

## Appendix E

# Wave Propagations in medium and vacuum

The classical wave such as sound can propagate through medium. However, it cannot propagate in vacuum as is well known. This is, of course, clear since the classical wave is the chain of the oscillations of the medium due to the pressure on the density.

On the other hand, quantum wave including photon can propagate in vacuum since it is a particle. Here, we clarify the difference in propagations between the classical wave and quantum wave. The most important point is that the classical wave should be always written in terms of real functions while photon or quantum wave should be described by the complex wave function of the shape  $e^{ikx}$  since it should be an eigenstate of the momentum.

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### E.1 What is wave ?

The sound can propagate through medium such as air or water. The wave can be described in terms of the amplitude  $\phi$  in one dimension

$$\phi(x, t) = A_0 \sin(\omega t - kx) \quad (\text{E.1})$$

where  $\omega$  and  $k$  denote the frequency and wave number, respectively. The dispersion relation of this wave can be written as

$$\omega = vk. \quad (\text{E.2})$$

Here, it is important to note that the amplitude is written as the real function, in contrast to the free wave function of electron in quantum mechanics. In fact, the free wave of electron can be described in one dimension as

$$\psi(x, t) = \frac{1}{\sqrt{V}} e^{i(\omega t - kx)} \quad (\text{E.3})$$

which is a complex function. The electron can propagate by itself and there is no medium necessary for the electron motion.

What is the difference between the real wave amplitude and the complex wave function? Here, we clarify this point in a simple way though this does not contain any new physics.

### E.1.1 A real wave function: Classical wave

If the amplitude is real such as (E.1), then it can only propagate in medium. This can be clearly seen since the energy of the wave can be transported in terms of the density oscillation which is a real as the physical quantity. In addition, the amplitude becomes zero at some point, and this is only possible when it corresponds to the oscillation of the medium. This means that the wave function of (E.1) has nothing to do with the probability of wave object. Instead, if it is the oscillation of the medium, then it is easy to understand why one finds the point where the amplitude vanishes to zero. The real amplitude is called a classical wave since it is indeed seen in the world of the classical physics.

### E.1.2 A complex wave function: Quantum wave

On the other hand, the free wave function of electron is a complex function, and there is no point where it can vanish to zero. Since this is just the wave function of electron, its probability of finding the wave is always a constant  $\frac{1}{V}$  at any space point of volume  $V$ .

## E.2 Classical wave

The sound propagates in the air, and its propagation should be transported in terms of density wave. The amplitude of this wave can be written in terms of the real function as given in eq.(E.1). This is quite reasonable since the density wave should be described by the real physical quantity. Instead, this requires the existence of the medium (air), and the wave can propagate as long as the air exists. Here, we first write the basic wave equation in one dimension

$$\frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} \quad (\text{E.4})$$

which is similar to the wave equation in quantum mechanics, though it is a real differential equation. Here,  $v$  denotes the speed of wave.

### E.2.1 Classical waves carry their energy ?

In this case, a question may arise as to what is a physical quantity which is carried by the classical wave like sound. It seems natural that the wave carries its energy (or wave length). In fact, the transportation of the energy should be carried out by the compression of the

density and successive oscillations of the medium. Therefore this is called compression wave.

### E.2.2 Longitudinal and transverse waves

Here, we discuss the terminology of the longitudinal and transverse waves, even though one should not stress its physics too much since there is no special physical meaning.

- **Longitudinal wave :** The sound propagates as the compressional wave, and the oscillations should be always in the direction of the wave motion. In this case, it is called longitudinal wave. This wave can be easily understood since one can make a picture of the density wave.
- **Transverse wave :** On the other hand, if the motion of the oscillations is in the perpendicular to the direction of the wave motion, then it is called transverse wave. The tidal wave may be the transverse wave, but its description may not be very simple since the density change may not directly be related to the wave itself.

## E.3 Quantum wave

Photon and quantum wave are quite different from the classical wave, and the quantum wave is a particle motion itself. No medium oscillation is involved. For example, a free electron moves with the velocity  $v$  in vacuum, and this motion is also called "wave". The reason why we call it wave is due to the fact that the equation of motion that describes electrons looks similar to the classical wave equation of motion. Further, the solution of the wave equation can be described as  $e^{ikx}$ , and thus it is the same as the wave behavior in terms of mathematics. But the physical meaning is completely different from the classical wave, and quantum wave is just the particle motion which behaves as the probabilistic motion.

### E.3.1 Quantum wave (electron motion)

The wave function of a free electron in one dimension can be described as

$$\psi(x, t) = \frac{1}{\sqrt{V}} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \quad (\text{E.5})$$

which is a solution of the Schrödinger equation of a free electron,

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi \quad (\text{E.6})$$

where  $k = \sqrt{2m\omega}$ , and  $V$  denotes the corresponding volume. Since the Schrödinger equation is quite similar to the wave equation in a classical sense, one calls the solution of the

Schrödinger equation as a wave. However, the physics of the quantum wave should be understood in terms of the quantum mechanics, and the relation to the classical wave should not be stressed. That is, the quantum wave is completely different from the classical wave, and one should treat the quantum wave as it is. In addition, the behavior and physics of the classical wave are very complicated and it is clear that we do not fully understand the behavior of the classical wave since it involves many body problems in physics.

### E.3.2 Photon

The electromagnetic wave is called photon which behaves like a particle and also like a wave. This photon can propagate in vacuum and thus it should be considered to be a particle. Photon can be described by the vector potential  $\mathbf{A}$ .

• **A is real ! :** However, this  $\mathbf{A}$  is obviously a real function, and therefore, it cannot propagate like a particle. This can be easily seen since the free Hamiltonian of photon commutes with the momentum operator  $\hat{\mathbf{p}} = -i\nabla$ , and therefore it can be a simultaneous eigenstate of the Hamiltonian. Thus, the  $\mathbf{A}$  should be an eigenstate of the momentum operator since the free state must be an eigenstate of momentum. However, any real function cannot be an eigenstate of the momentum operator, and thus the vector field in its present shape cannot describe the free particle state.

• **Free solution of vector field :** What should we do ? The only way of solving this puzzle is to quantize a photon field. First, the solution of  $\mathbf{A}$  can be written as

$$\mathbf{A}(x) = \sum_{\mathbf{k}, \lambda} \frac{1}{\sqrt{2\omega_{\mathbf{k}}V}} \boldsymbol{\epsilon}_{\mathbf{k}, \lambda} \left( c_{\mathbf{k}, \lambda}^{\dagger} e^{-i\mathbf{k}x} + c_{\mathbf{k}, \lambda} e^{i\mathbf{k}x} \right) \quad (\text{E.7})$$

with  $\mathbf{k}x \equiv \omega_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{r}$ . Here,  $\boldsymbol{\epsilon}_{\mathbf{k}, \lambda}$  denotes the polarization vector which will be discussed later more in detail. As one sees, the vector field is indeed a real function.

• **Quantization of vector field :** Now we impose the following quantization conditions on  $c_{\mathbf{k}, \lambda}^{\dagger}$  and  $c_{\mathbf{k}, \lambda}$

$$[c_{\mathbf{k}, \lambda}, c_{\mathbf{k}', \lambda'}^{\dagger}] = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\lambda, \lambda'}, \quad (\text{E.8})$$

$$[c_{\mathbf{k}, \lambda}, c_{\mathbf{k}', \lambda'}] = 0, \quad [c_{\mathbf{k}, \lambda}^{\dagger}, c_{\mathbf{k}', \lambda'}^{\dagger}] = 0. \quad (\text{E.9})$$

In this case,  $c_{\mathbf{k}, \lambda}^{\dagger}$ ,  $c_{\mathbf{k}, \lambda}$  become operators. Therefore, one should now consider the Fock space on which they can operate. This can be defined as

$$c_{\mathbf{k}, \lambda}|0\rangle = 0 \quad (\text{E.10})$$

$$c_{\mathbf{k}, \lambda}^{\dagger}|0\rangle = |\mathbf{k}, \lambda\rangle \quad (\text{E.11})$$

where  $|0\rangle$  denotes the vacuum state of the photon field. Therefore, if one operates the vector field on the vacuum state, then one obtains

$$\langle \mathbf{k}, \lambda | \mathbf{A}(x) | 0 \rangle = \frac{1}{\sqrt{2\omega_{\mathbf{k}}V}} \boldsymbol{\epsilon}_{\mathbf{k}, \lambda} e^{-i\mathbf{k}x}. \quad (\text{E.12})$$

As one sees, this new state is indeed the eigenstate of the momentum operator and should correspond to the observables. Therefore, photon can be described only after the vector field is quantized. Thus, photon is a particle whose dispersion relation becomes

$$\omega_{\mathbf{k}} = |\mathbf{k}|. \quad (\text{E.13})$$

## E.4 Polarization vector of photon

Until recently, there is a serious misunderstanding for the polarization vector  $\epsilon_{\mathbf{k},\lambda}^\mu$ . This is related to the fact that the equation of motion for the polarization vector is not solved, and thus there is one condition missing in the determination of the polarization vector.

### E.4.1 Equation of motion for polarization vector

Now the equation of motion for  $A^\mu = (A^0, \mathbf{A})$  without any source terms can be written from the Lagrange equation as

$$\partial_\mu F^{\mu\nu} = 0 \quad (\text{E.14})$$

where  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ . This can be rewritten as

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0. \quad (\text{E.15})$$

Now, the shape of the solution of this equation can be given as

$$A^\mu(x) = \sum_{\mathbf{k}} \sum_{\lambda} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \epsilon_{\mathbf{k},\lambda}^\mu \left[ c_{\mathbf{k},\lambda} e^{-ikx} + c_{\mathbf{k},\lambda}^\dagger e^{ikx} \right] \quad (\text{E.16})$$

and thus we insert it into eq.(E.15) and obtain

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \quad (\text{E.17})$$

Now the condition that there should exist non-zero solution of  $\epsilon_{\mathbf{k},\lambda}^\mu$  is obviously that the determinant of the matrix in the above equation should vanish to zero, namely

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \quad (\text{E.18})$$

This leads to  $k^2 = 0$ , which means  $k_0 \equiv \omega_{\mathbf{k}} = |\mathbf{k}|$ . This is indeed a proper dispersion relation for photon.

### E.4.2 Condition from equation of motion

Now we insert the condition of  $k^2 = 0$  into eq.(E.17), and obtain

$$k_\mu \epsilon^\mu = 0 \quad (\text{E.19})$$

which is a new constraint equation obtained from the basic equation of motion. Therefore, this condition (we call it ‘‘Lorentz condition’’) is most fundamental. It should be noted that the Lorentz gauge fixing is just the same as eq.(E.19). This means that the Lorentz gauge fixing is improper and forbidden for the case of no source term. In this sense, the best gauge fixing should be the Coulomb gauge fixing

$$\mathbf{k} \cdot \boldsymbol{\epsilon} = 0 \quad (\text{E.20})$$

from which one finds  $\epsilon_0 = 0$ , and this is indeed consistent with experiment.

• **Number of freedom of polarization vector :** Now we can understand the number of degree of freedom of the polarization vector. The Lorentz condition  $k_\mu \epsilon^\mu = 0$  should give one constraint on the polarization vector, and the Coulomb gauge fixing  $\mathbf{k} \cdot \boldsymbol{\epsilon} = 0$  gives another constraint. Therefore, the polarization vector has only two degrees of freedom, which is indeed an experimental fact.

• **State vector of photon :** The state vector of photon is already discussed. But here we should rewrite it again. This is written as

$$\langle \mathbf{k}, \lambda | A(x) | 0 \rangle = \frac{\boldsymbol{\epsilon}_{\mathbf{k}, \lambda}}{\sqrt{2\omega_{\mathbf{k}} V}} e^{-ikx}. \quad (\text{E.21})$$

In this case, the polarization vector  $\boldsymbol{\epsilon}_{\mathbf{k}, \lambda}$  has two components, and satisfies the following conditions

$$\boldsymbol{\epsilon}_{\mathbf{k}, \lambda} \cdot \boldsymbol{\epsilon}_{\mathbf{k}, \lambda'} = \delta_{\lambda, \lambda'}, \quad \mathbf{k} \cdot \boldsymbol{\epsilon}_{\mathbf{k}, \lambda} = 0. \quad (\text{E.22})$$

### E.4.3 Photon is a transverse wave ?

People often use the terminology of transverse photon. Is it a correct expression ? By now, one can understand that the quantum wave is a particle motion, and thus it has nothing to do with the oscillation of the medium. Therefore, it is meaningless to claim that photon is a transverse wave. The reason of this terminology may well come from the polarization vector  $\boldsymbol{\epsilon}_{\mathbf{k}, \lambda}$  which is orthogonal to the direction of photon momentum. However, as one can see, the polarization vector is an intrinsic property of photon, and it does not depend on space coordinates.

• **No rest frame of photon ! :** In addition, there is no rest frame of photon, and therefore, one cannot discuss its intrinsic property unless one fixes the frame. Even if one says that the polarization vector is orthogonal to the direction of the photon momentum, one has to be careful in which frame one discusses this property.

In this respect, it should be difficult to claim that photon behaves like a transverse wave. Therefore, one sees that photon should be described as a massless particle which has two degrees of freedom with the behavior of a boson. There is no correspondence between classical waves and photon, and even more, there is no necessity of making analogy of photon with the classical waves.

## E.5 Poynting vector and radiation

We have clarified that the propagation of the real function requires some medium which can make oscillations. Here, we discuss the Poynting vector how it appears in physics, and show that it cannot propagate in vacuum at all. Also, we present a brief description of the basic radiation mechanism how photon can be emitted.

### E.5.1 Field energy and radiation of photon

Before discussing the propagation of the Poynting vector, we should first discuss the mechanism of the radiation of photon in terms of classical electrodynamics. The interaction Hamiltonian can be written as

$$H_I = - \int \mathbf{j} \cdot \mathbf{A} d^3r \quad (\text{E.23})$$

which should be a starting point of all the discussions. Now, we make a time derivative of the interaction Hamiltonian and obtain

$$W \equiv \frac{dH_I}{dt} = - \int \left[ \frac{\partial \mathbf{j}}{\partial t} \cdot \mathbf{A} + \mathbf{j} \cdot \frac{\partial \mathbf{A}}{\partial t} \right] d^3r. \quad (\text{E.24})$$

Since we can safely set  $A^0 = 0$  in this treatment, we find

$$\mathbf{E} = - \frac{\partial \mathbf{A}}{\partial t}. \quad (\text{E.25})$$

Therefore, we can rewrite eq.(E.24) as

$$W = \int \mathbf{j} \cdot \mathbf{E} d^3r - \int \frac{\partial \mathbf{j}}{\partial t} \cdot \mathbf{A} d^3r. \quad (\text{E.26})$$

Defining the first term of eq.(E.24) as  $W_E$ , we can rewrite  $W_E$  as

$$W_E \equiv \int \mathbf{j} \cdot \mathbf{E} d^3r = - \frac{d}{dt} \left[ \int \left( \frac{1}{2\mu_0} |\mathbf{B}|^2 + \frac{\varepsilon_0}{2} |\mathbf{E}|^2 \right) d^3r \right] - \int \nabla \cdot \mathbf{S} d^3r \quad (\text{E.27})$$

which is just the energy of electromagnetic fields.

### E.5.2 Poynting vector

Here, the last term of eq.(E.27) is Poynting vector  $\mathbf{S}$  as defined by

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} \quad (\text{E.28})$$

which is connected to the energy flow of the electromagnetic field. This Poynting vector is a conserved quantity, and thus it has nothing to do with the electromagnetic wave. In addition, it is a real quantity, and thus there is no way that it can propagate in vacuum.

In addition, the Poynting vector cannot be a target of the field quantization, and thus it always remains classical since it is written in terms of  $\mathbf{E}$  and  $\mathbf{B}$ . However, there is still some misunderstanding in some of the textbooks on Electromagnetism, and therefore, one should be careful for the treatment of the Poynting vector.

• **Exercise problem:** Here, we present a simple exercise problem of circuit with condenser with  $C$  (disk radius of  $a$  and distance of  $d$ ) and resistance with  $R$ . The electric potential difference  $V$  is set on the circuit. In this case, the equation for the circuit can be written as

$$V = R \frac{dQ}{dt} + \frac{Q}{C}.$$

This can be easily solved with the initial condition of  $Q = 0$  at  $t = 0$ , and the solution becomes

$$Q = CV \left( 1 - e^{-\frac{t}{RC}} \right).$$

Therefore, the electric current  $J$  becomes

$$J = \frac{dQ}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}.$$

In this case, we find the electric field  $\mathbf{E}$  and the displacement current  $\mathbf{j}_d$

$$\mathbf{E} = \frac{Q}{\pi a^2} \mathbf{e}_z = \frac{VC}{\varepsilon_0 \pi a^2} \left( 1 - e^{-\frac{t}{RC}} \right) \mathbf{e}_z \quad (\text{E.29})$$

$$\mathbf{j}_d = \frac{\partial \mathbf{E}}{\partial t} = \frac{V}{R \pi a^2} e^{-\frac{t}{RC}} \mathbf{e}_z. \quad (\text{E.30})$$

Thus, the magnetic field  $\mathbf{B}$  becomes

$$\mathbf{B} = \frac{i_d r}{2} \mathbf{e}_\theta = \frac{r}{2 \pi a^2 R} e^{-\frac{t}{RC}} \mathbf{e}_\theta$$

where  $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 i_d \pi r^2$  is used. Therefore, the Poynting vector at the surface (with  $r = a$ ) of the cylindrical space of the disk condenser becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} = -\frac{V^2}{2 \pi a R d} e^{-\frac{t}{RC}} \left( 1 - e^{-\frac{t}{RC}} \right) \mathbf{e}_r.$$

It should be noted that the energy in the Poynting vector is always flowing into the cylindrical space. Therefore, the electric field energy is now accumulated in the cylindrical space. There is, of course, no electromagnetic wave radiation, and in fact, the Poynting vector is the flow of field energy, and has nothing to do with the electromagnetic wave.



### E.5.3 Emission of photon

The emission of photon should come from the second term of eq.(E.26) which can be defined as  $W_R$  and thus

$$W_R = - \int \frac{\partial \mathbf{j}}{\partial t} \cdot \mathbf{A} d^3r. \quad (\text{E.31})$$

In this case, we can calculate the  $\frac{\partial \mathbf{j}}{\partial t}$  term by employing the Zeeman effect Hamiltonian with a uniform magnetic field of  $\mathbf{B}_0$

$$H_Z = - \frac{e}{2m_e} \boldsymbol{\sigma} \cdot \mathbf{B}_0. \quad (\text{E.32})$$

The relevant Schrödinger equation for electron with its mass  $m_e$  becomes

$$i \frac{\partial \psi}{\partial t} = - \frac{e}{2m_e} \boldsymbol{\sigma} \cdot \mathbf{B}_0 \psi. \quad (\text{E.33})$$

Therefore, we find

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{e}{m_e} \left[ \frac{\partial \psi^\dagger}{\partial t} \hat{\mathbf{p}} \psi + \psi^\dagger \hat{\mathbf{p}} \frac{\partial \psi}{\partial t} \right] = - \frac{e^2}{2m_e^2} \nabla B_0(\mathbf{r}). \quad (\text{E.34})$$

In order to obtain the photon emission, one should quantize the field  $\mathbf{A}$  in eq.(E.31).

• **Field quantization :** The field quantization in electromagnetic interactions can be done only for the vector potential  $\mathbf{A}$ . The electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  are classical quantities which are defined before the field quantization.

## E.6 Gravitational wave

People often discuss the gravitational wave which is supposed to come from the Einstein equation. In this case, one sees that the equation for the metric tensor is all real, and thus the solution of this equation must be also real. Therefore, the gravitational wave, if at all exists, is a real function, and thus it cannot propagate in vacuum unless one believes the aether hypothesis.

• **No quantization of gravity :** In addition, there is no physical meaning to quantize the metric tensor and therefore, there is no chance that the gravitational wave propagates in vacuum.

### E.6.1 General relativity

Since we treat the gravitational wave, we should make a comment on the general relativity. Einstein invented the general relativity which is the second order differential equation for the metric tensor  $g^{\mu\nu}$ . A question may arise as to why the general relativity can be related to the gravitational theory. This reason is simply because Einstein claimed that he had proved the gravitational Poisson equation should be derived from the general relativity at the weak gravitational limit. However, in his proof, he assumed the following strange equation

$$g^{00} \simeq 1 + 2\phi \quad (\text{E.35})$$

where  $\phi$  denotes the gravitational field. Because of this equation (E.35), he could derive the gravitational Poisson equation

$$\nabla^2 \phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}) \quad (\text{E.36})$$

where  $G$  and  $\rho$  denote the gravitational constant and the density, respectively.

- **Eq.(E.35) is correct ? :** Here, we show that eq.(E.35) is not only strange but simply incorrect. In order to do so, we should examine the physical meaning of the equation  $g^{00} \simeq 1 + 2\phi$ . We should notice that 1 (unity) in the right hand side of eq.(E.35) is a simple number. This is clear since the metric tensor is just the coordinate system itself. However, the gravitational field  $\phi$  is a dynamical variable, and therefore this summation of two different categories is simply meaningless.

- **No connection between general relativity and gravity :** By now it should be clear that the general relativity has nothing to do with gravity. It is a theory for the coordinate system (metric tensor), but it is not a theory for nature.

#### Note :

The new gravitational theory is explained in detail in Chapter 6 in the text book of “Fundamental problems in quantum field theory” .

#### Reference :

Fundamental Problems in Quantum Field Theory  
T. Fujita and N. Kanda, Bentham Publishers, 2013