Appendix

1. 微分

(a) 微分の定義

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \qquad (*)$$

(b) 合成微分

$$\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx} \qquad (*)$$

**example:

$f(u) = u^2 \qquad u(x) = e^x$

公式 (*) より

$\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx} = 2ue^x = 2e^{2x}$

一方 直接計算すると

$f(x) = u^2 = e^{2x}$ より $\frac{df(u(x))}{dx} = 2e^{2x}$

(c) 偏微分

$$\frac{\partial f(x,y)}{\partial x} \equiv [\ y \text{ をとめて } x \text{ だけで微分}] \qquad (*)$$

**example:

$f(x,y) = x^n y^n$

$\frac{\partial f(x,y)}{\partial x} = nx^{n-1}y^n$

(d) 全微分

$f(x(t), y(t), t)$ のとき

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t} \qquad (*)$$

**example:

$f(x(t), y(t), t) = x^2 y^2 e^{at}$

$x = \sin t \quad y = \cos t$

公式 (*) より

$\frac{df}{dt} = 2xy^2 e^{at}\cos t - 2yx^2 e^{at}\sin t + ax^2y^2e^{at}$

$\qquad = \sin 2t \cos 2t e^{at} + \frac{1}{4}a(\sin 2t)^2 e^{at}$

一方 直接計算すると

$f(x(t), y(t), t) = \sin^2 t \cos^2 t e^{at} = \frac{1}{4}(\sin 2t)^2 e^{at}$

$\frac{df}{dt} = \sin 2t \cos 2t e^{at} + \frac{1}{4}a(\sin 2t)^2 e^{at}$

2. 合成変換

$(x,y) \to (u,v) \qquad$ namely $\qquad (u(x,y), v(x,y))$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad (*)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \qquad (*)$$

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad (*)$$

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} \qquad (*)$$

3. Grassmann 代数

a, b に対して

$$a * b = -b * a \qquad (*)$$

とする。また結合法則と分配法則が成立するとする。

微分 du, dv 等は Grassmann 代数で扱うと便利である。

$$\begin{aligned} du * dv &= (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy) * (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy) & (1) \\ &= (\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x})dx * dy \qquad (*) & (2) \\ J &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad (*) & (3) \end{aligned}$$

J を Jacobian という。

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad (*)$$

4. 変数変換

(a) example: $(x, y, z) \to (r, \theta, \phi)$

$$\begin{aligned} x &= r \sin\theta \cos\phi & (4) \\ y &= r \sin\theta \sin\phi & (5) \\ z &= r \cos\theta & (6) \\ dx &= \sin\theta\cos\phi dr + r\sin\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi & (7) \\ dy &= \sin\theta\sin\phi dr + r\sin\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi & (8) \\ dz &= \cos\theta dr - r\sin\theta d\theta & (9) \end{aligned}$$

これより Jacobian J を求めよ。

解: $J = r^2 \sin\theta$

(b) example: 微分演算子 Δ

$$\begin{aligned}\Delta &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad (*) \qquad (10)\\ \Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r\sin\theta^2}\frac{\partial^2}{\partial\phi^2} \qquad (*) \qquad (11)\\ \Delta &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2} \qquad (*) \qquad (12)\end{aligned}$$

(c) example: 微小距離 $(ds)^2$

$$\text{デカルト座標:} \qquad (ds)^2=(dx)^2+(dy)^2+(dz)^2 \qquad (*)$$

$$\text{極座標:} \qquad (ds)^2=(dr)^2+(rd\theta)^2+(r\sin\theta d\phi)^2 \qquad (*)$$

$$\text{円筒座標:} \qquad (ds)^2=(dr)^2+(rd\theta)^2+(dz)^2 \qquad (*)$$

(d) Taylor 展開

$$f(x+h)=f(x)+f^{'}(x)h+\frac{1}{2}f^{''}(x)h^2+\cdots+\frac{1}{n!}f^{(n)}(x)h^n+\cdots \qquad (*)$$

**example:

$$\begin{aligned}(1+x)^\alpha &= 1+\alpha x+\frac{1}{2}\alpha(\alpha-1)x^2+\cdots \qquad (*) \qquad (13)\\ \sqrt{1+x} &= 1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots \qquad (*) \qquad (14)\\ \frac{1}{1+x} &= 1-x+x^2-x^3+\cdots \qquad (*) \qquad (15)\\ \ln(1+x) &= x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots \qquad (*) \qquad (16)\\ e^x &= 1+x+\frac{1}{2}x^2+\cdots \qquad (*) \qquad (17)\\ \sin x &= x-\frac{1}{6}x^3+\cdots \qquad (*) \qquad (18)\\ \cos x &= 1-\frac{1}{2}x^2+\cdots \qquad (*) \qquad (19)\end{aligned}$$

**Euler の公式

$$e^{i\theta}=\cos\theta+i\sin\theta \qquad (*)$$

**Taylor 展開を使って Euler の公式を証明せよ。

5. 微分方程式

Definition:

$$\dot{u} = \frac{du}{dt}, \qquad u^{'} = \frac{du}{dx} \qquad (*)$$

(a) example :

$$\ddot{u} + \omega^2 u = 0 \qquad (*)$$

解: Put $u = e^{at}$. Then

$$(a^2 + \omega^2)e^{at} = 0$$

$a = \pm i\omega$

Therefore, u has two solutions.

$u = e^{i\omega}, e^{-i\omega}$

A general solution must be a linear combination of the two solutions.

$$u = Ae^{i\omega} + Be^{-i\omega}$$

From Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$),

$$u = A'\sin\omega t + B'\cos\omega t$$

Note: A, B, A' and B' are arbitary constants which should be determined from the initial condition.

(b) example :

$$\ddot{u} = f(u) = \frac{\partial g}{\partial u} \qquad (*)$$

解: Multiply $\dot{u}$ to the both sides above.

$$\dot{u}\ddot{u} = \dot{u}\frac{\partial g}{\partial u}$$

$$\frac{1}{2}\frac{d\dot{u}^2}{dt} = \frac{dg}{dt}$$

$$\frac{1}{2}\dot{u}^2 = g + C$$

$$\frac{du}{\sqrt{2(g(u) + C)}} = dt$$

(c) example :

Linear Differential Equation:

$$a_n \frac{d^n u}{dt^n} + \cdots + a_1 \frac{du}{dt} + a_0 u = 0$$

解:

Put $u = e^{\alpha t}$. Then

$$a_n \alpha^n + \cdots + a_1 \alpha + a_0 = 0$$

The solutions are

$u_1 = e^{\alpha_1 t}, \cdots, u_n = e^{\alpha_n t}$.

Therefore, general solutions are

$$u = c_1 e^{\alpha_1 t} + \cdots + c_n e^{\alpha_n}$$

Here, α_n are roots of the equation.

6. ベクトル

Definition: $\mathbf{a} = (a_1, a_2, \cdots, a_n)$

空間が 3 次元のとき

内積 $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$

複素数内積 $(\mathbf{a} \cdot \mathbf{b}) = a_x^* b_x + a_y^* b_y + a_z^* b_z$

外積

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (*) \tag{20}$$

$$= (a_y b_z - a_z b_y)\mathbf{e_x} + (a_z b_x - a_x b_z)\mathbf{e_y} + (a_x b_y - a_y b_x)\mathbf{e_z} \qquad (*) \tag{21}$$

絶対値 $\mid \mathbf{a} \mid = \sqrt{a_x^2 + a_y^2 + a_z^2}$

公式

* $\mathbf{a} \times \mathbf{a} = 0$
* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
* $\mathbf{a} \cdot \mathbf{b} = \mid \mathbf{a} \mid\mid \mathbf{b} \mid \cos\theta$
* $\mid \mathbf{a} \times \mathbf{b} \mid = \mid \mathbf{a} \mid\mid \mathbf{b} \mid \sin\theta$
* $(\mathbf{a} \times \mathbf{b})^2 = \mid \mathbf{a} \mid^2 \mid \mathbf{b} \mid^2 - (\mathbf{a} \cdot \mathbf{b})^2$
* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

7. 行列

* n 次正方行列 * $A=\{A_{ij}\}, i=1,\cdots,n \qquad j=1,\cdots,n$

複素共役 $(A^*)_{ij}=A^*_{ij}$

*転置行列 * $(A^t)_{ij}=A_{ji}$

エルミート $(A^\dagger)_{ij}=A^*_{ji}$

*エルミート行列 * $A=A^\dagger$

* 行列式 *

$$\det(A)\equiv\sum_P \epsilon_{(m_1\cdots m_n)}A_{1m_1}\cdots A_{nm_n} \qquad (*)$$

where $\epsilon_{(m_1\cdots m_n)}$ is $+1$ for even permutation and -1 for odd permutation.

公式

$$\det(AB)=\det(A)\det(B) \qquad (*)$$

*トレース Tr *

$$\mathrm{Tr}A\equiv\sum_{i=1}^{n}A_{ii} \qquad (*)$$

公式

$$\mathrm{Tr}(AB)=\mathrm{Tr}(BA) \qquad (*)$$

$$\det(A)=\exp(\mathrm{Tr}\ln A) \qquad (*)$$

8. 固有値と固有関数

$$A\mathbf{u} = a\mathbf{u}$$

a: 固有値　　$\mathbf{u}$: 固有関数

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix}$$

成分で書くと

$$\sum_{j=1}^{n} A_{ij} u_j = a u_i$$

固有値の求め方

$$\sum_{j=1}^{n} (A_{ij} u_j - a u_i) = 0 \qquad (*)$$

$$\sum_{j=1}^{n} (A_{ij} - a\delta_{ij}) u_i = 0 \qquad (*)$$

Here δ_{ij} is 1 for $i = j$ and 0 for $i \neq j$.
これは u_j に対する連立方程式である。$u_j \neq 0$ の解があるためには

$$\det(A_{ij} - a\delta_{ij}) = 0 \qquad (*)$$

常識
** Any eigenvalue of Hermite matrix must be real. **
解:

$$(\mathbf{u}, A\mathbf{u}) = a \mid \mathbf{u} \mid^2 = (A^\dagger \mathbf{u}, \mathbf{u}) = (A\mathbf{u}, \mathbf{u}) = (a\mathbf{u}, \mathbf{u}) = a^* \mid \mathbf{u} \mid^2$$

Therefore, we obtain $a = a^*$ which means a is real.
** The determinant of any unitary matrix is ± 1. **
解:

$$\det(U^\dagger U) = 1$$

On the other hand,

$$\det(U^\dagger) = \sum_P \epsilon_{(m_1\cdots m_n)} A^*_{m_1 1} \cdots A^*_{m_n n} \tag{22}$$

$$= (\sum_P \epsilon_{(m_1\cdots m_n)} A_{m_1 1} \cdots A_{m_n n})^* \tag{23}$$

$$= (\sum_P \epsilon_{(m_1\cdots m_n)} A_{1 m_1} \cdots A_{n m_n})^* = (\det(U))^* \tag{24}$$

$$\mid \det(U) \mid^2 = 1, \qquad \mid \det(U) \mid = \pm 1$$

9. 積分

$$\int_0^\infty e^{-ax}dx = \frac{1}{a} \qquad (*) \tag{25}$$

$$\int_0^\infty x^n e^{-ax}dx = (-1)^n \frac{\partial^n}{\partial a^n}\frac{1}{a} = \frac{n!}{a^{n+1}} \qquad (*) \tag{26}$$

$$\int_{-\infty}^\infty e^{-ax^2}dx = (\int_{-\infty}^\infty e^{-ax^2}dx \int_{-\infty}^\infty e^{-ay^2}dy)^{\frac{1}{2}} = (\int_0^\infty e^{-ar^2}rdr \int_0^{2\pi} d\theta)^{\frac{1}{2}} = \sqrt{\frac{\pi}{a}} \qquad (*) \tag{27}$$

$$\int_0^\infty x^{2n+1}e^{-ax^2}dx = \frac{1}{2}\frac{n!}{a^{n+1}} \qquad (*) \tag{28}$$

$$\int_0^\infty x^{2n}e^{-ax^2}dx = (-1)^n \frac{\partial^n}{\partial a^n}\frac{1}{2}\sqrt{\frac{\pi}{a}} = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}a^{n+\frac{1}{2}}} \qquad (*) \tag{29}$$

$$\int_\alpha^\beta \frac{g(x)dx}{[(x-\alpha)(x-\beta)]^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} 2g(\alpha + (\beta-\alpha)\sin^2\theta)d\theta \qquad (*) \tag{30}$$

解: Put $x = \alpha + (\beta - \alpha)\sin^2\theta$

$$\int \frac{f(x)dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int f(a\tan\theta)\cos\theta d\theta \qquad (*)$$

解: Put $x = a\tan\theta$

$$\int_0^\pi \sin^2\theta d\theta = \int_0^\pi \cos^2\theta d\theta = \frac{\pi}{2} \qquad (*)$$

10. 部分積分

Using the identity

$$(fg)' = f'g + fg'$$

Putting $f' = F$,

$$\int Fgdx = fg - \int fg'dx \qquad (*)$$

example

$$\int \ln x dx = x\ln x - x - C$$

11. 微分演算公式

(a) 直交座標系　x, y, z

i. gradient

$$\mathrm{grad}A_0 \equiv \nabla A_0 = \frac{\partial A_0}{\partial x}\mathbf{e}_x + \frac{\partial A_0}{\partial y}\mathbf{e}_y + \frac{\partial A_0}{\partial z}\mathbf{e}_z$$

ii. Laplacian

$$\nabla^2 A_0 \equiv \Delta A_0 = \frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_0}{\partial y^2} + \frac{\partial^2 A_0}{\partial z^2}$$

iii. 直交座標におけるベクトルポテンシャル

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z$$

iv. divergence

$$\mathrm{div}\mathbf{A} \equiv \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

v. rotation

$$\mathrm{rot}\mathbf{A} \equiv \nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z$$

(b) 円柱座標系　r, φ, z

i. gradient

$$\nabla A_0 = \frac{\partial A_0}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial A_0}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial A_0}{\partial z}\mathbf{e}_z$$

ii. Laplacian

$$\nabla^2 A_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_0}{\partial \varphi^2} + \frac{\partial^2 A_0}{\partial z^2}$$

iii. 円柱座標におけるベクトルポテンシャル

$$\mathbf{A} = A_r\mathbf{e}_r + A_\varphi\mathbf{e}_\varphi + A_z\mathbf{e}_z$$

$$A_r = A_x\cos\varphi + A_y\sin\varphi, \qquad A_\varphi = -A_x\sin\varphi + A_y\cos\varphi$$

iv. divergence

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

v. rotation

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\varphi + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}\right)\mathbf{e}_z$$

(c) 極座標系　r, θ, φ

i. gradient

$$\nabla A_0 = \frac{\partial A_0}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial A_0}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial A_0}{\partial \varphi}\mathbf{e}_\varphi$$

ii. Laplacian

$$\nabla^2 A_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_0}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial A_0}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_0}{\partial \varphi^2}$$

iii. 極座標におけるベクトルポテンシャル

$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\varphi\mathbf{e}_\varphi$$

$$A_r = A_x\sin\theta\cos\varphi + A_y\sin\theta\sin\varphi + A_z\cos\varphi$$

$$A_\theta = A_x\cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\varphi$$

$$A_\varphi = -A_x\sin\varphi + A_y\cos\varphi$$

iv. divergence

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

v. rotation

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\mathbf{e}_\theta$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\mathbf{e}_\varphi$$