{k^2 - i\varepsilon} \neq 0$$

where the left hand side should be zero due to the Lorentz condition. Further, it cannot satisfy the Coulomb gauge condition, and therefore whatever they invent, there is no way to claim that the Feynman propagator is a right one. However, as will be seen below, Feynman propagator can reproduce the same T-matrix of the fermion-fermion scattering as the one calculated from the correct propagator as long as the scattering particles are on the mass shell. Since the agreement of the T-matrices evaluated from the two propagators is entirely based on the free Dirac equation of the fermions involved in the scattering process, the Feynman propagator cannot be applied for physical processes involving fermions which are not free or off the mass shell.

2.4.3 Calculation of $\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle$

Here, we should evaluate the denominator of the propagator $\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle$ explicitly in order to avoid any confusions. First, we insert the free solution of the vector potential in eq.(2.44), and find

$$\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle = \sum_{\boldsymbol{k},\lambda}\sum_{\boldsymbol{k}',\lambda'}\frac{1}{\sqrt{4V^2\omega_k\omega_{k'}}}\epsilon^\mu_{\boldsymbol{k},\lambda}\epsilon^\nu_{\boldsymbol{k}',\lambda'}\times$$

$$\langle 0|T\left\{\left(c^\dagger_{\boldsymbol{k},\lambda}e^{-ikx_1}+c_{\boldsymbol{k},\lambda}e^{ikx_1}\right)\left(c^\dagger_{\boldsymbol{k}',\lambda'}e^{-ik'x_2}+c_{\boldsymbol{k}',\lambda'}e^{ik'x_2}\right)\right\}|0\rangle \tag{2.50}$$

which can be calculated to be

$$\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle = \sum_{\lambda=1}^{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\epsilon^\mu_{\boldsymbol{k},\lambda}\epsilon^\nu_{\boldsymbol{k},\lambda}\left(e^{ikx}\theta(t)+e^{-ikx}\theta(-t)\right) \tag{2.51}$$

where we define

$$x = x_1 - x_2, \quad \theta(t) = 1 \ \text{ for } \ t > 0, \quad \theta(t) = 0 \ \text{ for } \ t < 0.$$

By noting the following complex plane integrations

$$\int_{-\infty}^{\infty}\frac{dk_0}{(2\pi)}\frac{e^{ik_0t}}{k_0^2-\boldsymbol{k}^2-i\varepsilon} = \begin{cases} \frac{ie^{i\omega_k t}}{2\omega_k} & \text{for} \quad t>0 \\ \\ \frac{ie^{-i\omega_k t}}{2\omega_k} & \text{for} \quad t<0 \end{cases} \tag{2.52}$$

we can rewrite $\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle$ as

$$\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle = -i\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x_1-x_2)}}{k^2-i\varepsilon}\times\sum_{\lambda=1}^{2}\epsilon^\mu_{\boldsymbol{k},\lambda}\epsilon^\nu_{\boldsymbol{k},\lambda}. \tag{2.53}$$

This is just the propagator of photon. Now, the problem comes up when we evaluate the summation of the polarization vector.

2.4.4 Summation of Polarization States

Up to now, we have presented the expression of the propagator evaluation of photon without making any comments on the field quantization, Now, we should quantize only the vector field $\boldsymbol{A}$ which depends on time. The Coulomb field A^0 is already solved from the constraint equation, and thus it cannot appear in the S-matrix expansion. Therefore, we have the condition that $\epsilon_0 = 0$. In addition, we should respect the Coulomb gauge condition

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0.$$

Now, we are ready to construct the numerator of the propagator of photon, and we find

$$\sum_{\lambda=1}^{2} \epsilon^a_{\boldsymbol{k},\lambda} \epsilon^b_{\boldsymbol{k},\lambda} = \left(\delta^{ab} - \frac{k^a k^b}{\boldsymbol{k}^2} \right) \tag{2.54}$$

which is the only possible solution for the summation of the polarization vector. Note that this can satisfy the condition of $\boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0$, because the left hand side of eq.(2.54) multiplied by k^a becomes

$$k^a \sum_{\lambda=1}^{2} \epsilon^a_{\boldsymbol{k},\lambda} \epsilon^b_{\boldsymbol{k},\lambda} = 0$$

while the right hand side can be calculated as

$$k^a \left(\delta^{ab} - \frac{k^a k^b}{\boldsymbol{k}^2} \right) = k^b - \frac{\boldsymbol{k}^2 k^b}{\boldsymbol{k}^2} = 0$$

and thus eq.(2.54) can satisfy all the conditions we have for the polarization vectors. Therefore, the propagator of photon D^{ab} becomes

$$D^{ab}(k) = \frac{1}{k^2 - i\varepsilon} \left(\delta^{ab} - \frac{k^a k^b}{\boldsymbol{k}^2} \right). \tag{2.55}$$

2.4.5 Coulomb Propagator

The Coulomb part is solved exactly since it does not depend on time. Namely the equation of motion for the Coulomb part is a constraint equation which has nothing to do with the quantization of field. Note that the field quantization should always involve the time dependence of fields. Now, the equation of motion for the A^0 part can be written as

$$\boldsymbol{\nabla}^2 A^0 = -e\bar{\psi}\gamma^0\psi \equiv -ej^0(x) \tag{2.56}$$

which is a constraint equation. However, the right hand side is made of fermion fields, and the quantization of the fermion fields is already done. It should be noted that the Coulomb case is calculated from the first order perturbation theory since it arises from

$$H_C = e \int j^0(t, \boldsymbol{r}) A^0(\boldsymbol{r}) d^3r - \frac{1}{2} \int (\boldsymbol{\nabla} A^0)^2 d^3r.$$

In this case, the interaction Hamiltonian between two Dirac fields j_1^0 and j_2^0 becomes

$$H_C = \frac{e^2}{8\pi}\int \frac{j_1^0(x_1)j_2^0(x_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}d^3r_1d^3r_2$$

which can be rewritten in terms of the momentum representation as

$$H_C = e^2\int \frac{\tilde{j}_1^0(q)\tilde{j}_2^0(-q)}{\boldsymbol{q}^2}\frac{d^3q}{(2\pi)^3}. \tag{2.57}$$

On the other hand, the propagator of photon should be calculated from the S-matrix expansion in the second order perturbation theory. Therefore, the Coulomb propagator is completely different from the photon propagator which is calculated from the S-matrix expansion. However, the Coulomb field interaction should be always considered for the scattering process since fermions are already quantized.

2.4.6 Correct Propagator of Photon

The correct propagator of photon is given in eq.(2.55), but if we consider the scattering process such as electron-electron scattering in which the scattering particles are all on the mass shell, then we should add the Coulomb scattering in which the Coulomb propagator is employed. Therefore, the total propagators of photon together with the Coulomb scattering become

$$\begin{cases} D^{Coul}(k) = \frac{1}{\boldsymbol{k}^2} & A^0-\text{part} \\ D^{ab}(k) = \frac{1}{k^2-i\varepsilon}\left(\delta^{ab}-\frac{k^ak^b}{\boldsymbol{k}^2}\right) & \boldsymbol{A}-\text{part}. \end{cases} \tag{2.58}$$

2.5 Feynman Propagator vs. Correct Propagator

Here, we discuss the equivalence and/or difference between the T-matrices which are calculated from Feynman and correct propagators, The discussion of the equivalence between them is usually found in old field theory textbooks [3, 4]. However, this equivalence proof is valid only if the propagators appear in the scattering processes with free fermions. Therefore, if there is a loop involved such as the fermion self-energy, then the expected equivalence cannot be valid any more. Later in this section, we discuss some physical effects which may arise from the T-matrix difference between the Feynman and the correct propagators.

2.5.1 Scattering of Two Fermions

As an example, we present the scattering T-matrices between two fermions in which one fermion with its four momentum p_1 scatters with another fermion with its four momentum p_2, and after the scattering, we find two fermions with their momenta of p'_1 and p'_2. The four momentum transfer is defined as $q = p_1 - p'_1 = p'_2 - p_2$.

(a) Feynman Propagator

In the case of Feynman propagator as given in eq.(2.49), the T-matrix can be written in a straight forward way as

$$T^{(F)} = -\frac{e^2}{q^2}\left[\bar{u}(p'_1)\gamma^0 u(p_1)\bar{u}(p'_2)\gamma^0 u(p_2) - \bar{u}(p'_1)\boldsymbol{\gamma} u(p_1)\cdot\bar{u}(p'_2)\boldsymbol{\gamma} u(p_2)\right]. \quad (2.59)$$

(b) Correct Propagator

Now, we evaluate the T-matrix with the correct propagator of photon which is given by eq.(2.58). First, the T-matrix from the Coulomb part can be written as

$$T^{(C)} = \frac{e^2}{\boldsymbol{q}^2}\bar{u}(p'_1)\gamma^0 u(p_1)\bar{u}(p'_2)\gamma^0 u(p_2). \quad (2.60)$$

On the other hand, the T-matrix from the vector field $\boldsymbol{A}$ becomes

$$T^{(A)} = \frac{e^2}{q^2}\left[\bar{u}(p'_1)\boldsymbol{\gamma} u(p_1)\bar{u}(p'_2)\boldsymbol{\gamma} u(p_2) - \bar{u}(p'_1)\boldsymbol{\gamma}\cdot\boldsymbol{q} u(p_1)\frac{1}{\boldsymbol{q}^2}\bar{u}(p'_2)\boldsymbol{\gamma}\cdot\boldsymbol{q} u(p_2)\right]. \quad (2.61)$$

Now, we make use of the free Dirac equations for two fermions at the initial and final states

$$(\not{p}_1 - m_1)u(p_1) = 0,\ \bar{u}(p'_1)(\not{p}'_1 - m_1) = 0,\ (\not{p}_2 - m_2)u(p_2) = 0,\ \bar{u}(p'_2)(\not{p}'_2 - m_2) = 0$$

and thus we can rewrite

$$\bar{u}(p'_1)\boldsymbol{\gamma}\cdot\boldsymbol{q} u(p_1) = \bar{u}(p'_1)\gamma^0 u(p_1)q_1^0, \quad \bar{u}(p'_2)\boldsymbol{\gamma}\cdot\boldsymbol{q} u(p_2) = -\bar{u}(p'_2)\gamma^0 u(p_2)q_2^0$$

where $q_1^0 = E_1 - E_1{}'$ and $q_2^0 = E_2 - E_2{}'$. Therefore, $T^{(A)}$ becomes

$$T^{(A)} = \frac{e^2}{q^2}\left[\bar{u}(p'_1)\boldsymbol{\gamma} u(p_1)\cdot\bar{u}(p'_2)\boldsymbol{\gamma} u(p_2) + \bar{u}(p'_1)\gamma^0 u(p_1)\frac{q_1^0 q_2^0}{\boldsymbol{q}^2}\bar{u}(p'_2)\gamma^0 u(p_2)\right]. \quad (2.62)$$

Note that one may be tempted to assume that $q_1^0 = -q_2^0 = q^0$ at this point. However, the energy conservation can be used only at the final stage of the calculation, and therefore, the evaluation of the T-matrix should be done without using the energy conservation. It should be noted that the on-shell scattering processes like the fermion-fermion scattering must conserve the energy, and therefore one can employ the equation $q_1^0 = -q_2^0$ when one calculates the cross section. Now, it is easy to check that the sum of $T^{(C)}$ and $T^{(A)}$ becomes

$$T^{(C)} + T^{(A)} = -\frac{e^2}{q^2}\left[\bar{u}(p'_1)\gamma^0 u(p_1)\bar{u}(p'_2)\gamma^0 u(p_2) - \bar{u}(p'_1)\boldsymbol{\gamma} u(p_1)\cdot\bar{u}(p'_2)\boldsymbol{\gamma} u(p_2)\right]$$
$$+\frac{e^2(q^0 q^0 + q_1^0 q_2^0)}{q^2\boldsymbol{q}^2}\bar{u}(p'_1)\gamma^0 u(p_1)\bar{u}(p'_2)\gamma^0 u(p_2). \quad (2.63)$$

As can be seen, the T-matrix calculated from the correct propagator has an extra-term which is not found in the T-matrix evaluated from the Feynman propagator. Therefore, there exists a clear difference between the two T-matrices in the fermion-fermion scattering case.

Energy Conservation in T-matrix

Now, if one uses the energy conservation of $q_1^0 = -q_2^0 = q^0$, then the Feynman propagator can reproduce the right T-matrix for the fermion-fermion scattering cross section as one can find the equivalence proof in old textbooks [3, 4]. Indeed, the on-shell scattering case is justified because the energy conservation is taken into account for the whole system. This should be one of the strong reasons why people accepted the Feynman propagator.

2.5.2 Loop Diagrams (Fermion Self-energy)

As one sees from the comparison between the Feynman and correct propagators, the use of free Dirac equations play a very important role. Therefore, it is most likely that the two propagators should give the very big difference for the fermion self-energy type diagrams in which intermediate fermions do not satisfy the free Dirac equations.

(a) Feynman Propagator

Using the Feynman propagator, the self-energy of fermion can be easily written as

$$\Sigma^{(F)}(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma_\mu \frac{1}{\not p - \not k - m + i\varepsilon}\,\gamma^\mu \frac{1}{k^2 - i\varepsilon} = \frac{e^2}{8\pi^2}\,\ln\left(\frac{\Lambda}{m}\right)(-\not p + 4m) + \cdots \tag{2.64}$$

which is just the self-energy contribution normally found in the textbooks.

(b) Correct Propagator

The self-energy of fermion with the correct propagator has never been calculated up to now, but we should evaluate it since it is very important to examine whether this self-energy contribution can agree with the normal self-energy contribution with the Feynman propagator. First, the Coulomb part does not contribute to the fermion self-energy because of the equal time operations, and thus we should only calculate the contribution from the vector potential part which can be written as

$$\Sigma^{(A)}(p) = ie^2 \int \frac{d^4k}{(2\pi)^4}\gamma^a \frac{1}{\not p - \not k - m + i\varepsilon}\gamma^b \frac{\left(\delta^{ab} - \frac{k^a k^b}{\boldsymbol{k}^2}\right)}{k^2 - i\varepsilon}. \tag{2.65}$$

What we have to calculate and see is whether the $\Sigma^{(A)}(p)$ should be the same as $\Sigma^{(F)}(p)$ or not. From the calculations, one sees that it does not agree with the one calculated from the Feynman propagator. In this respect, there is no reason any more that one can employ the Feynman propagator for the calculation that involves the photon propagation unless all fermions are on the mass shell.