13-6 Maxwell方程式のLorentz不変性

$$
\begin{cases}
\nabla \cdot \mathbb{E} = \dfrac{\rho}{\varepsilon_0} & ① \\[2ex]
\nabla \cdot \mathbb{B} = 0 & ② \\[2ex]
\nabla \times \mathbb{E} = -\dfrac{\partial \mathbb{B}}{\partial t} & ③ \\[2ex]
\nabla \times \mathbb{B} = \mu_0 \boldsymbol{i} + \dfrac{1}{c^2}\dfrac{\partial \mathbb{E}}{\partial t} & ④
\end{cases}
$$

◎ ベクトルポテンシャル $\mathbb{A}$ と スカラーポテンシャル ϕ

$$
\begin{cases}
\mathbb{E} = -\nabla\phi - \dfrac{\partial \mathbb{A}}{\partial t} \\[2ex]
\mathbb{B} = \nabla \times \mathbb{A}
\end{cases}
$$

①式：

$$
\boxed{-\nabla^2\phi - \nabla\frac{\partial \mathbb{A}}{\partial t} = \frac{\rho}{\varepsilon_0}} \quad ⓐ
$$

② 式：　$\nabla \cdot \nabla \times \mathbb{A} = 0$　OK

③ 式：

$$\nabla \times \mathbb{E} = -\nabla \times \nabla \phi - \nabla \times \frac{\partial \mathbb{A}}{\partial t}$$

$$= -\frac{\partial}{\partial t} \nabla \times \mathbb{A} = -\frac{\partial \mathbb{B}}{\partial t}$$

OK

④ 式：

$$\nabla \times (\nabla \times \mathbb{A}) = \mu_0 \mathbb{i} + \frac{1}{c^2}\left(-\frac{\partial}{\partial t}\nabla\phi - \frac{\partial^2 \mathbb{A}}{\partial t^2}\right)$$

$$\therefore \boxed{\nabla(\nabla \cdot \mathbb{A}) - \nabla^2 \mathbb{A} = \mu_0 \mathbb{i} - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbb{A}}$$ ⓑ

- Gauge 不定性がある。ので1つ条件をおく必要がある。

$$\boxed{\nabla \cdot \mathbb{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0}$$

とおく。　Lorentz 条件という

ⓐ式：

$$\nabla^2\phi + \nabla\frac{\partial \mathbb{A}}{\partial t} = -\frac{\rho}{\varepsilon_0}$$

$$\therefore \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\therefore \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = \frac{\rho}{\varepsilon_0}$$

Lorentz invariant

ⓑ式：

$$\nabla(\nabla\mathbb{A}) - \nabla^2\mathbb{A} = \mu_0 i - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbb{A}$$

$$\cancel{-\frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}} - \nabla^2\mathbb{A} = \mu_0 i \cancel{- \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbb{A}$$

$$\therefore \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbb{A} = \mu_0 i$$

Lorentz invariant

◎ $\nabla\cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$

は Lorentz 不変か?

$$\begin{cases} A_\mu = \left(\frac{1}{c}\phi, A\right) \\ \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, \nabla\right) \end{cases}$$

として 4次元空間の内積をとると

$$\left(\frac{\partial}{\partial x_\mu}\cdot A_\mu\right) = \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla A$$

となる

内積は回転に対して不変

$$\boxed{x'_\mu = \sum a_{\mu\nu} x_\nu}$$

Lorentz 変換

但し $\sum_{\mu=0}^{3}\left(x'_\mu \cdot x'_\mu\right) \equiv x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2$

$$\sum_{\mu=0}^{3} (x'_\mu \cdot x'_\mu) = \sum_{\mu=0}^{3} (x_\mu \cdot x_\mu)$$

$$\sum_{\mu=0}^{3} \left(\sum_{\nu} a_{\mu\nu} x_{\nu} \right)\left(\sum_{\nu'} a_{\mu\nu'} x_{\nu'} \right)$$

$$= \sum_{\mu=0} \sum_{\nu\nu'} a_{\mu\nu} a_{\mu\nu'} \cdot (x_{\nu} \cdot x_{\nu'})$$

$$= \sum_{\mu=0} (x_\mu \cdot x_\mu)$$

$$= \sum_{\nu\nu'} \delta_{\nu\nu'} \quad (x_\nu \cdot x_{\nu'})$$

$$\therefore \boxed{\sum_{\mu=0} a_{\mu\nu} a_{\mu\nu'} = \delta_{\nu\nu'}}$$