Relativity : Mistakes in Famous Textbooks

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Preface

There are many articles which make elementary explanations of the theory of relativity since relativity may attract many people indeed. However, it should not be very easy to understand in depth the theory of relativity in a correct fashion. In particular, if authors may explain the theory of relativity following the science history, then there should be a great danger that the interpretation may well be made in the wrong direction.

In this short note, I should like to explain what is the theory of relativity and why it is so important in physics. The essence of the relativity is simple. Two inertial frames which are relatively moving with a constant velocity on the straight line should be equivalent to each other. This means that there is no special inertial system in physics, and therefore, any experiments which are carried out at one inertial frame should produce the same results as those given at the other inertial frame.

Among textbooks which explain the relativity, some of them might insist that the time in the highly moving inertial frame should be delayed compared to the rest frame. However, this is simply wrong, and later I should describe the reason why there is no time delay in any of inertial frames.

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Chapter 1

Relativity

Since the invariance of light velocity is established, people realized that the transformation of inertial frames must be the Lorentz transformation. This Lorentz transformation is derived such that the Maxwell equation should be invariant under the transformation among the inertial frames. Since the transformation is kinematics, there should be no effect on the dynamics from the Lorentz transformation.

1.1 Lorentz Transformation

We should prepare two inertial frames, $R(t, x, y, z)$ and $S(t', x', y', z')$. Now we assume that the frame S is moving with respect to the rest frame $R(t, x, y, z)$ in the $x-$axis with the velocity of v. In this case, the Lorentz transformation can be written as

$$x = \gamma(x' + vt'), \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \qquad y = y', \qquad z = z' \tag{1.1}$$

where γ is defined as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{1.2}$$

The Lorentz transformation is derived such that the Maxwell equations must have the same shape of differential equations. In eq.(1.1), if the velocity v is much smaller than the light velocity c, then we have

$$x \simeq x' + vt', \qquad t \simeq t', \qquad y = y', \qquad z = z' \tag{1.3}$$

which is just the Galilei transformation.

1.2 Lorentz Transformation of Differential Quantities

Here we should present the Lorentz transformation of differential quantities concerning the coordinates

$$x = \gamma(x' + vt'), \ t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{1.4}$$

and they can be written as

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right), \qquad \frac{\partial}{\partial t} = \gamma\left(v\frac{\partial}{\partial x'} - \frac{\partial}{\partial t'}\right). \tag{1.5}$$

Since y, z should not be affected, we do not write them here. Now if we define

$$p_x = -i\frac{\partial}{\partial x}, \qquad E = i\frac{\partial}{\partial t} \tag{1.6}$$

then we find

$$p_x = \gamma\left(p_x' + \frac{vE'}{c^2}\right), \qquad E = \gamma\left(E' + vp_x'\right) \tag{1.7}$$

which is just the Lorentz transformation of energy and momentum. Therefore, the scalar product of $px \equiv Et - \boldsymbol{p}\cdot\boldsymbol{r}$ is invariant under the Lorentz transformation

$$px = Et - \boldsymbol{p}\cdot\boldsymbol{r} = p'x' = E't' - \boldsymbol{p}'\cdot\boldsymbol{r}'. \tag{1.8}$$

1.3 Lorentz Invariance of Equation of Motions

The equation of motion should have the same shape of differential equations in any of inertial frames. Therefore, we want to check as to how the Newton equation and Maxwell equation should behave under the Lorentz transformation.

1.3.1 Newton Equation under Lorentz Transformation

As we see from the Lorentz transformation of $x = \gamma(x' + vt')$, $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$, x, t should be independent from each other. But here we assume that x should be a function of time. Therefore, the differential of x with respect to time t becomes

$$\frac{dx}{dt} = \frac{dx' + vdt'}{dt' + \frac{v}{c^2}dx'} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}. \tag{1.9}$$

Further the second derivative should be given as

$$\frac{d^2x}{dt^2} = \frac{1}{\gamma(dt' + \frac{v}{c^2}dx')}d\left(\frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}\right) = \frac{\frac{d^2x'}{dt'^2}}{\gamma^3\left(1 + \frac{v\frac{dx'}{dt'}}{c^2}\right)^3} \neq \frac{d^2x'}{dt'^2} \tag{1.10}$$

which is totally different from the Newton equation. Thus, the Newton equation cannot be invariant under the Lorentz transformation.

1.3.2 Maxwell Equation under Lorentz Transformation

The equation of motion for the electric field of $\boldsymbol{E}$ can be written when there is no current

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\boldsymbol{E} = 0. \tag{1.11}$$

Under the Lorentz transformation, we can prove

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \boldsymbol{\nabla}'^2 \tag{1.12}$$

and therefore, the Maxwell equation is invariant under the Lorentz transformation.

Chapter 2

Relativistic Classical Mechanics

In old days, people discussed the relativistic classical mechanics. This is a model which is the extension of classical mechanics to the kinematically relativistic version of mechanics. In terms of science history, this extension should be understandable. However, this is physically a meaningless mechanics which has never been applied to real physics.

2.1 Classical Mechanics

Here, there should be no point to explain any physical meaning of classical mechanics, but it should be important to note that the coordinate x in Newton equation is a function of time t. This is quite strange from the point of view of field theory since t and x should be independent from each other. This is, in fact, also true for the Lorentz transformation.

Then, a question may arise as to why the x should become a function of time in the Newton equation. This can be understood if one looks into the Ehrenfest theorem in detail. One sees that the time dependence of x should be a leftover of time dependence in the state vector $\psi(\boldsymbol{r}, t)$. In this sense, the x of a point particle in classical mechanics is identified as the x in the coordinate system.

2.2 Relativistic Classical Mechanics

The Newton equation can be extended to the relativistic equation of motion by making some modification of kinematics, and this mechanics is called relativistic classical mechanics. Here, however, we do not write this equation of motion since it has no physical meaning at all.

It is clear that we cannot derive the original equation from the equation which is obtained by some approximations. For example, if we take x as a small positive number, we find

$$(1+x)^\alpha = 1 + \alpha x + \cdots \tag{2.1}$$

as an approximation. However, it is impossible to guess the l.h.s of equation from the r.h.s. of equation. Thus, it is, of course, clear that the equation of quantum mechanics cannot be obtained from the classical mechanics.

2.3 Definition of Velocity

In classical mechanics, the velocity of particle is defined as

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}. \tag{2.2}$$

However, in the relativistic kinematics, we cannot define the velocity of particle in terms of the derivative of coordinate $\boldsymbol{r}$. This is simply because t and $\boldsymbol{r}$ are independent from each other in the relativistic kinematics as well as in the Lorentz transformation. Therefore, the velocity in the relativistic kinematics can be defined in terms of momentum as

$$\boldsymbol{v} = \frac{\boldsymbol{p}c^2}{E}. \tag{2.3}$$

In the nonrelativistic limit, we obtain, using $E \simeq mc^2$

$$\boldsymbol{v} \simeq \frac{\boldsymbol{p}}{m}. \tag{2.4}$$

This indicates that the velocity of particle in the relativistic kinematics should not be a fundamental physical quantity. In this respect, we cannot define the relativistic classical mechanics in a proper way.

2.3.1 Velocity in Quantum Mechanics

In quantum mechanics, the concept of velocity does not appear in fundamental physical quantities, and the momentum of particle appears as a basic quantity. If one wishes to know the velocity of particle, then one should calculate the expectation value of momentum as

$$\boldsymbol{v} \equiv \frac{1}{m} \int \psi^\dagger(\boldsymbol{r}) \hat{\boldsymbol{p}} \psi(\boldsymbol{r}) d^3r \tag{2.5}$$

where $\hat{\boldsymbol{p}}$ is defined as $\hat{\boldsymbol{p}} = -i\hbar \boldsymbol{\nabla}$ which denotes the momentum operator.

2.3.2   Velocity v in Lorentz Transformation

The velocity of particle is not related to a fundamental physical quantity, but the velocity v of inertial frame appears in the relativity. However, its physical meaning is not very clear yet.

2.4   Lorentz Contraction

People often discuss unphysical effects if they are based on the picture of relativistic classical mechanics. A typical example must be the Lorentz contraction. This is a claim that the length ℓ of the moving frame should be viewed to be contracted if one sees it in the rest frame. But people never explain how it is related to any physical observables.

In reality, however, only the center of mass of length ℓ should be transformed into the other inertial frame, and the Lorentz transformation cannot give any information on the internal structure of length ℓ. In this respect, there is no way to discuss the Lorentz contraction.

Further, the length ℓ should have a finite size, and this means that this ℓ should be a bound state of many atoms. However, in the quantum field theory, there is no way to solve even two body problems in a rigorous fashion. This means that we do not know how to solve the two body problem in the Dirac equation. One may ask a question as to whether the hydrogen atom should be solved in the Dirac equation. However, the hydrogen atom in the Dirac equation is solved with the approximation that proton is sufficiently heavy and thus it should be set to at rest.

2.5 High Energy Heavy Ion Reaction

When a moving inertial frame collides with some targets in the rest frame, is there any possible way to handle the reaction process? In fact, this is just the experiment of high energy heavy ion reactions which were performed around 1980. The energy of heavy ion should amount to around 1 GeV per nucleon, and this should indeed be treated relativistically. The projectile nucleus is ^{4}He, and at Max-Planck Institute, Hüfner and I started to make up some model which can analyze the data of heavy ion reactions.

However, we had some serious problems which are related to the treatment of the projectile wave function of ^{4}He in the rest frame. As one knows, the Lorentz transformation should be only for the center of mass system of ^{4}He, and thus there is no way to obtain the internal wave function of ^{4}He in the rest frame since ^{4}He is moving relativistically. The Lorentz transformation cannot tell us anything about the wave function which has some distribution in space.

As a result, we decided to make analysis of the reaction process in the projectile frame in which the ^{4}He nucleus is at rest. In this way, we are successful in obtaining some important information from the experimental data of this heavy ion reaction.

Chapter 3

Quantum Field Theory

By now, it is established that the basic theoretical framework is quantum field theory. In particular, the field theory of electron interacting with electromagnetic fields and gravitational field should be the most basic theoretical scheme, and the description of field theory can be found in detail in the textbooks [5, 6].

3.1 Lagrangian Density of QED and Gravity

Here we should write the basic Lagrangian density. This is the system of particle state ψ with the mass m which should interact with electromagnetic field of A_μ as well as the gravitational field $\mathcal{G}$. This Lagrangian density can be written as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m(1+g\mathcal{G})\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{v\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\ \partial^\mu\mathcal{G}$$

where $\mathcal{G}$ denotes a massless scalar field. $F^{\mu\nu}$ denotes the field strength and is defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

It should not be very easy to prove that this Lagrangian density is invariant under the Lorentz transformation. But one should examine it by oneself, and this is the starting point of studying the quantum field theory in depth.

3.2 Calculations in Quantum Field Theory

In quantum field theory, all the calculations are based on the perturbation theory since there is no chance to solve it exactly except free fields. Therefore, it should

be important to construct the Dirac vacuum since the solution of Dirac equation contains the negative energy states with the energy eigenvalues which should be physical. However, if there should be negative energy states present, then all the positive energy states should become unstable since the positive energy states should eventually decay into negative energy states.

In order to avoid this difficulty, Dirac proposed and defined the physical vacuum in which all the negative energy states should be occupied. In this case, the vacuum state becomes stable and we have a well-defined quantum field theory for fermions.

3.3   Two Body Problem in Quantum Field Theory

A hydrogen atom consists of proton and electron, and therefore, it is a real two body problem. However, if one wishes to solve the two body Dirac equation, one encounters the difficulty that there is no way to separate the center of mass system from the relative coordinate. Thus, one does not know how to solve the two body Dirac equation in the exact fashion.

Why is it so difficult to solve the Dirac equation for two body problems ? It should not be very easy to find the answer for the above question, but one thing must be clear that the difficulty should be related to the Dirac vacuum state. In terms of quantum field theory, the problem of hydrogen atom may not necessarily be the two body problem, but rather it should be a many body problem. One may say that one sees an electron rotating around proton, but in reality, there should be a small component of electron- positron pair state as a virtual mixture. This is the essence of quantum field theory, and it should naturally and always be a many body problem. Thus, there is no chance to solve it exactly. In physics, we try to understand the behavior of nature with simple equations of motion, and it must be true that physics has made a great success. However, even for the classical mechanics, we cannot solve many body problems. For example, the phenomena of turbulence should be too difficult to understand with physics law. It is extremely interesting to study physics, but we should also understand the limitation of physics application to nature.

Chapter 4

Is Time in Moving Frame Delayed?

From the Lorentz transformation eq.(1.1), it looks that time in the moving frame deviates from the rest frame. However, t and x are variables, and thus, they are not directly related to physical observables. Below we examine whether the time difference of Δt in the Gedanken experiment should be delayed or not.

4.1 Incorrect Gedanken Experiment

Here we first explain the time difference Δt in the Gedanken experiment which is often discussed in the science history. First, we consider a train (moving inertial frame) which is driving in the straight line with a constant velocity v. We assume that there should be big mirror wall in parallel to the straight line with its distance of ℓ.

4.1.1 Time Difference of Moving Frame from Rest Frame

First, an observer in the train emits laser beams against mirror wall. In this case, the observer in the train should not notice that the train is moving. Now this observer should detect the reflected laser beam and should measure the time difference $(2\Delta\tau)$. In this case, we see

$$\ell = c\Delta\tau. \tag{4.1}$$

On the other hand, an observer at the rest frame should detect the laser beam which reflects and travels through the triangle trajectory. In this case, the time difference $(2\Delta t)$ should be

$$\sqrt{(c\Delta t)^2 - \ell^2} = v\Delta t. \tag{4.2}$$

Therefore, we find

$$\sqrt{c^2 - v^2}\, \Delta t = c\Delta\tau \tag{4.3}$$

which gives us the following relation between the time differences of $\Delta\tau$ and Δt as

$$\Delta\tau = \sqrt{1 - \frac{v^2}{c^2}}\, \Delta t. \tag{4.4}$$

This suggests that the time difference in the moving frame seems to be somewhat smaller than that of the rest frame.

4.1.2 Time Difference of Rest Frame from Moving Frame

Now we should carry out the same type of Gedanken experiment from the observer at the moving frame. In this case, the rest frame is moving with the velocity of $-v$ for the observer of the moving frame. This can be easily seen if we solve the Lorentz transformation the other way around

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right), \quad y' = y, \quad z' = z. \tag{4.5}$$

Here we see that the rest frame is moving with its velocity of $(-v)$. But otherwise, everything is just the same as in the previous case. In this case, the observer in the rest frame emits laser beams against mirror wall, and the observer in the train should detect the reflected laser beam and should measure the time difference $(2\Delta ct)$. Thus, we find

$$\Delta t = \sqrt{1 - \frac{v^2}{c^2}}\, \Delta\tau. \tag{4.6}$$

4.1.3 Inconsistency of Time Difference

What is going on? The results of eqs. (4.4) and (4.6) contradict with each other. Since Δt and $\Delta\tau$ should be observables in the Gedanken experiment, there must be something wrong there.

4.2 Where is Incorrect Process in Gedanken Experiment?

What should be incorrect inductions in the Gedanken experiment? This can be easily seen if we look into eq. (1.1). After t, we took the coordinate of the train as $x' = x + vt$, which is wrong. The correct coordinate after t should be given by the Lorentz transformation as

$$x' = \gamma(x + vt). \tag{4.7}$$

Thus, we should replace in the following way

$$v\Delta t \Longrightarrow \gamma v\Delta t, \qquad c\Delta t \Longrightarrow \gamma c\Delta t. \tag{4.8}$$

Therefore, eq. (4.4) becomes

$$\begin{aligned}
\Delta\tau &= \sqrt{1 - \frac{v^2}{c^2}} \times \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t \\
&= \Delta t.
\end{aligned}$$

This shows that there is no time delay, and there is no inconsistency. This is just all what we see from the relativity.

4.2.1 No Time Delay in Moving Frame!

From the Gedanken experiment, we see that there is no time delay in the moving frame as compared to the rest frame. This is quite reasonable since the relativity only states that any inertial frames should produce the same results of all physical observables.

Chapter 5

Two Inertial Frames : Examples of Relativity

Here we should discuss possible observables when two inertial frames are involved in physical processes. It should be noted that this consideration is only related to the kinematics, and therefore, we cannot learn anything about dynamics of physical processes.

5.1 Doppler Effect of Light

When a star is moving away from the earth, then lights emitted from this star should be affected by the Lorentz transformation, and this is known as the Doppler effect. Let consider that a star is going away with its velocity v. The momentum p of light emitted at the star should become p' on the earth, and this relation is given by the Lorentz transformation as

$$p' = \gamma \left(p - \frac{vE}{c^2} \right) = \gamma \left(p - \frac{vp}{c} \right) = \frac{p \left(1 - \frac{v}{c} \right)}{\sqrt{\left(1 - \frac{v^2}{c^2} \right)}} = p \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}. \tag{5.1}$$

This shows that the momentum of light is decreased. If we express the above relation in terms of wave length, then we obtain

$$\lambda' = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \, \lambda. \tag{5.2}$$

Since the wave length of the observed light becomes longer, we call it "red shift". It should be noted that this naming has no physical meaning. It simply says that

red light has a longer wave length than that of blue light. The physical reason of the Doppler shift is because the energy and momentum make four dimensional vector, and therefore this is affected by the Lorentz transformation.

5.2 Life Time of Muon Produced in Atmosphere

High energy cosmic ray (protons) may collide with atmospheric N_2 or other molecule and may produce muons with the mass of $m_\mu = 105.6$ MeV/c^2. The life time τ_0 of this lepton is around $\tau_0 \simeq 2 \times 10^{-6}$ s. Therefore, muon is unstable. Now a question is as to whether the life time of muon may be affected by the Lorentz transformation or not. This problem is often discussed in science history, but here we should present a right description of muon as to how far it can travel in the air.

Now the life time τ_0 can be written in terms of decay width Γ as

$$\tau_0 = \frac{\hbar}{\Gamma}. \tag{5.3}$$

Here we note that Γ is a Lorentz invariant quantity. Therefore, the life time is also Lorentz invariant, and thus the life time of muon should be the same in any inertial frame.

5.2.1 Travel Distance L of Muon

Now we should calculate the travel distance L of muon after it is created from the collision of protons with atmosphere. This can be evaluated from the Lorentz transformation $x = \gamma(x' + vt')$ as

$$L = \gamma v \tau_0. \tag{5.4}$$

Here we take, as an example, muon with its energy of 1 GeV. In this case, the velocity of muon can be approximated by light velocity of c. The Lorentz factor γ should be $\gamma \simeq 10.6$. Therefore, the value of L becomes

$$L = \gamma v \tau_0 = 10.6 \times 3 \times 10^8 \times 2 \times 10^{-6} \simeq 6.3 \text{ km} \tag{5.5}$$

which is longer by γ than $v\tau_0$. This indicates that the muon produced in the atmosphere may well have some chance to be observed on the earth.

5.2.2   Accelerator Experiment

Unstable particles created by the large accelerator should travel the distance which is given by eq. (5.4). This is longer by a factor of γ than $v\tau_0$, but it has nothing to do with the delay of life time of unstable particles. It is simply due to the Lorentz transformation.

Chapter 6

Conclusions

In the theory of relativity, the transformation property among inertial frames can be given by the Lorentz transformation, and this corresponds to the transformation of a point particle. However, we find some incorrect description of time delay or Lorentz contraction in quite a few text books or short notes. We should stress that the Lorentz transformation is only concerned with the center of mass system of any complex objects, and that is all we know.

In particular, we prove that the time delay of moving frame never occurs in any inertial frames. The basic mistake should be originated from the fact that the coordinate of the moving frame with the velocity of v should be γvt after t second. In the science history, people want to modify v or t in order to understand γvt. However, from the point of theoretical scheme, we should understand γvt by itself in terms of Lorentz transformation.

6.1 Homework Problem

When a train (moving inertial frame) is running with its velocity of v, then the position x of the train after Δt second can be given by the Lorentz transformation as

$$x = \gamma(x' + v\Delta t), \qquad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{6.1}$$

Here we should consider a problem which is concerned with the emission of light in the train. In this case, a question is as to where the light should be found after Δt second in the rest frame.

In this short note, we give the reaching distance ℓ in the rest frame as

$$\ell = \gamma c \Delta t. \tag{6.2}$$

Prove the result of eq.(**6.2**).

[**Hints**] :
The light velocity c is not changed when it is emitted from the running train. But in order to understand the range of the light after the emission, we need to consider the Lorentz transformation in a proper manner.

Now, we should take a long train as the moving frame and emit light and observe it after Δt second in the train. In this case, the reaching distance in the moving frame should be simply

$$x' = c\Delta t. \tag{6.3}$$

Therefore, if we make the Lorentz transformation of eq.(6.1), then we find the reaching distance ℓ [(**6.2**)] of light in the rest frame.

Appendix A

Why Is General Relativity Meaningless?

The Einstein equation is a differential equation for the metric tensor of $g^{\mu\nu}$. This metric tensor is defined when the Lorentz invariant quantity $(ds)^2$ is expressed in terms of generalized formula as $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$. However, there is no special physical meaning in this generalization, and thus we cannot find any physics related to the metric tensor of $g^{\mu\nu}$. This problem of the general relativity has nothing to do with physics, but it is important in the science history. Therefore, we should explain why the general relativity was accepted to physicists for such a long time, even though it is a meaningless theory in physics.

A.1 Relativity Principle

Relativity principle should require that equations of motion in any inertial system should have the same form of differential equations, and, thus, all of the physical observables must be the same in every inertial system. This is the essence of the relativity, and nature can be understood in terms of four basic Lagrangian densities of electromagnetic, weak, strong eand gravitational interactions. Indeed, all the field theory models satisfy the relativistic invariance of Lorentz transformation.

A.1.1 Lorentz Transformation

Let us consider the moving frame $S(t', x', y', z')$ which is moving with linear motion of constant velocity v along $x-$axis with respect to the rest frame $R(t, x, y, z)$. In

this case, the requirement that the equation of motion must be equivalent to each other in both systems can be written in terms of Lorentz transformation

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \quad y = y', \quad z = z'. \tag{A.1}$$

A.1.2 Lorentz Invariance

This Lorentz transformation is the necessary and sufficient condition for relativity principle. However, if we consider only the invariance of Lorentz transformation, then there should be many other physical quantities. Here, we should discuss the small distance square of $(ds)^2$ in four dimensions, which is defined as

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2.$$

A.1.3 Minkowski Space

This $(ds)^2$ is introduced by Minkowski as a Lorentz invariant quantity

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{A.2}$$

which is indeed invariant under the Lorentz transformation of

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \quad y = y', \quad z = z'. \tag{A.3}$$

Minkowski extended mathematically $(ds)^2$ to

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \equiv g^{\mu\nu}dx_\mu dx_\nu \tag{A.4}$$

even though there is no physical reason for this generalization. In this case, dx^μ and dx_μ are introduced as

$$dx^\mu = (cdt, dx, dy, dz), \quad dx_\mu = (cdt, -dx, -dy, -dz). \tag{A.5}$$

Further, the metric tensor $g^{\mu\nu}$ is defined as

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This extension of $(ds)^2$ is not incorrect. However, the naming of $g^{\mu\nu}$ as metric tensor is wrong since it is a dimensionless quantity and, therefore, it cannot be taken as any measure of space and time.

A.2 Risk of Generalization

It indeed makes sense that $(ds)^2$ can be taken as a test of Lorentz invariance, and it is also understandable that $(ds)^2$ is expressed in terms of eq.(A.4). However, it should be important to realize that this generalization is physically meaningless since $(ds)^2$ itself is far from any essential quantity in physics.

A.2.1 Invariance of $(ds)^2$

Here, we should explain some important point of $(ds)^2$. This $(ds)^2$ is certainly Lorentz invariant, but it is the result of the Lorentz transformation, and not the condition. In fact, there should be many other transformations that can make $(ds)^2$ invariant. This point is quite important since it is related to the essence of relativity. The theory of relativity is a theoretical frame work in which any equation of motion must be the same in any inertial system. The Lorentz transformation satisfies this necessary and sufficient conditions. On the other hand, $(ds)^2$ can serve as a sufficient condition of the relativity requirement, but it is not necessary.

A.2.2 Generalized Expression of $(ds)^2$

For a long time, people believed that the generalized expression of $(ds)^2$

$$(ds)^2 = g^{\mu\nu} dx_\mu dx_\nu \tag{A.6}$$

must be basic and essential for $(ds)^2$. This is, of course, an illusion. However, most of physicists may well have been trapped for a long time in a blind state, and this is quite unfortunate.

A.2.3 Physical Meaning of $g^{\mu\nu}$

In physics, the expression of (A.2) is essential, and it is impossible to find any physical meaning for the metric tensor of $g^{\mu\nu}$. Indeed, $g^{\mu\nu}$ must be mathematically all right, but it has no physical meaning, and it is just useless.

A.3   General Relativity

The Einstein equation is the differential equation for this useless metric tensor $g^{\mu\nu}$ [7], and therefore, we cannot find any physical meaning in this equation.

In fact, even if the metric tensor $g^{\mu\nu}$ becomes some function of space and time, there is no effect on the relativity. In case the $(ds)^2$ which is expressed by $g^{\mu\nu}$ in eq.(A.6) has lost the Lorentz invariance, we should make use of $(ds)^2$ as expressed in eq.(A.2). Therefore, there is no physical effect of $g^{\mu\nu}$ in nature at all.

This clearly shows that the Einstein equation has nothing to do with physics, and it is simply a mathematical equation which may help young people learn geometrical differential equation as an exercise problem.

A.4   Negative Legacy

It is a shame that we could not clarify 30 years ago, for example, that the Einstein equation has nothing to do with physics. Many young people wasted their time by learning this general relativity which is completely meaningless in physics. This is quite unfortunate and serious.

Incidentally, there was a claim at one point that the Mercury perihelion shifts could be described by the metric tensor which is, by hand, connected to gravity. However, this shift is identified by the discontinuity of Mercury orbit, and, therefore, this prediction is both physically and mathematically meaningless. In this sense, this claim may well be one of the worst theoretical predictions in physics.

Appendix B

New Gravity Model

Quantum field theory is based on the free Dirac fields and four fundamental interactions. These are electromagnetic, weak, strong and gravitational interactions. In terms of coupling constant, the electromagnetic interaction must be a standard, and the strength of the coupling constant which is dimensionless is found to be

$$\alpha = \frac{1}{137}.$$
(B.1)

On the other hand, the strong interaction should be stronger by two orders of magnitude than the electromagnetic interaction while weak interaction must be weaker by a few orders of magnitude than the electromagnetic interaction. In this respect, the gravity is, by far, the weakest force among the four interactions. In fact, the gravity is by the order of $\sim 10^{-30}$ smaller than the electromagnetic interaction.

B.1 Introduction

Nevertheless, the gravity is very important in the universe for the formation of stars and galaxies since the force has a very long range, and it is always attractive. In fact, apart from strong interactions that should responsible for nuclear fusion in stars, the basic ingredients of forming stars and galaxies in the universe should be the gravitational interaction.

B.1.1   Why Gravity Has Large Effects on Star Formation?

The gravity is crucially important for the formation of stars even though the interaction strength is quite weak. There are two important aspects in the gravity when the stars should be formed. The first point is connected to the interaction range which is very long since it has the shape of $1/r$. The other point is that the gravity is always attractive and the strength of the force should be proportional to the masses of interacting objects. Therefore, as long as the corresponding body is massive, there should exist the attractive interactions from all other massive objects even though they are far away from each other. Because of the attractive nature, there should be no shielding in contrast to the electromagnetic cases.

B.1.2   Dirac Equation with Gravitational Potential

When the energy of a particle becomes as high as its mass, then we have to consider the relativistic equation of motion under the gravitational potential. In this case, the Newton equation is not appropriate for describing a relativistic motion, and thus, we have to find a new equation of motion. Since we know that the classical mechanics is derived from the Schrödinger equation, we should start from the relativistic equation in quantum mechanics. This is the Dirac equation, and therefore, we have to consider the Dirac equation with the gravitational interaction.

However, the Dirac equation with the gravitational potential has not been determined properly for a long time. This problem is connected to the ambiguity as to whether the gravitational potential should be taken as the fourth component of the vector type interaction or the mass term of scalar type interaction. This problem was not settled until recently, and thus, we should consider the gravitational field theory in some way or other. As will be discussed later, the new gravity model is, indeed, constructed in terms of a massless scalar field theory. Therefore, the corresponding Dirac equation with the gravitational potential is well established by now [6, 5].

B.2 Dirac Equation and Gravity

The Newton equation works very well under the gravitational potential, and indeed, the Kepler problem is best understood by solving the Newton equation.

● **Ehrenfest Theorem :**
This Newton equation itself is obtained from the Schrödinger equation by making some approximation such as Ehrenfest theorem. In this case, the time development of the expectation values of r and p in quantum mechanics lead to the Newton equation.

● **Foldy-Wouthuysen Transformation :**
The Schrödinger equation can be derived from the Dirac equation by making the Foldy-Wouthuysen transformation which is a unitary transformation. Therefore, the Dirac equation must be the starting point from which the Newton equation can be derived.

B.2.1 Dirac Equation and Gravitational Potential

As can be seen from the present discussion, it should be crucially important to have the Dirac equation with the gravitational potential properly taken into account. Otherwise, we cannot obtain the Newton equation with the gravitational potential. In other words, we should not start from the Newton equation with the gravitational potential since it is obtained only after some series of approximations should be properly made for quantum mechanics.

● **Dirac Equation with Coulomb Potential :**
Before going to the discussion of the Dirac equation with the gravity, we should first discuss the Dirac equation with the Coulomb potential of $V_c(r) = -\frac{Ze^2}{r}$. This is well-known and can be written as

$$\left(-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + m\beta - \frac{Ze^2}{r} \right) \Psi = E\Psi. \tag{B.2}$$

On the other hand, we should be careful in which way we put the gravitational potential of $V(r) = -\frac{GmM}{r}$ into the Dirac equation since there are two different ways, either the same way as the Coulomb case or putting the gravitational potential into the mass term.

● **Dirac Equation with Gravitational Potential :**
In fact, the right Dirac equation with the gravitational potential of
$V(r) = -\frac{GmM}{r}$ can be written by putting it into the scalar term as

$$\left[-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + \left(m - \frac{G_0 mM}{r} \right) \beta \right] \Psi = E\Psi. \tag{B.3}$$

This is obtained from the field theoretical construction of the gravity model. By now, we see that the scalar type potential of gravity must be the right gravitational potential, and we should discuss it more in detail below.

B.3   New Gravity Model

When we wish to construct the theory of gravity, the first thing we should work out should be to find the framework in which the gravitational potential can be properly taken into account in the Dirac equation. Without doing this procedure, there should be no way to consider the theory of gravity. In fact, the Dirac equation for a particle with its mass m in the gravitational potential can be written as

$$\left[-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + \left(m - \frac{GmM}{r} \right) \beta \right] \Psi = E\Psi \tag{B.4}$$

where M denotes the mass of the gravity center. In addition, if we make the non-relativistic reduction using the Foldy-Wouthuysen transformation, then we find the gravitational potential in classical mechanics

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2} \left(\frac{GmM}{r} \right)^2 \tag{B.5}$$

where the second term of the right hand side should be the additional potential which appears as the relativistic effect. This additional potential of gravity is a new gravitational potential, and this must be a new discovery ever since nineteenth century. It turns out that this new potential can explain the problem of leap second of the earth revolution period which will be discussed later.

• **Rough Estimation of Relativistic Effect :**
Historically, the first check of the relativistic effect was done by Michelson-Morley using the velocity of the earth revolution which should be the fastest object relevant to the observed speed on the earth. The result of Michelson-Morley experiment showed that the speed of light is not affected by the earth revolution, and this leads to the concept of the relativity principle. The relativistic effect in this case is

$$\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8} \tag{B.6}$$

where c and v denote the velocities of light and the earth revolution, respectively. It should be interesting to note that the leap second of the earth revolution period is found to be $(\Delta T/T \sim 2 \times 10^{-8})$ which is just the same order of magnitude as the relativistic effect.

B.3.1 Lagrangian Density

When we consider the theory of gravity, we should start from the scalar field theory since it gives always attractive interactions.

• **Lagrangian Density of Gravity :**
Here, we should write the Lagrangian density of a fermion field ψ interacting with the electromagnetic field A_μ and the gravitational field $\mathcal{G}$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m(1 + g\mathcal{G})\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\upsilon\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\ \partial^\mu\mathcal{G} \tag{B.7}$$

where m denotes the fermion mass. The gravitational field $\mathcal{G}$ is a massless scalar field. The reason why people did not consider the scalar field for the gravity should be mainly because the scalar field should not be renormalizable. However, there is no necessity of the field quantization of the gravitational field, and thus, there is no divergence at all.

• **Gravity Cannot Be Gauge Theory :**
For a long time, people believed that the basic field theory must be a gauge theory, even though there is no foundation for this belief. Indeed, the gauge theory has both attractive and repulsive interactions, and therefore, it is clear that this model of gauge field theory should not be suitable for the gravity.

By now, it is known that only the gauge theory of quantum electrodynamics using the Feynman propagator should give rise to some divergences in the calculation of physical observables such as vertex corrections. In fact, there is no divergence for the vertex corrections which are calculated from the massive vector field theory [6].

B.3.2   Equation for Gravitational Field

From the Lagrangian density, we can obtain the equation for the gravitational field from the Lagrange equation. Here, we can safely make the static approximation for the equation of motion, and obtain the equation for the gravitational field $\mathcal{G}_0$ as

$$\boldsymbol{\nabla}^2 \mathcal{G}_0 = mg\rho_g \tag{B.8}$$

where $m\rho_g$ corresponds to the matter density. The coupling constant g is related to the gravitational constant G as

$$G = \frac{g^2}{4\pi}.$$

This equation eq.(B.8) is indeed the Poisson equation for gravity.

B.3.3   Dirac Equation with Gravitational Potential

From the Lagrangian density with gravity and electromagnetic interactions, we can derive the Dirac equation

$$\left[-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + m\beta \left(1 + g\mathcal{G}\right) - \frac{Ze^2}{r} \right] \Psi = E\Psi. \tag{B.9}$$

Further, in case the gravitational force is produced by nucleus with its mass of M, the Dirac equation becomes

$$\left[-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + \left(m - \frac{GmM}{r}\right)\beta - \frac{Ze^2}{r} \right] \Psi = E\Psi \tag{B.10}$$

which is just the equation discussed in the previous section.

B.3.4   Foldy-Wouthuysen Transformation of Dirac Hamiltonian

The Dirac equation with the gravitational interaction

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta\right]\Psi = E\Psi \tag{B.11}$$

can be reduced to the non-relativistic equation in quantum mechanics. This can be done in terms of Foldy-Wouthuysen transformation which is a unitary transformation. Therefore, the transformation procedure is very reliable indeed.

• Foldy-Wouthuysen Transformation :
Here, we start from the Hamiltonian with the gravitational potential

$$H = -i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta. \tag{B.12}$$

This Hamiltonian can be rewritten in terms of the Foldy-Wouthuysen transformation which is somewhat a complicated and tedious procedure involved, though it can be done in a straightforward way [?]. In this case, the non-relativistic Hamiltonian should be obtained as

$$H = m + \frac{\boldsymbol{p}^2}{2m} - \frac{GmM}{r} + \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2 - \frac{1}{2m^2}\frac{GMm}{r^3}(\boldsymbol{s}\cdot\boldsymbol{L}) \tag{B.13}$$

which is kept only up to the order of $\left(\dfrac{\boldsymbol{p}}{m}\right)^2\dfrac{GM}{r}$.

B.3.5   Classical Limit of Hamiltonian with Gravity

Here, we should calculate the classical equation of motion from the non-relativistic Hamiltonian in quantum mechanics. In this case, the Hamiltonian which is only relevant to the present discussion can be written as

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{GmM}{r} + \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2. \tag{B.14}$$

This can be reduced to the Newton equation by making the expectation values of operators in quantum theory in terms of the Ehrenfest theorem. In this case, we approximate the products by the factorization in the following way

$$\left\langle \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2 \right\rangle = \left\langle \frac{1}{2m^2}\frac{GmM}{r} \right\rangle \left\langle \boldsymbol{p}^2 \right\rangle \tag{B.15}$$

which must be a good approximation in the classical mechanics application. In addition, we make use of the Virial theorem

$$\left\langle \frac{\boldsymbol{p}^2}{m} \right\rangle = -\left\langle V \right\rangle. \tag{B.16}$$

Therefore, we finally obtain the following additional potential

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{B.17}$$

which is a new gravitational potential in classical mechanics. The derivation of the additional potential is similar to the Zeeman effects in that both interactions appear in the non-relativistic reduction as the higher order terms of coupling constant.

B.4 Predictions of New Gravity Model

By now, a new gravity model is constructed, and as a byproduct, there appears the additional gravitational potential. This is a very small term, but its effect can be measurable. Indeed, this is the relativistic effect which becomes

$$\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8} \tag{B.18}$$

for the earth revolution around the sun. On the other hand, the leap second of the earth revolution is found to be

$$\left(\frac{\Delta T}{T}\right) \sim 2 \times 10^{-8} \tag{B.19}$$

which is just the same order of magnitude as the relativistic effect. Therefore, as we see later, it is natural that the leap second value can be understood by the additional potential of the new gravity model.

B.4.1 Period Shifts in Additional Potential

In the new gravity model, there appears the additional potential in addition to the normal gravitational potential. In the case of the earth revolution around the sun, this potential is written as

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{B.20}$$

where the second term is the additional potential [6]. Here, G and c denote the gravitational constant and the velocity of light, respectively. m and M correspond to the masses of the earth and the sun, respectively.

● Non-integrable Potential :
It should be important to note that the additional potential should be a non-integrable, and therefore, the treatment should be done in terms of the perturbation theory. In this case, the Newton equation with the perturbative procedure of the additional potential can be solved, and the period T of the revolution is written as

$$\omega T \simeq 2\pi(1 + 2\eta) \tag{B.21}$$

where η is given as

$$\eta = \frac{G^2 M^2}{c^2 R^4 \omega^2}. \tag{B.22}$$

Here, R is the average radius of the earth orbit. The angular velocity ω is related to the period T by

$$\omega = \frac{2\pi}{T}. \tag{B.23}$$

The period shift due to the additional potential becomes

$$\frac{\Delta T}{T} = 2\eta \tag{B.24}$$

which is the delay of the period of the revolution [6, 5]

B.4.2   Period Shifts of Earth Revolution (Leap Second)

In the earth revolution, the orbit radius, the mass of the sun and the angular velocity can be written as

$$R = 1.496 \times 10^{11} \text{ m}, \quad M = 1.989 \times 10^{30} \text{ kg}, \quad \omega = 1.991 \times 10^{-7}. \tag{B.25}$$

In this case, the period shift becomes

$$\frac{\Delta T}{T} = 2\eta \simeq 1.981 \times 10^{-8}. \tag{B.26}$$

Therefore, the period of the earth revolution per year amounts to

$$\Delta T_{N.G.} = 0.621 \quad [\text{s/year}] \tag{B.27}$$

which is a delay. This suggests that the corrections must be necessary in terms of the leap second.

● Leap Second :
In fact, the leap second corrections have been made for more than 40 years. The first leap second correction started from June 1972, and for 40 years, people made corrections of 25 second. Therefore, the average leap second per year becomes

$$\Delta T_{N.G.}^{Obs} \simeq 0.625 \pm 0.013 \quad [\text{s/year}] \tag{B.28}$$

which agrees perfectly with the prediction of eq.(B.27).

● Definition of Newcomb Time :
Newcomb defined the time series of second in terms of the earth revolution period. However, the recent measurement of time in terms of atomic clock turns out to deviate from the Newcomb time [8]. This deviation should be due to the relativistic effects, and indeed this deviation can be understood by the additional potential of gravity.

B.4.3   Mercury Perihelion Shifts

For a long time, people believed that the Mercury perihelion shifts can be understood by the higher order effects of general relativity. However, it is proved that there should be no perihelion shifts for one period of the earth revolution.

Instead, there should be the Mercury perihelion shifts which may arise from the effects of other planets such as Jupiter if we can measure the perihelion shifts for some long period of revolutions. Concerning the Mercury perihelion shifts, however, the measurements as well as the calculations of the effects from other planets should be carried out more carefully. After the calculation of Newcomb in the 19 century, no careful calculation on the perihelion shifts has been done until now.

B.4.4   Retreat of Moon

The moon is also affected by the additional potential of gravity from the earth. The shifts of the moon orbit can be expressed just in the same way as the earth revolution. In this case, η can be written as

$$\eta = \frac{G^2 M^2}{c^2 R^4 \omega^2}. \tag{B.29}$$

Here, R is the radius of the moon orbit. M and ω denote the mass of the earth and the angular velocity, respectively. They are written as

$$R = 3.844 \times 10^8 \ \text{m}, \quad M = 5.974 \times 10^{24} \ \text{kg}, \quad \omega = 2.725 \times 10^{-6} \tag{B.30}$$

Therefore, the period shift becomes

$$\frac{\Delta T}{T} = 2.14 \times 10^{-11}. \tag{B.31}$$

Now, we should carry out the calculation as to how the orbit can be shifted, and the shift of the angle can be written as

$$\Delta \theta = 4\pi \eta. \tag{B.32}$$

Thus, the orbit shift $\Delta \ell_m$ can be written as

$$\Delta \ell_m = R \Delta \theta \simeq 0.052 \ \text{m} \tag{B.33}$$

and therefore, the shift per year becomes

$$\Delta \ell_{m \ (one \ year)} = \Delta \ell_m \times \frac{3.156 \times 10^7}{2.36 \times 10^6} \simeq 69.5 \ \text{cm}. \tag{B.34}$$

● **Calculated Results of Retreat of Moon :**
Since the orbit of the moon is ellipse, the orbit shift can be seen as if it were retreated [9]. The orbit is described by

$$r = \frac{R}{1 + \varepsilon \cos \theta}. \tag{B.35}$$

In addition, the eccentricity is quite small ($\varepsilon = 0.055$) and therefore, we can rewrite the above equation as

$$r \simeq R(1 - \varepsilon \cos \theta). \tag{B.36}$$

Thus, the orbit shift Δr at $\theta \simeq \frac{\pi}{2}$ becomes per year

$$\Delta r \simeq R\Delta\theta \; \varepsilon \simeq \Delta\ell_{m \; (one \; year)} \; \varepsilon \; \simeq \; 3.8 \; \text{cm} \tag{B.37}$$

On the other hand, the observed value of the retreat shift of the moon orbit is

$$\Delta r_m^{obs} \simeq 3.8 \; \text{cm} \tag{B.38}$$

which agrees very well with the prediction.

● **Retreat Shift is not Real! :**
It should be noted that this observation is only possible by making use of the Doppler shift measurement. This is not a direct measurement of the moon orbit distance which is not possible due to the uncertainty of the accuracy of light velocity

$$c = (2.99792458 \; \pm 0.000000012) \times 10^8 \; \text{cm/s}. \tag{B.39}$$

The accuracy of the orbit shift $\Delta r_m^{obs} \simeq 3.8$ cm is at the order of 10^{-10} while the light velocity is measured only up to 10^{-8} accuracy. This means that the shift of the orbit radius is just the instantaneous and apparent effect.

B.5 Summary

The new gravity theory of eq.(B.7)) can naturally lead to the Dirac equation of eq.(B.3). This is very important in modern physics since we have now the Dirac equation with the gravitational potential properly taken into account. This Dirac equation can be reduced to the non-relativistic Hamiltonian which then gives rise to the Newton equation with the gravitational potential, and this new equation should contain a new gravitational potential as the additional potential.

● Massless Scalar Field :
The fact that the gravity is described by the massless scalar field can give rise to some important effects on the non-relativistic reduction. This is in contrast to the Coulomb case, but rather similar to the non-relativistic reduction of the vector potential case. In the non-relativistic reduction of the vector potential term in the Hamiltonian, we find new terms such as Zeeman effects or spin-orbit interactions. In the same way, in the non-relativistic reduction of the scalar potential term in the Hamiltonian, we find the new additional potential. In fact, this new additional potential can reproduce the leap second of the earth revolution.

● Inertial Mass and Gravitational Mass :
From experiments, it is known that the inertial mass and gravitational mass are just the same. This equivalence of two masses is taken to be one of the grounds in constructing the general relativity. On the other hand, this equivalence is derived as a natural consequence in the new gravity model. This is one of the strong reasons why this new gravity model is a correct theory of gravity.

Appendix C

Non-integrable Potential

When the non-integrable potential appears as the small perturbation on the Newton equation, what should be the best way to take into account this small potential effect?

C.1   Non-integrable Potential

Here we discuss the physical effects of the non-integrable potential. The additional potential from the new gravity model has the shape of $\frac{B_0}{r^2}$, and, therefore, we can write the non-integrable potentials into the simple shape in the following way

$$V_a(r) = \frac{q}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{C.1}$$

where

$$q = \begin{cases} -6 & \text{for} \;\; \text{General} \;\; \text{Relativity} \\ 1 & \text{for} \;\; \text{New Gravity} \end{cases} . \tag{C.2}$$

In this case, the differential equation for the orbit with the additional potential becomes

$$\frac{dr}{d\varphi} \;\; = \;\; \frac{\dot{r}}{\dot{\varphi}} = r^2 \sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2}. \tag{C.3}$$

This equation can be solved exactly and the effect due to the correction appears in $\cos\varphi$ term and is written as

$$r = \frac{A_g}{1 + \varepsilon \cos\left(\frac{L_g}{\ell}\varphi\right)} \tag{C.4}$$

where A_g and L_g are given as

$$A_g = \frac{L_g^2}{GMm^2}, \quad L_g \equiv \sqrt{\ell^2 + \frac{qG^2M^2m^2}{c^2}} \equiv \ell\sqrt{1+\eta} \simeq \ell\left(1+\frac{1}{2}\eta\right). \quad (C.5)$$

Here, the η is defined as

$$\eta \equiv \frac{qG^2M^2}{c^2R^4\omega^2} \quad (C.6)$$

which is a very small number. It is around 10^{-8} for the planet motion such as the earth or Mercury.

C.1.1 Effects of Non-integrable Potential on Solution

The solution of eq.(C.4) has a serious problem in that the orbit is not closed. This is quite well known that the potential with the non-integrable shape such as $V_c(r) = \frac{C}{r^2}$ gives rise to the orbit which is not closed. It is, of course, clear that this type of orbits should not happen in nature.

The abnormal behavior of the solution eq.(C.4) can also be seen from the following term

$$\cos\left(\frac{L_g}{\ell}\varphi\right) \simeq \cos(\varphi + \frac{1}{2}\eta\varphi). \quad (C.7)$$

It should be interesting to see that this term cannot be described in terms of the cartesian coordinates of $x = r\cos\varphi$, $y = r\sin\varphi$. In fact, $\cos(\varphi + \frac{1}{2}\eta\varphi)$ term becomes

$$\cos(\varphi + \frac{1}{2}\eta\varphi) = \frac{x}{r}\cos\frac{1}{2}\eta\varphi - \frac{y}{r}\sin\frac{1}{2}\eta\varphi \quad (C.8)$$

and there is no way to transform the $\cos\frac{1}{2}\eta\varphi$ term into x, y coordinates even though we started from this cartesian coordinate. This is very serious since the solution expressed by polar coordinates cannot be written any more in the cartesian coordinates. This is related to the fact that the orbit is not closed due to the non-integrable potential effects.

C.1.2 Discontinuity of Orbit

The effect of the non-integral potential can be further seen as the discontinuity of the orbit trajectory since the orbit is not closed. In order to see this discontinuity

of the orbit, we first start from the orbit solution with the non-integral potential, which is eq.(C.4)

$$r = \frac{A_g}{1 + \varepsilon \cos\left(1 + \frac{1}{2}\eta\right)\varphi}.$$

In this case, we find the radius r at $\varphi = 0$ and $\varphi = 2\pi$ as

$$r = \frac{A_g}{1 + \varepsilon}, \qquad \varphi = 0 \tag{C.9}$$

$$r = \frac{A_g}{1 + \varepsilon \cos \pi\eta}, \qquad \varphi = 2\pi. \tag{C.10}$$

Therefore the difference Δr becomes

$$\Delta r \equiv r_{(\varphi=2\pi)} - r_{(\varphi=0)} \simeq \frac{1}{2} A_g \pi^2 \eta^2 \varepsilon \simeq 0.15 \text{ cm} \tag{C.11}$$

for the Mercury orbit case of the general relativity as an example. This means that the orbit is discontinuous when φ becomes 2π. This is not acceptable for the classical mechanics, and indeed it disagrees with the observation. In addition, eq.(C.4) cannot generate the perihelion shift, and this can be easily seen from the orbit trajectory of eq.(C.4).

C.2 Perturbative Treatment of Non-integrable Potential

Here we should present a perturbative treatment of the non-integrable potential. This must be the only way to reliably treat the non-integrability in classical mechanics.

C.2.1 Integrable Expression

The equation for the orbit determination becomes

$$
\begin{aligned}
\frac{dr}{d\varphi} &= \frac{\dot{r}}{\dot{\varphi}} = r^2 \sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2} \\
&= r^2 \sqrt{1+\eta} \sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}}.
\end{aligned}
\tag{C.12}
$$

Therefore, we can rewrite the above equation as

$$
\sqrt{1+\eta}\, d\varphi = \frac{dr}{r^2 \sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}}}.
\tag{C.13}
$$

Here we note that $\eta = \frac{q}{\ell^2 c^2}\left(GmM\right)^2$ is a very small number which is of the order $\eta \sim 10^{-8}$. Now in order to keep the effect of the non-integrable potential in terms of integrable expression, we should make an approximation as

$$
\sqrt{1+\eta}\, d\varphi \simeq d\varphi.
\tag{C.14}
$$

The reason why we should make this approximation is because we should consider the dynamical effect as the perturbation while the η in the right hand side of eq.(C.13) should only change the value of constants such as E or α in the differential equation. In this way, the equation to determine the orbit becomes

$$
\frac{dr}{d\varphi} = r^2 \sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}}
\tag{C.15}
$$

which gives the right orbit solution. Now the orbit is closed, and the solution can be written as

$$
r = \frac{A_g}{1 + \varepsilon \cos\varphi}
\tag{C.16}
$$

where A_g is given as

$$A_g = \frac{\ell^2}{GMm^2}(1 + \eta).$$
<div align="right">(C.17)</div>

Note that the ε is also changed due to the η term, but here we can safely neglect this effect since it does not play any role for physical observables. Therefore, the effect of the additional potential is to change the radius A_g of the orbit even though this change is very small indeed. Now eq.(C.16) clearly shows that there is no perihelion shift, and this is very reasonable since the additional potential cannot shift the main axis of the orbit.

C.2.2 Higher Order Effect of Perturbation

Here we should estimate the higher order effect of the perturbation in eq.(C.13). Denoting the solution of eq.(C.16) by $r^{(0)}$

$$r^{(0)} = \frac{A_g}{1 + \varepsilon \cos \varphi}$$

and the perturbative part of the radius by r' ($r = r^{(0)} + r'$), we can write the equation for r' as

$$\frac{dr'}{d\varphi} = \frac{1}{2}\eta(r^{(0)})^2 \sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r^{(0)}} - \frac{1}{(r^{(0)})^2}}$$
<div align="right">(C.18)</div>

where the right side depends only on φ. Here, we should make a rough estimation and only consider the case in which the eccentricity ε is zero. In this case, the right side does not depend on the variable ε, and thus we can prove that the right side is zero. Therefore, the higher order correction of r' should be proportional to the eccentricity ε and can be written as

$$r' \simeq C_0 \eta \varepsilon A_g$$
<div align="right">(C.19)</div>

where C_0 should be some numerical constant. For the earth revolution, the value of ε is very small ($\varepsilon \simeq 0.0167$) and thus we can safely ignore this higher order perturbative effect.

Appendix D

Planet Effects on Mercury Perihelion

In this Appendix, we discuss the Mercury perihelion shifts which should come from the gravitational interactions between Mercury and other planets such as Jupiter or Saturn. This calculation can be carried out in the perturbation theory of the Newton dynamics, which is rather new to the classical mechanics. Here, we should compare the numerical results with those calculated by Newcomb in 1898.

D.1 Planet Effects on Mercury Perihelion

The motion of the other planets should affect on the Mercury orbits. However, this is the three body problems, and thus it is not easy to solve the equation of motion in an exact fashion. Here, we develop the perturbative treatment of the other planet motions. Suppose Mercury and the planet (Jupiter) are orbiting around the sun, and in this case, the Lagrangian can be written as

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{GmM}{r} + \frac{1}{2}m_w\dot{\boldsymbol{r}}_w{}^2 + \frac{Gm_wM}{r_w} + \frac{Gmm_w}{|\boldsymbol{r} - \boldsymbol{r}_w|} \qquad (\text{D.1})$$

where $(m, \boldsymbol{r})$ and $(m_w, \boldsymbol{r}_w)$ denote the mass and coordinate of Mercury and the planet, respectively. The last term in the right side of eq.(D.1) is the gravitational potential between Mercury and the planet, and therefore, it should be much smaller than the gravitational force from the sun.

D.1.1   The Same Plane of Planet Motions

Here, we assume that the motion of Mercury and the planet must be in the same plane, and therefore we rewrite the Lagrangian in terms of polar coordinates in two dimensions

$$
\begin{aligned}
L &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GmM}{r} + \frac{1}{2}m_w(\dot{r_w}^2 + r_w^2\dot{\varphi_w}^2) + \frac{Gm_wM}{r_w} \\
&+ \frac{Gmm_w}{\sqrt{r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w)}}.
\end{aligned}
\tag{D.2}
$$

In this case, the Lagrange equation for Mercury can be written as

$$
m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2} - \frac{Gmm_w(r - r_w\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}
\tag{D.3}
$$

$$
\frac{d}{dt}(mr^2\dot{\varphi}) = -\frac{GmMrr_w\sin(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}
\tag{D.4}
$$

$$
m_w\ddot{r_w} = m_wr_w\dot{\varphi}^2 - \frac{Gm_wM}{r_w^2} - \frac{Gmm_w(r_w - r\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}
\tag{D.5}
$$

$$
\frac{d}{dt}(m_wr_w^2\dot{\varphi}) = -\frac{Gm_wMrr_w\sin(\varphi_w - \varphi))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}.
\tag{D.6}
$$

D.1.2   Motion of Mercury

If we ignore the interaction between Mercury and the planet, then the Mercury orbit is just given as the Kepler problem, and the equations of motion become

$$
m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2}
\tag{D.7}
$$

$$
\frac{d}{dt}(mr^2\dot{\varphi}) = 0.
\tag{D.8}
$$

Here, the solution of the orbit trajectory is given as

$$
r = \frac{A}{1 + \varepsilon\cos\varphi}
\tag{D.9}
$$

where A and ε are written as

$$
A = \frac{\ell^2}{m\alpha}, \qquad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}} \quad \text{with} \quad \alpha = GMm
\tag{D.10}
$$

which should be taken as the unperturbed solution of the revolution orbit.

D.2 Approximate Estimation of Planet Effects

Now we should make a perturbative calculation of the many body Kepler problem by assuming that the interaction between Mercury and the planet is sufficiently small. In this case, we can estimate the effects of other planets on the Mercury orbit. Here we write again the equation of motion for Mercury including the gravity from the other planet

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(r - r_w\cos(\varphi - \varphi_w))}{\left(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w)\right)^{\frac{3}{2}}}. \tag{D.11}$$

Now we replace r, r_w by the average orbit radius R, R_w in the last term of the right side, and thus, the equation becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(R - R_w\cos(\varphi - \varphi_w))}{\left(R^2 + R_w^2 - 2RR_w\cos(\varphi - \varphi_w)\right)^{\frac{3}{2}}}. \tag{D.12}$$

Below we present some approximate solution of eq.(D.12).

D.2.1 Legendre Expansion

First we define the last term of eq.(D.12) by F as

$$F(x) \equiv -\frac{Gm_w(R - R_w x)}{\left(R^2 + R_w^2 - 2RR_w x\right)^{\frac{3}{2}}}, \quad \text{with} \quad x = \cos(\varphi - \varphi_w) \tag{D.13}$$

and we make the Legendre expansion

$$F(x) = -\frac{Gm_w R}{\left(R^2 + R_w^2\right)^{\frac{3}{2}}} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{\left(R^2 + R_w^2\right)^{\frac{5}{2}}} x + \cdots . \tag{D.14}$$

Therefore we obtain the equation of motion

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{\left(R^2 + R_w^2\right)^{\frac{5}{2}}} \cos(\varphi - \varphi_w) \tag{D.15}$$

where the constant term is irrelevant and thus we do not write it above.

D.2.2 Iteration Method

Now we employ the iteration method in order to solve eq.(D.15). First we make use of the solution of the Kepler problem

$$\varphi = \varphi^{(0)} + \omega t \tag{D.16}$$
$$\varphi_w = \varphi_w^{(0)} + \omega_w t \tag{D.17}$$

and thus eq.(D.15) becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{Gm_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{D.18}$$

where b and β should be given as

$$b = \varphi^{(0)} - \varphi_w^{(0)}, \quad \beta = \omega - \omega_w. \tag{D.19}$$

D.2.3 Particular Solution

In order to solve eq.(D.18), we assume that the last term is sufficiently small and therefore r may be written in the following shape as

$$r = r^{(0)} + K \frac{Gm_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{D.20}$$

where $r^{(0)}$ denotes the Kepler solution of $r^{(0)} = \frac{A}{1+\varepsilon \cos \varphi}$. Now we insert the solution of eq.(D.20) into eq.(D.18), and we find the solution of K as

$$K = -\frac{1}{\beta^2}. \tag{D.21}$$

Therefore, we obtain the approximate solution as

$$r = r^{(0)} - \frac{Gm_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}} \beta^2} \cos(b + \beta t). \tag{D.22}$$

D.3 Effects of Other Planets on Mercury Perihelion

Therefore we should put the Kepler solution for $r^{(0)}$ and thus the Mercury orbit can be written as

$$
\begin{aligned}
r &= \frac{A}{1 + \varepsilon \cos \varphi} - \frac{Gm_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}} \beta^2} \cos(b + \beta t) \\
&\simeq \frac{A}{1 + \varepsilon \cos \varphi + \frac{Gm_w R_w (R_w^2 - 2R^2)}{R(R^2 + R_w^2)^{\frac{5}{2}} (\omega - \omega_w)^2} \cos(b + \beta t)}
\end{aligned}
\tag{D.23}
$$

where we take $A \simeq R$ and also $\beta = \omega - \omega_w$. Here as for ε_w, we take

$$
\varepsilon_w \equiv \frac{Gm_w}{RR_w^2 (\omega - \omega_w)^2} \frac{\left(1 - \frac{2R^2}{R_w^2}\right)}{\left(1 + \frac{R^2}{R_w^2}\right)^{\frac{5}{2}}}
\tag{D.24}
$$

and using $b + \beta t = \varphi - \varphi_w$, we obtain

$$
r \simeq \frac{A}{1 + \varepsilon \cos \varphi + \varepsilon_w \cos(\varphi - \varphi_w)}.
\tag{D.25}
$$

This equation suggests that the Mercury perihelion may well be affected by the planet motions.

D.3.1 Numerical Evaluations

Now we calculate the Mercury perihelion shifts due to the planet motions such as Jupiter or Venus. In order to do so, we first rewrite $\varepsilon \cos \varphi + \varepsilon_w \cos(\varphi - \varphi_w)$ terms as

$$
\varepsilon \cos \varphi + \varepsilon_w \cos(\varphi - \varphi_w) = c_1 \cos \varphi + c_2 \sin \varphi = \sqrt{c_1^2 + c_2^2} \cos(\varphi + \delta)
\tag{D.26}
$$

where c_1 and c_2 are defined as

$$
\begin{aligned}
c_1 &= \varepsilon + \varepsilon_w \cos \varphi_w \\
c_2 &= \varepsilon_w \sin \varphi_w.
\end{aligned}
$$

(D.27)

(D.28)

Here $\cos \delta$ can be written as

$$
\cos \delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}.
\tag{D.29}
$$

Further, ε_w is much smaller than ε and thus eq.(D.29) becomes

$$\cos \delta = \frac{\varepsilon + \varepsilon_w \cos \varphi_w}{\sqrt{(\varepsilon + \varepsilon_w \cos \varphi_w)^2 + (\varepsilon_w \sin \varphi_w)^2}} \simeq 1 - \frac{1}{2} \left(\frac{\varepsilon_w}{\varepsilon} \right)^2 \sin^2 \varphi_w. \tag{D.30}$$

D.3.2   Average over One Period of Planet Motion

Now we should make the average over one period of planet motion and therefore we find

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \varphi_w \, d\varphi_w = \frac{1}{2}. \tag{D.31}$$

Thus, δ becomes

$$\begin{aligned} \delta &\simeq \frac{\varepsilon_w}{\sqrt{2}\,\varepsilon} \simeq \frac{1}{\sqrt{2}\,\varepsilon} \frac{GM}{R_w^2} \frac{1}{R(\omega - \omega_w)^2} \left(\frac{m_w}{M} \right) \frac{\left(1 - \frac{2R^2}{R_w^2} \right)}{\left(1 + \frac{R^2}{R_w^2} \right)^{\frac{5}{2}}} \\ &\simeq \frac{R_w \, \omega_w^2}{\sqrt{2}\,\varepsilon R\,(\omega - \omega_w)^2} \left(\frac{m_w}{M} \right) \frac{\left(1 - \frac{2R^2}{R_w^2} \right)}{\left(1 + \frac{R^2}{R_w^2} \right)^{\frac{5}{2}}} \end{aligned} \tag{D.32}$$

where the planet orbits are taken to be just the circle, for simplicity.

D.3.3 Numerical Results

In order to calculate the effects of the planet motions on the δ, we first write the properties of planets in Table 1. Here, numbers are shown in units of the earth.

<div align="center">

Table 1

</div>

	Mercury	Venus	Mars	Jupiter	Saturn	Earth	Sun
Orbit Radius	0.387	0.723	1.524	5.203	9.55	1.0	
Mass	0.055	0.815	0.107	317.8	95.2	1.0	332946.0
Period	0.241	0.615	1.881	11.86	29.5	1.0	
ω	4.15	1.626	0.532	0.0843	0.0339	1.0	

In Table 2, we present the calculations of the values δ for one hundred years of averaging and the calculations are compared with the calculated results by Newcomb.

<div align="center">

Table 2   The values of δ for one hundred years

</div>

Planets	Venus	Earth	Mars	Jupiter	Saturn	Sum of Planets
δ by eq.(D.32)	49.7	27.4	0.77	32.1	1.14	111.1
δ by Newcomb	56.8	18.8	0.51	31.7	1.5	109.3

As one sees, the agreement between the present calculation and Newcomb results is surprisingly good [8]. Here we do not verify the calculation of Newcomb for the other planet effects on the Mercury perihelion shifts, and instead we simply employ his calculated results.

D.3.4 Comparison with Experiments

The observed values of the Mercury perihelion shifts are often quoted in some of the old textbooks. However, it should be very difficult to find some reliable numbers of the Mercury perihelion shifts since these values are determined for 100 years of observation period in 19 century. In this respect, the comparison between the calculation and observation should be a homework problem for readers.

Bibliography

[1] J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics", (McGraw-Hill Book Company,1964)

[2] J.J. Sakurai, "Advanced Quantum Mechanics", (addison-Wesley,1967)

[3] Fields and Particles
K. Nishijima, W.A. Benjamin, INC, 1969

[4] Momentum distribution after fragmentation in nucleus nucleus collisions at high energy
T. Fujita and J. Hüfner, Nucl. Phys. A343 (1980) 493

[5] Symmetry and Its Breaking in Quantum Field Theory
T. Fujita, Nova Science Publishers, 2011 (2nd edition)

[6] Fundamental Problems in Quantum Field Theory
T. Fujita and N. Kanda, Bentham Publishers, 2013

[7] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," Annalen der Physik vol. 49, pp. 769−822, März. 1916.

[8] Simon Newcomb, "Tables of the Four Inner Planets", 2nd ed. (Washington: Bureau of Equipment, Navy Dept., 1898).

[9] B.G. Bills and R.D. Ray. (1999), "Lunar Orbital Evolution: A Synthesis of Recent Results" ,
Geophysical Research Letters 26 (19) 3045-3048