Chapter 5

Weak Interactions

Abstract: In this chapter, we first present a brief review of the weak interaction theory. In particular, we discuss why the conserved vector current model had to be modified to a new theory. After that, we clarify the physics of the spontaneous symmetry breaking and then discuss the intrinsic problem of the Higgs mechanism in the Weinberg-Salam model. In addition, we present the calculation of the vertex correction due to the weak vector bosons and show that there is no logarithmic divergence in this vertex corrections. Therefore, there is no need of the renormalization procedure in the weak interaction models with massive vector bosons.

Keywords: CVC theory, vertex corrections of weak vector boson, Weinberg-Salam model, spontaneous symmetry breaking, Higgs mechanism, right propagator of massive vector boson, Lorentz condition

5.1 Introduction

The physics of weak interactions started from the Fermi model of the four fermion interaction Hamiltonian. This model pointed to the essentially correct physics picture of the weak decay processes. However, the four fermion interaction model has a quadratic divergence in the second order perturbation calculations even with a very small coupling constant. Therefore, the model cannot be accepted for a correct theory unless one makes some modifications, even though this model is applied to physical processes with the first order perturbation theory and has made a great success.

At the same time, there were several strong experimental evidences that the four fermion interaction model should be mediated by very heavy bosons, and indeed, the experimental discovery of the weak vector bosons ($W^{\pm},\ Z^0$) was followed. In the mean time, Weinberg and Salam proposed a weak interaction model which is based on the $SU(2) \otimes U(1)$ non-abelian gauge theory. The reason why they employ the gauge theory is simply because they believed that the gauge theory should be renormalizable, though without any foundations.

However, the problem is that this standard model has two serious mistakes. The first one is related to the non-abelian nature of gauge fields in the model Lagrangian density. As we discuss in the previous chapter, the charges of the non-abelian gauge fields are gauge dependent, and therefore they are not physical observables at all. This means that these gauge fields cannot become free particles unless one makes mistakes somewhere within the theoretical framework. The second mistake in the standard model is connected to their treatment of the Higgs mechanism. There, the local gauge invariance is broken by hand in order to give a finite mass to the gauge field at the Lagrangian density level, and this is a wrong procedure. This is mainly based on the fact that the symmetry breaking physics is completely misunderstood at the time of the construction of the theory, and indeed the symmetry breaking physics is only concerned with the property of the interacting vacuum state, and it cannot induce any change of the gauge field properties in the Hamiltonian since the symmetry breaking has nothing to do with the field operators. In reality, the chiral symmetry is never broken spontaneously, as we see below.

In this chapter, we review what is the basic problem of the standard model of the weak interactions. In short, the problem of the Weinberg-Salam model is concerned with the symmetry nature which should be kept at any time in the Lagrangian density, even though the state (here the vacuum state of the interacting field theory model) can find the symmetry property which is different from the one found in the free field theory model. There is nothing surprising since the true vacuum of the interacting Hamiltonian may well have a non-vanishing charge associated with the symmetry of the Hamiltonian while the free field theory model may have zero charge of the symmetry group. On the other hand, the Weinberg-Salam model had to break the symmetry itself at the Lagrangian density level because it started from the local gauge theory whose fields must be always massless, and this is more than a serious defect of the model Hamiltonian, but it is physically a wrong procedure. This clearly indicates that, instead of the Weinberg-Salam model, one should find a new model Hamiltonian with three massive vector bosons from the beginning, and it turns out that this is indeed renormalizable. In fact, there is no logarithmic divergence in the calculations of any physical observables in the new model, and thus one does not have to worry about the renormalization procedure.

5.2 Critical Review of Weinberg-Salam Model

The Weinberg-Salam model has basically two important ingredients. The first one is concerned with the fermion and vector field coupling that leads to the four fermion interaction model in the second order perturbative calculations. This is a very reasonable assumption, and indeed one sees that the model can reproduce almost all of the experimental observations. The second part is the Higgs mechanism which has, in fact, a serious problem in connection with the *unitary gauge* fixing. In this mechanism, the condition of $\phi = \phi^\dagger$ is imposed on the Higgs fields. However, this does not correspond to a proper gauge fixing. Instead, this is simply a procedure for giving a finite but very large mass to a gauge field by breaking the local gauge invariance by hand. This suggests that the starting Lagrangian density of the weak interactions should be reconsidered, and indeed we should start from the three massive vector boson fields from the beginning. The massive vector fields should couple to the fermion currents as the initial ingredients. Here, it is shown that the new renormalization scheme with massive vector bosons has no intrinsic problem, and the massive vector boson fields do not give rise to any divergences for physical observables and therefore we do not need any renormalization procedure.

5.2.1 Spontaneous Symmetry Breaking

Before going to the discussion of the Higgs mechanism, we should clarify the physics of the spontaneous symmetry breaking. The whole idea of the symmetry breaking has been critically examined in the recent textbook [31, 45, 46], and the physics of the spontaneous symmetry breaking is, by now, well understood in terms of the standard knowledge of quantum field theory. In particular, if one wishes to understand the vacuum state in a field theory model of fermions, then one has to understand the structure of the negative energy states of the corresponding field theory model.

The terminology of the spontaneous symmetry breaking is misleading, and one should say that it is incorrectly used. It does not express the right physics of the symmetry breaking [47, 48, 49]. This is simply because the breaking of the symmetry cannot, of course, occur in the Hamiltonian of isolated system [50, 51]. If the symmetry breaking is concerned with the comparison of the vacuum states between the free field theory and the interacting field theory models, then we see that the chiral charge associated with the chiral symmetry transformation in the interacting vacuum state may well have a finite but different charge from the vacuum state of the free field theory which indeed has a zero chiral charge. For the total Hamiltonian $H = H_0 + H_I$, we have the vacuum state $|vac\rangle_{exact}$ which is an eigenstate of H, and the vacuum state may well have the eigenvalue of the chiral charge operator $\hat{Q}_5$ as [31]

$$e^{i\alpha\hat{Q}_5}|vac\rangle_{exact} = e^{in\alpha}|vac\rangle_{exact}$$

where n is ± 1 for the Thirring model. On the other hand, the free vacuum state $|vac\rangle_{free}$ which is an eigenstate of H_0 should have

$$e^{i\alpha\hat{Q}_5}|vac\rangle_{free} = |vac\rangle_{free}.$$

Here, one can see that there is nothing special in this symmetry arguments. The most important of all is that there is no symmetry breaking in the Hamiltonian of H. Only the exact vacuum state has a finite chiral charge, in contrast to the zero chiral charge of the free vacuum state.

On the other hand, some people completely misunderstood this physics of symmetry breaking and thought that the vacuum state of the interacting Hamiltonian itself broke the chiral symmetry [49]. This should arise from the two kinds of misunderstanding in their calculations. The first point is that they made use of the approximation scheme of Bogoliubov transformation, and this approximation method happens to induce a deceptive term which looks like a mass term though its mass is infinite [31]. The second misunderstanding is concerned with the concept of the cutoff momentum, and in fact, their result of the mass term is expressed by the cutoff momentum Λ which should be set to infinity at the end of the calculation. In this respect, it is clear that one cannot discuss its physics by rewriting the Lagrangian density into a new shape. As one knows, the property of the vacuum state should be determined from the eigenstate of the total Hamiltonian in the corresponding field theory model.

In summary, the symmetry of the Hamiltonian can never be spontaneously broken, and the eigenstate of the Hamiltonian should keep the symmetry property, unless the symmetry breaking terms should be added to the Hamiltonian by hand. As we discuss below, the physics of the Higgs mechanism has nothing to do with the property of the vacuum state, and therefore it is not related to the symmetry breaking physics at all [52].

5.2.2 Higgs Mechanism

As we show below, the whole procedure of the Higgs mechanism cannot be justified at all. This is mainly connected to the misunderstanding of the gauge fixing where one degree of freedom of the gauge fields must be reduced in order to solve the equations of motion of the gauge fields. Therefore, one cannot insert the condition of the gauge fixing into the Lagrangian density. This is clear since the Lagrangian density only plays a role for producing the equation of motions. Indeed, the Lagrangian density itself is not directly a physical observable, and the Hamiltonian constructed from the Lagrangian density is most important after the fields are quantized. For the field quantization, one has to make use of the gauge fixing condition which can determine the gauge field A_μ together with the equation of motions. This means that only the final Hamiltonian density is relevant to the description of physical observables, and thus the success of the Glashow-Weinberg-Salam model [53, 54, 55] is entirely due to the final version of the weak Hamiltonian which is not at all the gauge field theory but is a model field theory of the massive vector fields which couple to the fermion currents. The success of the standard model is, of course, due to the fact that it can be reduced to the theory of conserved vector current (CVC).

In this respect, it is very important to examine the renormalizability of the final version of the weak Hamiltonian. Here, we show that the renormalizability of the model field theory can be indeed justified. This is basically due to the fact that there is no divergence in the vertex corrections of fermions due to the massive vector boson propagations once we employ a proper propagator of the massive vector bosons. Here, we briefly review how we can obtain the new propagator of the massive vector boson, and the correct shape of the propagator of the massive vector bosons should be given as [56]

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}. \tag{5.0}$$

This shape is determined by solving the equations of motion for the massive vector bosons. As long as we employ the above propagator, we find that the anomalous magnetic moment of electron due to weak Z^0 bosons does not have any divergences and it is indeed very small number which is consistent with experiment. Thus, one can see that the physical observables with the massive vector boson propagations are all finite and that there are neither conceptual nor technical problems in the renormalization scheme of the massive vector bosons interacting with fermions. Namely, there is no need of the wave function renormalization.

5.2.3 Gauge Fixing

Now we discuss the basic problem of the Higgs mechanism [46]. The Lagrangian density of the Higgs mechanism is given as

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}u_0\left(|\phi|^2 - \lambda^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.1}$$

where

$$D^\mu = \partial^\mu + igA^\mu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{5.2}$$

Here, we only consider the U(1) case since it is sufficient for the present discussions. The above Lagrangian density is indeed gauge invariant, and in this respect, the scalar field may interact with gauge fields in eq.(5.1). However, it should be noted that there is no experimental indication that the fundamental scalar field can interact with any gauge fields in terms of the Lagrangian density of eq.(5.1). In this sense, this is only a toy model. Now, the equations of motion for the scalar field ϕ become

$$\partial_\mu(\partial^\mu + igA^\mu)\phi = -u_0\phi\left(|\phi|^2 - \lambda^2\right) - igA_\mu(\partial^\mu + igA^\mu)\phi \tag{5.3}$$

$$\partial_\mu(\partial^\mu - igA^\mu)\phi^\dagger = -u_0\phi^\dagger\left(|\phi|^2 - \lambda^2\right) + igA_\mu(\partial^\mu - igA^\mu)\phi^\dagger. \tag{5.4}$$

On the other hand, the equation of motion for the gauge field A_μ can be written as

$$\partial_\mu F^{\mu\nu} = gJ^\nu \tag{5.5}$$

where

$$J^\mu = \frac{1}{2}i\left\{\phi^\dagger(\partial^\mu + igA^\mu)\phi - \phi(\partial^\mu - igA^\mu)\phi^\dagger\right\}. \tag{5.6}$$

One can also check that the current J^μ is conserved, that is

$$\partial_\mu J^\mu = 0. \tag{5.7}$$

This Lagrangian density of eq.(5.1) has been employed for the discussion of the Higgs mechanism.

5.2.4 Gauge Freedom and Number of Independent Equations

Now, we should count the number of the degrees of freedom and the number of equations. For the scalar field, we have two independent functions ϕ and $\phi^\dagger$. Concerning the gauge fields A^μ, we have four since there are A^0, A^1, A^2, A^3 fields. Thus, the number of the independent fields is six. On the other hand, the number of equation is five since the equation for the scalar fields is two and the number of the gauge fields is three. This number of three can be easily understood, even though it looks that the independent number of equations in eq.(5.5) is four, but due to the current conservation the number of the independent equations becomes three. This means that the number of the independent functions is six while the number of equations is five, and they are not equal. This is the gauge freedom, and therefore in order to solve the equations of motion, one has to put an additional condition for the gauge field A_μ like the Coulomb gauge which means $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. In this respect, the gauge fixing is simply to reduce the redundant functional variable of the gauge field A_μ to solve the equations of motion, and nothing more than that.

5.2.5 Unitary Gauge Fixing

In the Higgs mechanism, the central role is played by the gauge fixing of the unitary gauge. The unitary gauge means that one takes

$$\phi = \phi^\dagger. \tag{5.8}$$

This is the constraint on the scalar field ϕ even though there is no gauge freedom in this respect. For the scalar field, the phase can be changed, but this does not mean that one can erase one degree of freedom. One should transform the scalar field in the gauge transformation as

$$\phi' = e^{-ig\chi}\phi$$

but one must keep the number of degree of freedom after the gauge transformation. Whatever one fixes the gauge χ, one cannot change the shape of the scalar field ϕ since it is a functional variable and must be determined from the equations of motion. The gauge freedom is, of course, found in the vector potential A_μ as we discussed above. In this sense, one sees that the unitary gauge fixing is a simple mistake [57]. The basic reason why people overlooked this simple-minded mistake must be due to their obscure presentation of the Higgs mechanism. Also, it should be related to the fact that, at the time of presenting the Higgs mechanism, the spontaneous symmetry breaking physics was not understood properly since the vacuum of the corresponding field theory was far beyond the proper understanding. Indeed, the Goldstone boson after the spontaneous symmetry breaking was taken to be almost a mysterious object since there was no experiment which suggests any existence of the Goldstone boson. Instead, a wrong theory prevailed among physicists. Therefore, they could assume a very unphysical procedure of the Higgs mechanism and people pretended that they could understand it all.

5.2.6 Non-abelian Gauge Field

Now, one should be careful for the renormalizability of the non-abelian gauge field theory. As one can easily convince oneself, the non-abelian gauge theory has an intrinsic problem of the perturbation theory [58]. This is connected to the fact that the color charge in the non-abelian gauge field depends on the gauge transformation, and therefore it cannot be physical observables. This means that the free gauge field which has a color charge is gauge dependent, and thus one cannot develop the perturbation theory in a normal way. In QCD, this is exhibited as the experimental fact that both free quarks and free gluons are not observed in nature. The absence of free fields is a kinematical constraint and thus it is beyond any dynamics. Therefore, one cannot discuss the renormalizability of the non-abelian gauge field theory models due to the lack of the perturbation scheme in this model field theory [31, 58]. Therefore, the problem of the renormalizability in the non-abelian field theory model is a meaningless subject since the perturbation theory is not defined in this model field theory.

5.2.7 Summary of Higgs Mechanism

The intrinsic problem of the Higgs mechanism is discussed in terms of the gauge fixing condition. This is also related to the misunderstanding of the spontaneous symmetry breaking physics. Here, we have shown that the Higgs mechanism cannot be justified since the gauge invariance of the Lagrangian density is violated by hand. However, we believe that the final version of the weak Hamiltonian should be correct, and therefore we should discuss the renormalization scheme of the massive vector bosons in detail. As we discuss above, the basic reason why the standard model Hamiltonian becomes a reasonable model is due to the fact that they make mistakes twice and thus it gets back to the right Hamiltonian which can describe the nature. The first mistake is related to the non-abelian character of the gauge field theory model while the second mistake is concerned with the breaking of the local gauge invariance in terms of Higgs mechanism, and it is, of course, an incorrect treatment. Therefore, if we remove the Higgs fields and the non-abelian nature of the massive vector bosons from the Weinberg-Salam model, then the final Hamiltonian of the standard model should be physically acceptable.

At this point, we should make a comment on the present status of the Higgs particle search. At present (January, 2013), there is no indication of the existence of the Higgs particle in spite of the fact that the total period of the experimental efforts of the Higgs search must be almost more then three decades. The main difference between the W and Z bosons and Higgs particle searches can be understood in the following way. The Higgs particle search started from the theoretical requirements (though incorrect) without having any firm experimental motivations of its existence, while the W-boson cases had many experimental indications of their existence before they were discovered by the UA1 and UA2 collaborations of the CERN-SPS experiments in 1983.

5.3 Theory of Conserved Vector Current

It should be important to construct the Lagrangian density which can describe the weak interaction processes. The basic starting point is, of course, the conserved vector current (CVC) theory which can describe most of the observed weak decay processes quite well. This CVC theory should be derived from the second order perturbation theory by exchanging the weak vector bosons between corresponding fermions.

5.3.1 Lagrangian Density of CVC Theory

The theory of the weak interactions is developed in terms of the four fermion interaction model [59] by Fermi and, after some time, Feynman and Gell-Mann extended it to the conserved vector current (CVC) theory, which is quite successful for describing experiments [60, 61, 62, 63]. The Lagrangian density of the CVC theory can be written as

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c.$$

where G_F denotes the weak coupling constant $G_F \simeq 1.2 \times 10^{-5} \frac{1}{M_p^2}$. Also J^μ is composed of the leptonic and hadronic currents and is written as

$$J^\mu = j_\ell^\mu + j_h^\mu$$

where both of the currents can be expressed as

$$j_\ell^\mu = \bar{\psi}_{\nu_e} \gamma^\mu (1-\gamma^5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma^\mu (1-\gamma^5) \psi_\mu + \cdots$$

$$j_h^\mu = \cos\theta \bar{\psi}_u \gamma^\mu (1-\gamma^5) \psi_d + \sin\theta \bar{\psi}_u \gamma^\mu (1-\gamma^5) \psi_s + \cdots .$$

It should be important to note that the current-current interaction model can describe many experimental data to a very high accuracy, and this is, indeed, a well-known fact before the discovery of the weak vector bosons of $W^\pm, Z^0$.

5.3.2 Renormalizability of CVC Theory

However, this model Hamiltonian of CVC theory should have a serious problem related to the divergence in the second order perturbation theory. Since the coupling constant G_F is very small compared to the fine structure constant, one can expect that the second order perturbation must be reliable. On the contrary, however, the second order calculation has a quadratic divergence since the coupling constant G_F has the dimension of the inverse square of the energy. Therefore, it is clear that this theoretical framework should have an intrinsic problem of the divergence, and thus it should be very important to construct a theory which should not have any divergence.

5.3.3 Renormalizability of Non-Abelian Gauge Theory

Now, in order to construct a theory which is renormalizable, it was believed that the gauge field theory should be renormalizable at the time when people discovered the CVC theory. Therefore, it is natural that the non-abelian gauge theory of $SU(2)\otimes U(1)$ was proposed by Weinberg-Salam. However, one sees by now that the non-abelian gauge field has a charge associated with its gauge group, but the charge is not a physical observable since it is gauge dependent. Therefore, there is no way to develop any perturbation theory in this non-abelian gauge field theory. This means that the non-abelian gauge theory has an intrinsic problem before going to the renormalization scheme. [64]

5.4 Lagrangian Density of Weak Interactions

Even though the Higgs mechanism itself has an intrinsic problem, the final Hamiltonian density may well be physically meaningful. This is clear since, from this Hamiltonian density, one can construct the CVC theory which describes the experimental observables quite well.

5.4.1 Massive Vector Field Theory

In this respect, we may write the simplest Lagrangian density for two flavor leptons which couple to the SU(2) vector fields W_μ^a

$$\mathcal{L} = \bar{\Psi}_\ell (i\partial_\mu \gamma^\mu - m)\Psi_\ell - gJ_\mu^a W^{\mu,a} + \frac{1}{2}M^2 W_\mu^a W^{\mu,a} - \frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a} \tag{5.9}$$

where M denotes the mass of the vector boson. Here, we do not write the hadronic part, for simplicity. The lepton wave function Ψ_ℓ has two components

$$\Psi_\ell = \begin{pmatrix} \psi_e \\ \psi_\nu \end{pmatrix}. \tag{5.10}$$

Correspondingly, the mass matrix can be written as

$$m = \begin{pmatrix} m_e & 0 \\ 0 & m_\nu \end{pmatrix}. \tag{5.11}$$

The fermion current J_μ^a and the field strength $G_{\mu\nu}^a$ are defined as

$$J_\mu^a = \bar{\Psi}_\ell \gamma_\mu (1-\gamma_5)\tau^a \Psi_\ell, \qquad G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a. \tag{5.12}$$

This Lagrangian density is almost the same as the standard model Lagrangian density, apart from the Higgs fields and the abelien nature. In fact, there is no experiment in weak process which cannot be described by the Lagrangian density of eq.(5.9). The only thing which, people thought, may be a defect in the above Lagrangian density is concerned with the renormalization of the theory. As we see below, the problem of the renormalization is completely solved by employing the right propagator of the massive vector bosons. This means that we find that there is no logarithmic divergence in the evaluation of the vertex corrections due to the propagations of the massive vector bosons. Therefore, we do not need any renormalization procedure since all the physical observables are calculated to be finite.

5.5 Propagator of Massive Vector Boson

Here, we briefly review the derivation of the new propagator of the massive vector boson which has recently been evaluated properly in terms of the polarization vector [56]. The correct shape of the boson propagator is found to be the one given as

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}. \tag{5.13}$$

This is quite important since this does not generate any quadratic divergences in the self-energy diagrams of fermions any more while the old propagator in the textbooks

$$D^{\mu\nu}_{old}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2 - i\varepsilon}$$

gives rise to the quadratic divergence [31, 15]. This old propagator is obtained by making use of the Green's function method. However, the summation of the polarization vectors cannot be connected to the Green's function as we discuss below, and thus the employment of the old propagator is incorrect if one should treat the physical processes which involve the loop integral.

5.5.1 Lorentz Conditions of $k_\mu \epsilon^\mu = 0$

Here, we briefly explain how we can obtain eq.(5.13). The free Lagrangian density for the vector field Z^μ with its mass M is written as

$$\mathcal{L}_Z = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 Z_\mu Z^\mu$$

with $G^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$. In this case, the equation of motion becomes

$$\partial_\mu(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + M^2 Z^\nu = 0. \tag{5.14}$$

Since the free massive vector boson field should have the following shape of the solution

$$Z^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{3} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\, \epsilon^\mu_{k,\lambda} \left[c_{\boldsymbol{k},\lambda} e^{ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} \right] \tag{5.15}$$

Here, we can insert this solution into eq.(5.14) and obtain the following equation for the polarization vector ϵ^μ

$$(k^2 - M^2)\epsilon^\mu - (k_\nu \epsilon^\nu)k^\mu = 0. \tag{5.16}$$

The condition that there should exist a non-zero solution for the ϵ^μ requires that the determinant of the matrix should be zero, namely

$$\det\{(k^2 - M^2)g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{5.17}$$

This equation can be easily solved, and we find the following equation

$$k^2 - M^2 = 0 \tag{5.18}$$

which is the only physical solution of eq.(5.17). Therefore we insert this solution into eq.(5.16) and obtain the equation for the polarization vector ϵ^μ

$$k_\mu \epsilon^\mu = 0 \tag{5.19}$$

which should always hold. Here, we should note that this process of determining the condition on the wave function of ϵ^μ is just the same as solving the free Dirac equation. Obviously this is the most important process of determining the wave functions in quantum mechanics, and surprisingly, this has been missing in the treatment of determining not only the massive vector boson propagator but also the photon propagator as well. Also, one can notice that the condition of eq.(5.19) is just the same as the Lorentz gauge fixing condition in quantum electrodynamics (QED), and this is often employed as the gauge fixing. However, one sees by now that the Lorentz condition itself can be obtained from the equation of motion, and therefore it is more fundamental than the gauge fixing, even though the theory of massive bosons has no gauge freedom. This indicates that the Lorentz gauge fixing in QED should not be a proper gauge fixing procedure since the Lorentz gauge fixing cannot give a further constraint on the polarization vector in the perturbation theory of QED. In addition, the number of degrees of freedom for the gauge fields can be understood properly since photon must have the two degrees of freedom due to the two constraint equations (the Lorentz condition and the gauge fixing condition).

5.5.2 Right Propagator of Massive Vector Boson

Now, we can evaluate the propagator of the massive vector field in the S-matrix expression. The second order perturbation of the S-matrix for the bosonic part can be written in terms of the T-product of the boson fields and it becomes

$$\langle 0|T\{Z^\mu(x_1)Z^\nu(x_2)\}|0\rangle = i\sum_{\lambda=1}^{3}\int \frac{d^4k}{(2\pi)^4}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}\frac{e^{ik(x_1-x_2)}}{k^2 - M^2 - i\varepsilon}. \tag{5.20}$$

After the summation over the polarization states, we find the following shape for $\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}$ as

$$\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda} = -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{5.21}$$

which satisfies the Lorentz invariance and the condition of the polarization vector $k_\mu \epsilon^\mu = 0$. One sees that this is the only possible solution. From eq.(5.21), one finds that the right propagator of the massive vector boson should be the one given in eq.(5.13)

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}.$$

Here it may be important to note that the polarization vector $\epsilon^\mu_{k,\lambda}$ should depend only on the four momentum k^μ, and it cannot depend on the boson mass at this expression. Later, one may replace the k^2 term by M^2 in case the vector boson is found at the external line. But in the propagator, the replacement of the k^2 term by M^2 is forbidden.

5.5.3 Renormalization Scheme of Massive Vector Fields

In 1970's, people found that some experiments indicate there might be heavy vector bosons exchanged between leptons and baryons in the weak processes. Therefore, people wanted to start from the massive vector bosons. However, it was somehow believed among educated physicists that only gauge field theories must be renormalizable. We do not know where this belief came from. In fact, there is no strong reason that only the gauge field theory is renormalizable. On the contrary, we know by now that only QED may well have a strange divergence in the vertex corrections.

5.6 Vertex Corrections by Weak Vector Bosons

Now we can calculate the vertex correction $\Lambda^\rho(p', p)$ of electromagnetic interaction due to the Z^0 boson. The Lagrangian density for the Z^0 boson which couples to the electron field ψ_e should be written as

$$\mathcal{L}_{Z^0} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} M^2 Z_\mu Z^\mu - g_z \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_e Z^\mu \tag{5.22}$$

where the free Lagrangian density part of electron is not written here for simplicity. This vertex correction is a physical process which can be directly related to the physical observables, and therefore we should be concerned with its divergences. The vertex correction $\Lambda^\rho(p', p)$ can be written by evaluating the corresponding Feynman diagrams as [56]

$$\Lambda^\rho(p', p) = -ig_z^2 e \int \frac{d^4k}{(2\pi)^4} \left(\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon} \right) \gamma_\mu \gamma^5 \frac{1}{\not{p}' - \not{k} - m_e} \gamma^\rho \frac{1}{\not{p} - \not{k} - m_e} \gamma_\nu \gamma^5 \tag{5.23}$$

where only the term corresponding to the $\gamma^5 \gamma_\mu$ is written for simplicity.

5.6.1 No Divergences

First, we show that the apparent logarithmic divergent terms in eq.(5.23) vanish to zero, and this can be easily proved since we can find

$$\Lambda^\rho(p,p) = -ieg_z^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 2xdx \frac{\left(\gamma_\mu \not{k} \gamma^\rho \not{k} \gamma^\mu - \frac{\not{k}\not{k}\gamma^\rho \not{k}\not{k}}{k^2}\right)}{(k^2 - s - i\varepsilon)^3} = 0 \tag{5.24}$$

where $s = M^2(1-x) + m_e^2 x^2$. Therefore, there is no logarithmic divergence for the vertex correction from the weak massive vector boson propagations. This is very important in that the physical processes do not have any divergences when we make use of the proper propagator of the massive vector boson.

5.6.2 Electron $g-2$ by Z^0 Boson

The finite part of the vertex correction due to the Z^0 boson can be easily calculated and, therefore, the electron $g-2$ should be modified by the weak interaction to

$$\frac{g-2}{2} \simeq \frac{7\alpha_z}{12\pi} \left(\frac{m_e}{M}\right)^2 \simeq 2 \times 10^{-14} \tag{5.25}$$

where

$$\alpha_z = \frac{g_z^2}{4\pi} \simeq 2.73 \times 10^{-3}.$$

This is a very small effect, and therefore, it is consistent with the electron $g-2$ experiment. We should note that, if we employed the standard propagator of the massive vector boson as given in the field theory textbooks [15], then we would have obtained a very large effect on the electron $g-2$, even if we had successfully treated the problem of the quadratic and logarithmic divergences in some way or the other, by renormalizing them into the fermion self-energy contributions. This strongly suggests from the point of view of the renormalization scheme that the propagator of the massive vector field should be the one given by eq.(5.13).

5.6.3 Muon $g-2$ by Z^0 Boson

Here, we should also give a calculated value of the muon $g-2$ due to the Z^0 boson since it is just the same formula as eq.(5.24) except the mass of lepton. The result becomes

$$\left(\frac{g-2}{2}\right)_\mu \simeq \frac{7\alpha_z}{12\pi} \left(\frac{m_\mu}{M}\right)^2 \simeq 8.6 \times 10^{-10} \tag{5.26}$$

which is much larger than the electron case. This is, however, still too small to be observed by the muon $g-2$ experiments at the present stage.

Appendix C

Lorentz Conditions

Here, we clarify that the Lorentz condition of $k_\mu \epsilon^\mu = 0$ should be obtained from the equation of motion, and therefore it is more fundamental than the requirement of the gauge fixing condition in QED. For the massive vector bosons, the Lorentz condition plays a fundamental role for determining the polarization sum of the vector boson.

C.1 Gauge Field of Photon

We write the Lagrangian density for the free gauge field as

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{C.1}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In this case, the equation of motion becomes

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \tag{C.2}$$

Since the free photon field should have the following solution

$$A^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{\epsilon^\mu(k,\lambda)}{\sqrt{2V\omega_{\boldsymbol{k}}}} \left[c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda} e^{ikx} \right] \tag{C.3}$$

we can insert this solution into eq.(C.2) and obtain the following equation for $\epsilon^\mu(k,\lambda)$

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{C.4}$$

This equation can be written in terms of the matrix equation for the polarization vector ϵ^μ

$$\sum_{\nu=0}^{3} \{k^2 g^{\mu\nu} - k^\mu k^\nu\} \epsilon_\nu = 0 \tag{C.5}$$

where we write the summation explicitly. In order that the ϵ^μ should have a non-zero solution, the determinant of the matrix should vanish to zero

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{C.6}$$

Now it is easy to prove that $k^2 = 0$ is the only physical solution of eq.(C.6) since one finds

$$\det\{-k^\mu k^\nu\} = 0.$$

Therefore, putting the solution of $k^2 = 0$ into eq.(C.4), we obtain

$$k_\mu \epsilon^\mu = 0 \tag{C.7}$$

which becomes the solution for the polarization vector. Here, we should note that this process of determining the condition on the wave function of ϵ^μ is just the same as solving the free Dirac equation. Obviously this is the most important process of determining the wave functions in quantum mechanics, and surprisingly, this has been missing in the treatment of determining not only the massive vector boson propagator but also the photon propagator as well.

This constraint equation of eq.(C.7) is obtained from the equation of motion, even though it is just the same equation as Lorentz gauge fixing condition. As can be seen by now, the gauge fixing condition is still left for use. In fact, if we take the Coulomb gauge fixing of $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, then we find $\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0$ which leads to the condition of $\epsilon_0 = 0$. Therefore, we now see that the photon field has only two degrees of freedom which can be naturally obtained from the equation of motion and the gauge fixing condition.

In addition, one realizes that the Lorentz gauge fixing is not allowed in the free field gauge theory since the same equation of the Lorentz gauge fixing is already obtained from the equation of motion. Namely, it cannot give a further constraint on the polarization vector. In this respect, one sees that the Coulomb gauge fixing gives a proper condition on the polarization vector.

C.2 Massive Vector Fields

The massive vector field can be treated just in the same manner as above. We first write the free Lagrangian density for the vector boson field Z^μ with its mass M

$$\mathcal{L}_W = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} M^2 Z_\mu Z^\mu \tag{C.8}$$

with $G^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$. In this case, the equation of motion becomes

$$\partial_\mu (\partial^\mu Z^\nu - \partial^\nu Z^\mu) + M^2 Z^\nu = 0. \tag{C.9}$$

Since the free massive boson field should have the following shape of the solution

$$Z^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{3} \frac{\epsilon^\mu(k,\lambda)}{\sqrt{2V\omega_{\boldsymbol{k}}}} \left[c_{\boldsymbol{k},\lambda} e^{ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} \right] \tag{C.10}$$

we can insert this solution into eq.(C.9) and obtain the following equation for the polarization vector ϵ^μ

$$(k^2 - M^2)\epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \qquad (C.11)$$

In the same way as above, we can prove that

$$k^2 - M^2 = 0$$

should hold, and this is the only physical solution of eq.(C.11). Therefore we obtain the following equation for the polarization vector ϵ^μ

$$k_\mu \epsilon^\mu = 0 \qquad (C.12)$$

which should always hold. This is just the same equation as Lorentz gauge fixing condition in QED. However, there is no gauge freedom for the massive vector boson, and therefore the degrees of freedom of the polarization vector ϵ^μ for the massive vector boson is three, in contrast to the gauge field. Now, we understand that the massive vector field should have a spin of $s = 1$ which has indeed three components as we saw above. In this sense, the photon field is special in that it has a spin of $s = 1$ with only two degrees of freedom. This should be directly related to the massless nature of photon which is required from the gauge invariance of the Lagrangian density of the vector field.

Appendix D

Basic Notations in Field Theory

In field theory, one often employs special notations which are by now commonly used. In this Appendix, we explain some of the notations which are particularly useful in field theory calculations.

D.1 Natural Units and Constants

Here, we employ the natural units because of its simplicity

$$c = 1, \quad \hbar = 1. \tag{D.1.1}$$

If one wishes to get the right dimensions out, one should use

$$\hbar c = 197.33 \ \ \mathrm{MeV} \cdot \mathrm{fm}. \tag{D.1.2}$$

For example, pion mass is $m_\pi \simeq 140$ MeV/c^2. Its Compton wave length is

$$\frac{1}{m_\pi} = \frac{\hbar c}{m_\pi c^2} = \frac{197 \text{ MeV} \cdot \text{ fm}}{140 \text{ MeV}} \simeq 1.4 \text{ fm}.$$

Fine structure constant: $\alpha = e^2 = \dfrac{e^2}{\hbar c} = \dfrac{e^2}{4\pi} = \dfrac{e^2}{4\pi\hbar c} = \dfrac{1}{137.036}$.

Some constants:
$$\begin{cases} \text{Electron mass}: \ \ m_e = 0.511 \ \ \mathrm{MeV}/c^2 \\ \text{Muon mass}: \ \ m_\mu = 105.66 \ \ \mathrm{MeV}/c^2 \\ \text{Proton mass}: \ \ M_p = 938.28 \ \ \mathrm{MeV}/c^2 \\ \text{Bohr radius}: \ \ a_0 = \dfrac{1}{m_e e^2} = 0.529 \times \ \ 10^{-8} \ \ \mathrm{cm} \end{cases}$$

Gravitational constant: $G = 5.906 \times 10^{-39}\ \frac{1}{M_p^2}$

Weak coupling Constant: $G_F = 1.166 \times 10^{-5}\ (\mathrm{GeV})^{-2}$

Magnetic moments : $\begin{pmatrix} \text{Electron :} & \mu_e = 1.00115965219 & \dfrac{e\hbar}{2m_e c} \\ \text{Muon :} & \mu_\mu = 1.001165920 & \dfrac{e\hbar}{2m_\mu c} \end{pmatrix}$

Weak bosons : $\begin{cases} W^{\pm} - \text{boson} : M_W = 80.4\ \text{GeV}/c^2, & \alpha_W \simeq 4.3 \times 10^{-3} \\ Z^0 - \text{boson} : M_z = 91.2\ \text{GeV}/c^2, & \alpha_Z \simeq 2.73 \times 10^{-3} \end{cases}$

D.2 Hermite Conjugate and Complex Conjugate

For a complex c-number A

$$A = a + bi \quad (a, b:\ \text{real}). \tag{D.2.1}$$

Its complex conjugate A^* is defined as

$$A^* = a - bi. \tag{D.2.2}$$

Matrix A

If A is a matrix, one defines the hermite conjugate $A^\dagger$

$$(A^\dagger)_{ij} = A^*_{ji}. \tag{D.2.3}$$

Differential Operator $\hat{A}$

If $\hat{A}$ is a differential operator, then the hermite conjugate can be defined only when the Hilbert space and its scalar product are defined. For example, suppose $\hat{A}$ is written as

$$\hat{A} = i\frac{\partial}{\partial x}\,. \tag{D.2.4}$$

In this case, its hermite conjugate $\hat{A}^\dagger$ becomes

$$\hat{A}^\dagger = -i\left(\frac{\partial}{\partial x}\right)^T = i\,\frac{\partial}{\partial x} = \hat{A} \tag{D.2.5}$$

which means $\hat{A}$ is Hermitian. This can be easily seen in a concrete fashion since

$$\langle \psi | \hat{A}\psi \rangle = \int_{-\infty}^{\infty} \psi^\dagger(x) i\,\frac{\partial}{\partial x}\,\psi(x)\,dx = -i\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\,\psi^\dagger(x)\right)\psi(x)\,dx = \langle \hat{A}\psi|\psi\rangle, \tag{D.2.6}$$

where $\psi(\pm\infty) = 0$ is assumed. The complex conjugate of $\hat{A}$ is simply

$$\hat{A}^* = -i\,\frac{\partial}{\partial x} \neq \hat{A}. \tag{D.2.7}$$

Field ψ

If the $\psi(x)$ is a c-number field, then the hermite conjugate $\psi^\dagger(x)$ is just the same as the complex conjugate $\psi^*(x)$. However, when the field $\psi(x)$ is quantized, then one should always take the hermite conjugate $\psi^\dagger(x)$. When one takes the complex conjugate of the field as $\psi^*(x)$, one may examine the time reversal invariance.

D.3 Scalar and Vector Products (Three Dimensions) :

Scalar Product

For two vectors in three dimensions

$$\boldsymbol{r} = (x, y, z) \equiv (x_1, x_2, x_3), \quad \boldsymbol{p} = (p_x, p_y, p_z) \equiv (p_1, p_2, p_3) \tag{D.3.1}$$

the scalar product is defined

$$\boldsymbol{r} \cdot \boldsymbol{p} = \sum_{k=1}^{3} x_k p_k \equiv x_k p_k, \tag{D.3.2}$$

where, in the last step, we omit the summation notation if the index k is repeated twice.

Vector Product

The vector product is defined as

$$\boldsymbol{r} \times \boldsymbol{p} \equiv (x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3, x_1 p_2 - x_2 p_1). \tag{D.3.3}$$

This can be rewritten in terms of components,

$$(\boldsymbol{r} \times \boldsymbol{p})_i = \epsilon_{ijk} x_j p_k, \tag{D.3.4}$$

where ϵ_{ijk} denotes anti-symmetric symbol with

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \quad \text{otherwise} = 0.$$

D.4 Scalar Product (Four Dimensions)

For two vectors in four dimensions,

$$x^\mu \equiv (t, x, y, z) = (x_0, \boldsymbol{r}), \quad p^\mu \equiv (E, p_x, p_y, p_z) = (p_0, \boldsymbol{p}) \tag{D.4.1}$$

the scalar product is defined

$$x \cdot p \equiv Et - \boldsymbol{r} \cdot \boldsymbol{p} = x_0 p_0 - x_k p_k. \tag{D.4.2}$$

This can be also written as

$$x_\mu p^\mu \equiv x_0 p^0 + x_1 p^1 + x_2 p^2 + x_3 p^3 = Et - \boldsymbol{r} \cdot \boldsymbol{p} = x \cdot p, \tag{D.4.3}$$

where x_μ and p_μ are defined as

$$x_\mu \equiv (x_0, -\boldsymbol{r}), \quad p_\mu \equiv (p_0, -\boldsymbol{p}). \tag{D.4.4}$$

Here, the repeated indices of the Greek letters mean the four dimensional summation $\mu = 0, 1, 2, 3$. The repeated indices of the roman letters always denote the three dimensional summation throughout the text.

Metric Tensor

It is sometimes convenient to introduce the metric tensor $g^{\mu\nu}$ which has the following properties

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{D.4.5}$$

In this case, the scalar product can be rewritten as

$$x \cdot p = x^\mu p^\nu g_{\mu\nu} = Et - \boldsymbol{r} \cdot \boldsymbol{p}. \tag{D.4.6}$$

D.5 Four Dimensional Derivatives ∂_μ

The derivative ∂_μ is introduced for convenience

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla} \right), \tag{D.5.1}$$

where the lower index has the positive space part. Therefore, the derivative ∂^μ becomes

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla} \right). \tag{D.5.2}$$

D.5.1 $\hat{p}^\mu$ and Differential Operator

Since the operator $\hat{p}^\mu$ becomes a differential operator as

$$\hat{p}^\mu = (\hat{E}, \hat{\boldsymbol{p}}) = \left(i\frac{\partial}{\partial t},\ -i\boldsymbol{\nabla} \right) = i\partial^\mu$$

the negative sign, therefore, appears in the space part. For example, if one defines the current j^μ in four dimension as

$$j^\mu = (\rho, \boldsymbol{j}),$$

then the current conservation is written as

$$\partial_\mu j^\mu = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = \frac{1}{i}\,\hat{p}_\mu j^\mu = 0. \qquad (D.5.3)$$

D.5.2 Laplacian and d'Alembertian Operators

The Laplacian and d'Alembertian operators, Δ and $\Box$ are defined as

$$\Delta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,,$$
$$\Box \equiv \partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta.$$

D.6 γ-Matrix

Here, we present explicit expressions of the γ-matrices in two and four dimensions. Before presenting the representation of the γ-matrices, we first give the explicit representation of Pauli matrices.

D.6.1 Pauli Matrix

Pauli matrices are given as

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (D.6.1)$$

Below we write some properties of the Pauli matrices.

Hermiticity

$$\sigma_1^\dagger = \sigma_1, \quad \sigma_2^\dagger = \sigma_2, \quad \sigma_3^\dagger = \sigma_3.$$

Complex Conjugate

$$\sigma_1^* = \sigma_1, \quad \sigma_2^* = -\sigma_2, \quad \sigma_3^* = \sigma_3.$$

Transposed

$$\sigma_1^T = \sigma_1, \quad \sigma_2^T = -\sigma_2, \quad \sigma_3^T = \sigma_3 \quad (\sigma_k^T = \sigma_k^*).$$

Useful Relations

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{D.6.2}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{D.6.3}$$

D.6.2 Representation of γ-matrix

(a) Two dimensional representations of γ-matrices

Dirac : $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

Chiral : $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

(b) Four dimensional representations of gamma matrices

Dirac : $\gamma^0 = \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$$

Chiral : $\gamma^0 = \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}.$$

where $\quad \mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

D.6.3 Useful Relations of γ-Matrix

Here, we summarize some useful relations of the γ-matrices.

Anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^5, \gamma^\nu\} = 0. \tag{D.6.4}$$

Hermiticity

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \ \ (\gamma_0^\dagger = \gamma_0, \ \ \gamma_k^\dagger = -\gamma_k), \qquad \gamma_5^\dagger = \gamma_5. \tag{D.6.5}$$

Complex Conjugate

$$\gamma_0^* = \gamma^0, \quad \gamma_1^* = \gamma_1, \quad \gamma_2^* = -\gamma_2, \quad \gamma_3^* = \gamma_3, \quad \gamma_5^* = \gamma_5. \qquad (D.6.6)$$

Transposed

$$\gamma_\mu^T = \gamma^0 \gamma_\mu^\dagger \gamma^0, \quad \gamma_5^T = \gamma_5. \qquad (D.6.7)$$

D.7 Transformation of State and Operator

When one transforms a quantum state $|\psi\rangle$ by a unitary transformation U which satisfies

$$U^\dagger U = 1$$

one writes the transformed state as

$$|\psi'\rangle = U|\psi\rangle. \qquad (D.7.1)$$

The unitarity is important since the norm must be conserved, that is,

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1.$$

In this case, an arbitrary operator $\mathcal{O}$ is transformed as

$$\mathcal{O}' = U\mathcal{O}U^{-1}. \qquad (D.7.2)$$

This can be obtained since the expectation value of the operator $\mathcal{O}$ must be the same between two systems, that is,

$$\langle\psi|\mathcal{O}|\psi\rangle = \langle\psi'|\mathcal{O}'|\psi'\rangle. \qquad (D.7.3)$$

Since

$$\langle\psi'|\mathcal{O}'|\psi'\rangle = \langle\psi|U^\dagger\mathcal{O}'U|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

one finds

$$U^\dagger\mathcal{O}'U = \mathcal{O}$$

which is just eq.(D.7.2).

D.8 Fermion Current

We summarize the fermion currents and their properties of the Lorentz transformation. We also give their nonrelativistic expressions since the basic behaviors must be kept in the nonrelativistic expressions. Here, the approximate expressions are obtained by making use of the plane wave solutions for the Dirac wave function.

$$\text{Fermion currents}: \begin{cases} \text{Scalar}: & \bar{\psi}\psi \simeq 1 \\ \text{Pseudoscalar}: & \bar{\psi}\gamma^5\psi \simeq \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} \\ \text{Vector}: & \bar{\psi}\gamma^\mu\psi \simeq \left(1, \dfrac{\boldsymbol{p}}{m}\right) \\ \text{Axialvector}: & \bar{\psi}\gamma^\mu\gamma^5\psi \simeq \left(\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}, \boldsymbol{\sigma}\right) \end{cases} \tag{D.8.1}$$

Therefore, under the parity $\hat{P}$ and time reversal $\hat{T}$ transformation, the currents behave

$$\text{Parity } \hat{P} \quad : \begin{cases} \bar{\psi}'\psi' = \bar{\psi}\hat{P}^{-1}\hat{P}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_5\hat{P}\psi = -\bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\hat{P}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi \end{cases} \tag{D.8.2}$$

$$\text{Time Reversal } \hat{T}: \begin{cases} \bar{\psi}'\psi' = \bar{\psi}\hat{T}^{-1}\hat{T}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_5\hat{T}\psi = \bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\hat{T}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\gamma_5\hat{T}\psi = -\bar{\psi}\gamma_k\gamma_5\psi \end{cases} \tag{D.8.3}$$

D.9 Trace in Physics

D.9.1 Definition

The trace of $N \times N$ matrix A is defined as

$$\mathrm{Tr}[A] = \sum_{i=1}^{N} A_{ii}. \tag{D.9.1}$$

It is easy to prove

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]. \tag{D.9.2}$$

D.9.2 Trace in Quantum Mechanics

The trace of the Hamiltonian H becomes

$$\mathrm{Tr}[H] = \mathrm{Tr}[UHU^{-1}] = \sum_{n=1} E_n, \tag{D.9.3}$$

where U is a unitary operator, and E_n denotes the energy eigenvalue of the Hamiltonian.

D.9.3 Trace in $SU(N)$

In $SU(N)$, the element U^a can be described in terms of the generator T^a

$$U^a = e^{i\alpha T^a} \tag{D.9.4}$$

where the generator must be hermitian and traceless since

$$\det U^a = \exp\Big(\mathrm{Tr}\,[\ln U^a]\Big) = \exp\Big(i\alpha\,\mathrm{Tr}\,[T^a]\Big) = 1 \tag{D.9.5a}$$

$$\mathrm{Tr}\,[T^a] = 0. \tag{D.9.5b}$$

The generators of $SU(N)$ group satisfy the following commutation relations

$$[T^a, T^b] = iC^{abc}T^c, \tag{D.9.6}$$

where C^{abc} denotes a structure constant. The generators are normalized such that

$$\mathrm{Tr}\,[T^aT^b] = \frac{1}{2}\,\delta^{ab}. \tag{D.9.7}$$

D.9.4 Trace of γ-Matrices and $\not{p}$

Trace of γ-matrices :

$$\mathrm{Tr}\,[1] = 4, \quad \mathrm{Tr}\,[\gamma_\mu] = 0, \quad \mathrm{Tr}\,[\gamma_5] = 0. \tag{D.9.8}$$

Symbol $\not{p}$: $\not{p} \equiv p_\mu\gamma^\mu$

Useful Relations:

$$\gamma_\mu\not{p}\gamma^\mu = -2\not{p} \tag{D.9.9}$$

$$\not{p}\not{q} = p\cdot q - i\sigma_{\mu\nu}p^\mu q^\nu \tag{D.9.10}$$

$$\mathrm{Tr}\,[\not{p}\not{q}] = 4p\cdot q \tag{D.9.11}$$

$$\mathrm{Tr}\,[\gamma_5\not{p}\not{q}] = 0 \tag{D.9.12}$$

$$\mathrm{Tr}\,[\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = 4\Big\{(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\Big\} \tag{D.9.13}$$

$$\mathrm{Tr}\,[\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = -4i\varepsilon_{\alpha\beta\gamma\delta}\,p_1^\alpha\,p_2^\beta\,p_3^\gamma\,p_4^\delta \tag{D.9.14}$$

$$\begin{aligned}\mathrm{Tr}\,[\gamma^5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}] = -4i\,[&g_{\mu_1\mu_2}\varepsilon_{\mu_3\mu_4\mu_5\mu_6} - g_{\mu_1\mu_3}\varepsilon_{\mu_2\mu_4\mu_5\mu_6}\\ &+g_{\mu_2\mu_3}\varepsilon_{\mu_1\mu_4\mu_5\mu_6} + g_{\mu_4\mu_5}\varepsilon_{\mu_1\mu_2\mu_3\mu_6} - g_{\mu_4\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_5} + g_{\mu_5\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}]\end{aligned} \tag{D.9.15}$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'} = - \begin{vmatrix} \delta^{\mu}{}_{\mu'} & \delta^{\mu}{}_{\nu'} & \delta^{\mu}{}_{\alpha'} & \delta^{\mu}{}_{\beta'} \\ \delta^{\nu}{}_{\mu'} & \delta^{\nu}{}_{\nu'} & \delta^{\nu}{}_{\alpha'} & \delta^{\nu}{}_{\beta'} \\ \delta^{\alpha}{}_{\mu'} & \delta^{\alpha}{}_{\nu'} & \delta^{\alpha}{}_{\alpha'} & \delta^{\alpha}{}_{\beta'} \\ \delta^{\beta}{}_{\mu'} & \delta^{\beta}{}_{\nu'} & \delta^{\beta}{}_{\alpha'} & \delta^{\beta}{}_{\beta'} \end{vmatrix} \qquad (D.9.16)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu'\alpha'\beta'} = - \begin{vmatrix} \delta^{\nu}{}_{\nu'} & \delta^{\nu}{}_{\alpha'} & \delta^{\nu}{}_{\beta'} \\ \delta^{\alpha}{}_{\nu'} & \delta^{\alpha}{}_{\alpha'} & \delta^{\alpha}{}_{\beta'} \\ \delta^{\beta}{}_{\nu'} & \delta^{\beta}{}_{\alpha'} & \delta^{\beta}{}_{\beta'} \end{vmatrix} \qquad (D.9.17)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha'\beta'} = -2 \begin{vmatrix} \delta^{\alpha}{}_{\alpha'} & \delta^{\alpha}{}_{\beta'} \\ \delta^{\beta}{}_{\alpha'} & \delta^{\beta}{}_{\beta'} \end{vmatrix} \qquad (D.9.18)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta'} = -6\delta^{\beta}{}_{\beta'} \qquad (D.9.19)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} = -24 \qquad (D.9.20)$$

D.10 Lagrange Equation

In classical field theory, the equation of motion is most important, and it is derived from the Lagrange equation. Therefore, we review briefly how we can obtain the equation of motion from the Lagrangian density.

D.10.1 Lagrange Equation in Classical Mechanics

Before going to the field theory treatment, we first discuss the Lagrange equation (Newton equation) in classical mechanics. In order to obtain the Lagrange equation by the variational principle in classical mechanics, one starts from the action S as defined

$$S = \int L(q, \dot{q})\, dt, \qquad (D.10.1)$$

where the Lagrangian $L(q, \dot{q})$ depends on the general coordinate q and its velocity $\dot{q}$. At the time of deriving equation of motion by the variational principle, q and $\dot{q}$ are independent as the function of t. This is clear since, in the action S, the functional dependence of $q(t)$ is unknown and therefore one cannot make any derivative of $q(t)$ with respect to time t. Once the equation of motion is established, then one can obtain $\dot{q}$ by time differentiation of $q(t)$ which is a solution of the equation of motion. The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of q and $\dot{q}$.

$$\delta S = \int \delta L(q, \dot{q})\, dt = \int \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q} \right) dt$$
$$= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt = 0, \qquad (D.10.2)$$

where the surface terms should vanish. Thus one obtains the Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \qquad (D.10.3)$$

Hamiltonian in Classical Mechanics

The Lagrangian must be invariant under the infinitesimal time displacement ϵ of $q(t)$ as

$$q(t+\epsilon) \to q(t) + \dot{q}\epsilon, \quad \dot{q}(t+\epsilon) \to \dot{q}(t) + \ddot{q}\epsilon + \dot{q}\,\frac{d\epsilon}{dt}\,. \qquad (D.10.4)$$

Therefore, one finds

$$\delta L(q,\dot{q}) = L(q(t+\epsilon), \dot{q}(t+\epsilon)) - L(q,\dot{q}) = \frac{\partial L}{\partial q}\,\dot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\,\ddot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\,\dot{q}\frac{d\epsilon}{dt} = 0. \quad (D.10.5)$$

Since the surface term vanishes, one obtains

$$\delta L(q,\dot{q}) = \left[\frac{\partial L}{\partial q}\,\dot{q} + \frac{\partial L}{\partial \dot{q}}\,\ddot{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right)\right]\epsilon = \left[\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{q}}\,\dot{q}\right)\right]\epsilon = 0 \qquad (D.10.6)$$

where the term in bracket is a conserved quantity, and thus the Hamiltonian H is defined as

$$H \equiv \frac{\partial L}{\partial \dot{q}}\,\dot{q} - L. \qquad (D.10.7)$$

D.10.2 Lagrange Equation for Fields

The Lagrange equation for fields can be obtained almost in the same way as the particle case. For fields, we should start from the Lagrangian density $\mathcal{L}$ and the action is written as

$$S = \int \mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt, \qquad (D.10.8)$$

where $\psi(x)$, $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x_k}$ are independent functional variables. Hereafter, we use the notation of $\dot{\psi}(x) \equiv \frac{\partial \psi}{\partial t}$. The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of ψ, $\dot{\psi}$ and $\frac{\partial \psi}{\partial x_k}$,

$$\delta S = \int \delta\mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt = \int \left(\frac{\partial \mathcal{L}}{\partial \psi}\,\delta\psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\delta\dot{\psi} + \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,\delta\left(\frac{\partial \psi}{\partial x_k}\right)\right) d^3r\, dt$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial}{\partial x_k}\,\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\right)\delta\psi\, d^3r\, dt = 0, \qquad (D.10.9)$$

where the surface terms are assumed to vanish. Therefore, one obtains

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k}\,\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,, \qquad (D.10.10)$$

which can be expressed in the relativistic covariant way as

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right). \qquad (D.10.11)$$

D.11 Noether Current

If the Lagrangian density is invariant under the transformation of the field with a continuous variable, then there is always a conserved current associated with this symmetry. This is called *Noether current* and can be derived from the invariance of the Lagrangian density and the Lagrange equation.

D.11.1 Global Gauge Symmetry

The Lagrangian density which is discussed in this textbook should have the following functional dependence in general

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi + \mathcal{L}_I \left\{ \bar{\psi}\psi, \bar{\psi}\gamma_5\psi, \bar{\psi}\gamma_\mu\psi \right\}$$

which is obviously invariant under the global gauge transformation

$$\psi' = e^{i\alpha}\psi, \quad {\psi'}^\dagger = e^{-i\alpha}\psi^\dagger, \qquad (D.11.1)$$

where α ia a real constant. Therefore, the Noether current is conserved in this system. To derive the Noether current conservation for the global gauge transformation, one can consider the infinitesimal global transformation, that is, $|\alpha| \ll 1$

$$\psi' = \psi + \delta\psi, \quad \delta\psi = i\alpha\psi. \qquad (D.11.2a)$$

$${\psi'}^\dagger = \psi^\dagger + \delta\psi^\dagger, \quad \delta\psi^\dagger = -i\alpha\psi^\dagger. \qquad (D.11.2b)$$

Invariance of Lagrangian Density

Now, it is easy to find

$$\delta\mathcal{L} = \mathcal{L}(\psi', {\psi'}^\dagger, \partial_\mu\psi', \partial_\mu{\psi'}^\dagger) - \mathcal{L}(\psi, \psi^\dagger, \partial_\mu\psi, \partial_\mu\psi^\dagger) = 0 \qquad (D.11.3a)$$

which becomes

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \psi}\,\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\left(\partial_\mu\psi\right) + \frac{\partial \mathcal{L}}{\partial \psi^\dagger}\delta\psi^\dagger + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\delta\left(\partial_\mu\psi^\dagger\right) \\
&= i\alpha\left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\right)\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\mu\psi - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\right)\psi^\dagger - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\partial_\mu\psi^\dagger\right] \\
&= i\alpha\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\psi^\dagger\right] = 0 \qquad (D.11.3b)
\end{aligned}$$

where the equation of motion for ψ is employed.

Current Conservation

Therefore, one defines the current j^μ as

$$j^\mu \equiv -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\psi^\dagger\right] \qquad (D.11.4)$$

and one has the current conservation

$$\partial_\mu j^\mu = 0. \qquad (D.11.5)$$

For Dirac fields, one finds the conserved current

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \qquad (D.11.6)$$

D.11.2 Chiral Symmetry

When the Lagrangian density is invariant under the chiral transformation,

$$\psi' = e^{i\alpha\gamma_5}\psi \qquad (D.11.7)$$

then there is another Noether current. Here, $\delta\psi$ as defined in eq.(D.11.2) becomes

$$\delta\psi = i\alpha\gamma_5\psi. \qquad (D.11.8)$$

Therefore, a corresponding conserved current for massless Dirac fields becomes

$$j_5^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\gamma_5\psi = \bar{\psi}\gamma^\mu\gamma_5\psi \qquad (D.11.9)$$

and we have

$$\partial_\mu j_5^\mu = 0. \qquad (D.11.10)$$

The conservation of the axial vector current holds for massless field theory models.

D.12 Hamiltonian Density

The Hamiltonian density $\mathcal{H}$ is constructed from the Lagrangian density $\mathcal{L}$. If the Lagrangian density is invariant under the translation a^μ, then there is a conserved quantity which is the energy momentum tensor $\mathcal{T}^{\mu\nu}$. The Hamiltonian density is constructed from the energy momentum tensor of $\mathcal{T}^{00}$.

D.12.1 Hamiltonian Density from Energy Momentum Tensor

Now, the Lagrangian density is given as $\mathcal{L}\left(\psi_i, \partial_0\psi_i, \frac{\partial\psi_i}{\partial x_k}\right)$. If one considers the following infinitesimal translation a^μ of the field ψ_i and $\psi_i^\dagger$

$$\psi_i' = \psi_i + \delta\psi_i, \quad \delta\psi_i = (\partial_\nu\psi_i)a^\nu,$$

$$\psi_i^{\dagger\prime} = \psi_i^\dagger + \delta\psi_i^\dagger, \quad \delta\psi_i^\dagger = (\partial_\nu\psi_i^\dagger)a^\nu,$$

then the Lagrangian density should be invariant

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi_i', \partial_\mu\psi_i') - \mathcal{L}(\psi_i, \partial_\mu\psi_i)$$

$$= \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) + \frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,\delta\psi_i^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\delta(\partial_\mu\psi_i^\dagger)\right] = 0. \quad (D.12.1)$$

Making use of the Lagrange equation, one obtains

$$\delta\mathcal{L} = \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,(\partial_\nu\psi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,(\partial_\mu\partial_\nu\psi_i) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial_\nu\psi_i\right)\right]a^\nu$$

$$+ \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,(\partial_\nu\psi_i^\dagger) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,(\partial_\mu\partial_\nu\psi_i^\dagger) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial_\nu\psi_i^\dagger\right)\right]a^\nu$$

$$= \partial_\mu\left[\mathcal{L}g^{\mu\nu} - \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger\right)\right]a_\nu = 0. \quad (D.12.2)$$

Energy Momentum Tensor $\mathcal{T}^{\mu\nu}$

Therefore, if one defines the energy momentum tensor $\mathcal{T}^{\mu\nu}$ by

$$\mathcal{T}^{\mu\nu} \equiv \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger\right) - \mathcal{L}g^{\mu\nu} \quad (D.12.3)$$

then, $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0.$$

This leads to the definition of the Hamiltonian density $\mathcal{H}$ in terms of $\mathcal{T}^{00}$

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i)}\,\partial^0\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i^\dagger)}\,\partial^0\psi_i^\dagger\right) - \mathcal{L}. \quad (D.12.4)$$

D.12.2 Hamiltonian Density for Free Dirac Fields

For a free Dirac field with its mass m, the Lagrangian density becomes

$$\mathcal{L} = \psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma^0 \right]_{ij} \psi_j. \qquad (D.12.5)$$

Therefore, we find the Hamiltonian density as

$$\mathcal{H} = \mathcal{T}^{00} = \bar{\psi}_i \left[-i\gamma_k \partial_k + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \qquad (D.12.6)$$

Hamiltonian for Free Dirac Fields

The Hamiltonian H is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi \, d^3r. \qquad (D.12.7)$$

In classical field theory, this Hamiltonian is not an operator but is just the field energy itself. However, this field energy cannot be evaluated unless one knows the shape of the field $\psi(x)$ itself. Therefore, one should determine the shape of the field $\psi(x)$ by the equation of motion in the classical field theory.

D.12.3 Role of Hamiltonian

The classical field Hamiltonian itself is not useful. This is similar to the classical mechanics case in which one has to derive the Hamilton equations in order to calculate physical properties of the system, and the Hamilton equations are equivalent to the Lagrange equations in classical mechanics.

Classical Field Theory

In classical field theory, the situation is just the same as the classical mechanics case. If one stays in the classical field theory, then one should derive the field equation from the Hamiltonian by the functional variational principle.

Quantized Field Theory

The Hamiltonian of the field theory becomes important when the fields are quantized. In this case, the Hamiltonian becomes an operator, and thus one has to solve the eigenvalue problem for the quantized Hamiltonian $\hat{H}$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \qquad (D.12.8)$$

where $|\Psi\rangle$ is called *Fock state* and should be written in terms of the creation and annihilation operators of fermion and anti-fermion. The space spanned by the Fock states is called *Fock space*. In normal circumstances of the field theory models such as QED and QCD, it is practically impossible to find the eigenstate of the quantized Hamiltonian. The difficulty of the quantized field theory comes mainly from two reasons. Firstly, one has to construct the vacuum state which is composed of infinite many negative energy particles interacting with each other. The vacuum state should be the eigenstate of the Hamiltonian

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle,$$

where E_Ω denotes the energy of the vacuum and it is in general infinity with the negative sign. The vacuum state $|\Omega\rangle$ is composed of infinitely many negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger(s)}_{\boldsymbol{p}}|0\rangle\rangle,$$

where $|0\rangle\rangle$ denotes the null vacuum state. In the realistic calculations, the number of the negative energy particles must be set to a finite value, and this should be reasonable since physical observables should not depend on the deep negative energy particles.

D.13 Variational Principle in Hamiltonian

Now, one can derive the equation of motion by requiring that the Hamiltonian should be minimized with respect to the functional variation of the state $\psi(\boldsymbol{r})$.

D.13.1 Schrödinger Field

When one minimizes the Hamiltonian

$$H = \int \left[-\frac{1}{2m}\psi^\dagger \boldsymbol{\nabla}^2\psi + \psi^\dagger U \psi \right] d^3r \qquad (D.13.1)$$

with respect to $\psi(\boldsymbol{r})$, then one can obtain the static Schrödinger equation.

Functional Derivative

First, one defines the functional derivative for an arbitrary function $\psi_i(\boldsymbol{r})$ by

$$\frac{\delta\psi_i(\boldsymbol{r}')}{\delta\psi_j(\boldsymbol{r})} = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'). \qquad (D.13.2)$$

This is the most important equation for the functional derivative, and once one accepts this definition of the functional derivative, then one can evaluate the functional variation just in the same way as normal derivative of the function $\psi_i(\boldsymbol{r})$.

Functional Variation of Hamiltonian

For the condition on $\psi(\boldsymbol{r})$, one requires that it should be normalized according to

$$\int \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, d^3r = 1. \qquad (D.13.3)$$

In order to minimize the Hamiltonian with the above condition, one can make use of the Lagrange multiplier and make a functional derivative of the following quantity with respect to $\psi^\dagger(\boldsymbol{r})$

$$H[\psi] = \int \left[-\frac{1}{2m}\, \psi^\dagger(\boldsymbol{r}')\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + \psi^\dagger(\boldsymbol{r}')U\psi(\boldsymbol{r}')\right] d^3r' \\ -E\left(\int \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}')\, d^3r' - 1\right), \qquad (D.13.4)$$

where E denotes a Lagrange multiplier and just a constant. In this case, one obtains

$$\frac{\delta H[\psi]}{\delta\psi^\dagger(\boldsymbol{r})} = \int \delta(\boldsymbol{r}-\boldsymbol{r}')\left[-\frac{1}{2m}\,\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + U\psi(\boldsymbol{r}') - E\psi(\boldsymbol{r}')\right] d^3r' = 0. \qquad (D.13.5)$$

Therefore, one finds

$$-\frac{1}{2m}\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) + U\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \qquad (D.13.6)$$

which is just the static Schrödinger equation.

D.13.2 Dirac Field

The Dirac equation for free field can be obtained by the variational principle of the Hamiltonian eq.(D.12.7). Below, we derive the static Dirac equation in a concrete fashion by the functional variation of the Hamiltonian.

Functional Variation of Hamiltonian

For the condition on $\psi_i(\boldsymbol{r})$, one requires that it should be normalized according to

$$\int \psi_i^\dagger\psi_i(\boldsymbol{r})\, d^3r = 1. \qquad (D.13.7)$$

Now, the Hamiltonian should be minimized with the condition of eq.(D.13.7)

$$H[\psi_i] = \int \psi_i^\dagger(\boldsymbol{r})\left[-i(\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})_{ij} + m(\gamma^0)_{ij}\right]\psi_j(\boldsymbol{r})\, d^3r \\ -E\left(\int \psi_i^\dagger(\boldsymbol{r})\psi_i(\boldsymbol{r})\, d^3r - 1\right), \qquad (D.13.8)$$

where E is just a constant of the Lagrange multiplier. By minimizing the Hamiltonian with respect to $\psi_i^\dagger(\boldsymbol{r})$, one obtains

$$(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta)\,\psi(\boldsymbol{r}) - E\psi(\boldsymbol{r}) = 0 \qquad (D.13.9)$$

which is just the static Dirac equation for free field.

Appendix E

Wave Propagations in medium and vacuum

The classical wave such as sound can propagate through medium. However, it cannot propagate in vacuum as is well known. This is, of course, clear since the classical wave is the chain of the oscillations of the medium due to the pressure on the density.

On the other hand, quantum wave including photon can propagate in vacuum since it is a particle. Here, we clarify the difference in propagations between the classical wave and quantum wave. The most important point is that the classical wave should be always written in terms of real functions while photon or quantum wave should be described by the complex wave function of the shape e^{ikx} since it should be an eigenstate of the momentum.

This part is written as Appendix to the field theory text book “Fundamental problems in quantum field theory” published in Bentham publishers in 2013.

E.1 What is wave ?

The sound can propagate through medium such as air or water. The wave can be described in terms of the amplitude ϕ in one dimension

$$\phi(x,t) = A_0 \sin(\omega t - kx) \tag{E.1}$$

where ω and k denote the frequency and wave number, respectively. The dispersion relation of this wave can be written as

$$\omega = vk. \tag{E.2}$$

Here, it is important to note that the amplitude is written as the real function, in contrast to the free wave function of electron in quantum mechanics. In fact, the free wave of electron can be described in one dimension as

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - kx)} \tag{E.3}$$

which is a complex function. The electron can propagate by itself and there is no medium necessary for the electron motion.

What is the difference between the real wave amplitude and the complex wave function? Here, we clarify this point in a simple way though this does not contain any new physics.

E.1.1 A real wave function: Classical wave

If the amplitude is real such as (E.1), then it can only propagate in medium. This can be clearly seen since the energy of the wave can be transported in terms of the density oscillation which is a real as the physical quantity. In addition, the amplitude becomes zero at some point, and this is only possible when it corresponds to the oscillation of the medium. This means that the wave function of (E.1) has nothing to do with the probability of wave object. Instead, if it is the oscillation of the medium, then it is easy to understand why one finds the point where the amplitude vanishes to zero. The real amplitude is called a classical wave since it is indeed seen in the world of the classical physics.

E.1.2 A complex wave function: Quantum wave

On the other hand, the free wave function of electron is a complex function, and there is no point where it can vanish to zero. Since this is just the wave function of electron, its probability of finding the wave is always a constant $\frac{1}{V}$ at any space point of volume V.

E.2 Classical wave

The sound propagates in the air, and its propagation should be transported in terms of density wave. The amplitude of this wave can be written in terms of the real function as given in eq.(E.1). This is quite reasonable since the density wave should be described by the real physical quantity. Instead, this requires the existence of the medium (air), and the wave can propagate as long as the air exists. Here, we first write the basic wave equation in one dimension

$$\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} \tag{E.4}$$

which is similar to the wave equation in quantum mechanics, though it is a real differential equation. Here, v denotes the speed of wave.

E.2.1 Classical waves carry their energy ?

In this case, a question may arise as to what is a physical quantity which is carried by the classical wave like sound. It seems natural that the wave carries its energy (or wave length). In fact, the transportation of the energy should be carried out by the compression of the density and successive oscillations of the medium. Therefore this is called compression wave.

E.2.2 Longitudinal and transverse waves

Here, we discuss the terminology of the longitudinal and transverse waves, even though one should not stress its physics too much since there is no special physical meaning.

• **Longitudinal wave :** The sound propagates as the compressional wave, and the oscillations should be always in the direction of the wave motion. In this case, it is called longitudinal wave. This wave can be easily understood since one can make a picture of the density wave.

• **Transverse wave :** On the other hand, if the motion of the oscillations is in the perpendicular to the direction of the wave motion, then it is called transverse wave. The tidal wave may be the transverse wave, but its description may not be very simple since the density change may not directly be related to the wave itself.

E.3 Quantum wave

Photon and quantum wave are quite different from the classical wave, and the quantum wave is a particle motion itself. No medium oscillation is involved. For example, a free electron moves with the velocity v in vacuum, and this motion is also called "wave". The reason why we call it wave is due to the fact that the equation of motion that describes electrons looks similar to the classical wave equation of motion. Further, the solution of the wave equation can be described as e^{ikx}, and thus it is the same as the wave behavior in terms of mathematics. But the physical meaning is completely different from the classical wave, and quantum wave is just the particle motion which behaves as the probabilistic motion.

E.3.1 Quantum wave (electron motion)

The wave function of a free electron in one dimension can be described as

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \tag{E.5}$$

which is a solution of the Schrödinger equation of a free electron,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\boldsymbol{\nabla}^2\psi \tag{E.6}$$

where $k = \sqrt{2m\omega}$, and V denotes the corresponding volume. Since the Schrödinger equation is quite similar to the wave equation in a classical sense, one calls the solution of the

Schrödinger equation as a wave. However, the physics of the quantum wave should be understood in terms of the quantum mechanics, and the relation to the classical wave should not be stressed. That is, the quantum wave is completely different from the classical wave, and one should treat the quantum wave as it is. In addition, the behavior and physics of the classical wave are very complicated and it is clear that we do not fully understand the behavior of the classical wave since it involves many body problems in physics.

E.3.2 Photon

The electromagnetic wave is called photon which behaves like a particle and also like a wave. This photon can propagate in vacuum and thus it should be considered to be a particle. Photon can be described by the vector potential $\boldsymbol{A}$.

• **$\boldsymbol{A}$ is real ! :** However, this $\boldsymbol{A}$ is obviously a real function, and therefore, it cannot propagate like a particle. This can be easily seen since the free Hamiltonian of photon commutes with the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$, and therefore it can be a simultaneous eigenstate of the Hamiltonian. Thus, the $\boldsymbol{A}$ should be an eigenstate of the momentum operator since the free state must be an eigenstate of momentum. However, any real function cannot be an eigenstate of the momentum operator, and thus the vector field in its present shape cannot describe the free particle state.

• **Free solution of vector field :** What should we do ? The only way of solving this puzzle is to quantize a photon field. First, the solution of $\boldsymbol{A}$ can be written as

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k},\lambda} \frac{1}{\sqrt{2\omega_k V}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \left(c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda} e^{ikx} \right) \tag{E.7}$$

with $kx \equiv \omega_k t - \boldsymbol{k}\cdot\boldsymbol{r}$. Here, $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ denotes the polarization vector which will be discussed later more in detail. As one sees, the vector field is indeed a real function.

• **Quantization of vector field :** Now we impose the following quantization conditions on $c^\dagger_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}$

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'}, \tag{E.8}$$

$$[c_{\boldsymbol{k},\lambda},\ c_{\boldsymbol{k}',\lambda'}] = 0, \qquad [c^\dagger_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = 0. \tag{E.9}$$

In this case, $c^\dagger_{\boldsymbol{k},\lambda},\ c_{\boldsymbol{k},\lambda}$ become operators. Therefore, one should now consider the Fock space on which they can operate. This can be defined as

$$c_{\boldsymbol{k},\lambda}|0\rangle = 0 \tag{E.10}$$

$$c^\dagger_{\boldsymbol{k},\lambda}|0\rangle = |\boldsymbol{k},\lambda\rangle \tag{E.11}$$

where $|0\rangle$ denotes the vacuum state of the photon field. Therefore, if one operates the vector field on the vacuum state, then one obtains

$$\langle \boldsymbol{k},\lambda|\boldsymbol{A}(x)|0\rangle = \frac{1}{\sqrt{2\omega_k V}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} e^{-ikx}. \tag{E.12}$$

As one sees, this new state is indeed the eigenstate of the momentum operator and should correspond to the observables. Therefore, photon can be described only after the vector field is quantized. Thus, photon is a particle whose dispersion relation becomes

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|. \tag{E.13}$$

E.4 Polarization vector of photon

Until recently, there is a serious misunderstanding for the polarization vector $\epsilon^{\mu}_{\boldsymbol{k},\lambda}$. This is related to the fact that the equation of motion for the polarization vector is not solved, and thus there is one condition missing in the determination of the polarization vector.

E.4.1 Equation of motion for polarization vector

Now the equation of motion for $A^{\mu} = (A^0, \boldsymbol{A})$ without any source terms can be written from the Lagrange equation as

$$\partial_\mu F^{\mu\nu} = 0 \tag{E.14}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This can be rewritten as

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0. \tag{E.15}$$

Now, the shape of the solution of this equation can be given as

$$A^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \epsilon^{\mu}_{\boldsymbol{k},\lambda} \left[c_{\boldsymbol{k},\lambda} e^{-ikx} + c^{\dagger}_{\boldsymbol{k},\lambda} e^{ikx} \right] \tag{E.16}$$

and thus we insert it into eq.(E.15) and obtain

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{E.17}$$

Now the condition that there should exist non-zero solution of $\epsilon^{\mu}_{\boldsymbol{k},\lambda}$ is obviously that the determinant of the matrix in the above equation should vanish to zero, namely

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{E.18}$$

This leads to $k^2 = 0$, which means $k_0 \equiv \omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. This is indeed a proper dispersion relation for photon.

E.4.2 Condition from equation of motion

Now we insert the condition of $k^2 = 0$ into eq.(E.17), and obtain

$$k_\mu \epsilon^\mu = 0 \tag{E.19}$$

which is a new constraint equation obtained from the basic equation of motion. Therefore, this condition (we call it "Lorentz condition") is most fundamental. It should be noted that the Lorentz gauge fixing is just the same as eq.(E.19). This means that the Lorentz gauge fixing is improper and forbidden for the case of no source term. In this sense, the best gauge fixing should be the Coulomb gauge fixing

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0 \tag{E.20}$$

from which one finds $\epsilon_0 = 0$, and this is indeed consistent with experiment.

• **Number of freedom of polarization vector :** Now we can understand the number of degree of freedom of the polarization vector. The Lorentz condition $k_\mu \epsilon^\mu = 0$ should give one constraint on the polarization vector, and the Coulomb gauge fixing $\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0$ gives another constraint. Therefore, the polarization vector has only two degrees of freedom, which is indeed an experimental fact.

• **State vector of photon :** The state vector of photon is already discussed. But here we should rewrite it again. This is written as

$$\langle \boldsymbol{k}, \lambda | \boldsymbol{A}(x) | 0 \rangle = \frac{\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}}{\sqrt{2\omega_k V}} e^{-ikx}. \tag{E.21}$$

In this case, the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ has two components, and satisfies the following conditions

$$\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda'} = \delta_{\lambda,\lambda'}, \quad \boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0. \tag{E.22}$$

E.4.3 Photon is a transverse wave ?

People often use the terminology of transverse photon. Is it a correct expression ? By now, one can understand that the quantum wave is a particle motion, and thus it has nothing to do with the oscillation of the medium. Therefore, it is meaningless to claim that photon is a transverse wave. The reason of this terminology may well come from the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ which is orthogonal to the direction of photon momentum. However, as one can see, the polarization vector is an intrinsic property of photon, and it does not depend on space coordinates.

• **No rest frame of photon ! :** In addition, there is no rest frame of photon, and therefore, one cannot discuss its intrinsic property unless one fixes the frame. Even if one says that the polarization vector is orthogonal to the direction of the photon momentum, one has to be careful in which frame one discusses this property.

In this respect, it should be difficult to claim that photon behaves like a transverse wave. Therefore, one sees that photon should be described as a massless particle which has two degrees of freedom with the behavior of a boson. There is no correspondence between classical waves and photon, and even more, there is no necessity of making analogy of photon with the classical waves.

E.5 Poynting vector and radiation

We have clarified that the propagation of the real function requires some medium which can make oscillations. Here, we discuss the Poynting vector how it appears in physics, and show that it cannot propagate in vacuum at all. Also, we present a brief description of the basic radiation mechanism how photon can be emitted.

E.5.1 Field energy and radiation of photon

Before discussing the propagation of the Poynting vector, we should first discuss the mechanism of the radiation of photon in terms of classical electrodynamics. The interaction Hamiltonian can be written as

$$H_I = -\int \boldsymbol{j} \cdot \boldsymbol{A}\, d^3r \tag{E.23}$$

which should be a starting point of all the discussions. Now, we make a time derivative of the interaction Hamiltonian and obtain

$$W \equiv \frac{dH_I}{dt} = -\int \left[\frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} + \boldsymbol{j} \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right] d^3r. \tag{E.24}$$

Since we can safely set $A^0 = 0$ in this treatment, we find

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}. \tag{E.25}$$

Therefore, we can rewrite eq.(E.24) as

$$W = \int \boldsymbol{j} \cdot \boldsymbol{E}\, d^3r - \int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A}\, d^3r. \tag{E.26}$$

Defining the first term of eq.(E.24) as W_E, we can rewrite W_E as

$$W_E \equiv \int \boldsymbol{j} \cdot \boldsymbol{E}\, d^3r = -\frac{d}{dt}\left[\int \left(\frac{1}{2\mu_0}|\boldsymbol{B}|^2 + \frac{\varepsilon_0}{2}|\boldsymbol{E}|^2 \right) d^3r \right] - \int \boldsymbol{\nabla} \cdot \boldsymbol{S}\, d^3r \tag{E.27}$$

which is just the energy of electromagnetic fields.

E.5.2 Poynting vector

Here, the last term of eq.(E.27) is Poynting vector $\boldsymbol{S}$ as defined by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} \tag{E.28}$$

which is connected to the energy flow of the electromagnetic field. This Poynting vector is a conserved quantity, and thus it has nothing to do with the electromagnetic wave. In addition, it is a real quantity, and thus there is no way that it can propagate in vacuum.

In addition, the Poynting vector cannot be a target of the field quantization, and thus it always remains classical since it is written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. However, there is still some misunderstanding in some of the textbooks on Electromagnetism, and therefore, one should be careful for the treatment of the Poynting vector.

• **Exercise problem:** Here, we present a simple exercise problem of circuit with condenser with C (disk radius of a and distance of d) and resistance with R. The electric potential difference V is set on the circuit. In this case, the equation for the circuit can be written as

$$V = R\frac{dQ}{dt} + \frac{Q}{C}.$$

This can be easily solved with the initial condition of $Q = 0$ at $t = 0$, and the solution becomes

$$Q = CV\left(1 - e^{-\frac{t}{RC}}\right).$$

Therefore, the electric current J becomes

$$J = \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}.$$

In this case, we find the electric field $\boldsymbol{E}$ and the displacement current $\boldsymbol{j}_d$

$$\boldsymbol{E} = \frac{Q}{\pi a^2}\boldsymbol{e}_z = \frac{VC}{\varepsilon_0 \pi a^2}\left(1 - e^{-\frac{t}{RC}}\right)\boldsymbol{e}_z \tag{E.29}$$

$$\boldsymbol{j}_d = \frac{\partial \boldsymbol{E}}{\partial t} = \frac{V}{R\pi a^2}e^{-\frac{t}{RC}}\boldsymbol{e}_z. \tag{E.30}$$

Thus, the magnetic field $\boldsymbol{B}$ becomes

$$\boldsymbol{B} = \frac{i_d\, r}{2}\boldsymbol{e}_\theta = \frac{r}{2\pi a^2 R}\, e^{-\frac{t}{RC}}\, \boldsymbol{e}_\theta$$

where $\int_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 i_d \pi r^2$ is used. Therefore, the Poynting vector at the surface (with $r = a$) of the cylindrical space of the disk condenser becomes

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} = -\frac{V^2}{2\pi a R d}\, e^{-\frac{t}{RC}}\left(1 - e^{-\frac{t}{RC}}\right)\boldsymbol{e}_r.$$

It should be noted that the energy in the Poynting vector is always flowing into the cylindrical space. Therefore, the electric field energy is now accumlated in the cylindrical space. There is, of course, no electromagnetic wave radiation, and in fact, the Poynting vector is the flow of field energy, and has nothing to do with the electromagnetic wave.

E.5.3 Emission of photon

The emission of photon should come from the second term of eq.(E.26) which can be defined as W_R and thus

$$W_R = -\int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A}\, d^3r. \tag{E.31}$$

In this case, we can calculate the $\frac{\partial \boldsymbol{j}}{\partial t}$ term by employing the Zeeman effect Hamiltonian with a uniform magnetic field of $\boldsymbol{B}_0$

$$H_Z = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0. \tag{E.32}$$

The relevant Schrödinger equation for electron with its mass m_e becomes

$$i\frac{\partial \psi}{\partial t} = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0\, \psi. \tag{E.33}$$

Therefore, we find

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e}{m_e}\left[\frac{\partial \psi^\dagger}{\partial t}\hat{\boldsymbol{p}}\psi + \psi^\dagger \hat{\boldsymbol{p}}\frac{\partial \psi}{\partial t}\right] = -\frac{e^2}{2m_e^2}\boldsymbol{\nabla} B_0(\boldsymbol{r}). \tag{E.34}$$

In order to obtain the photon emission, one should quantize the field $\boldsymbol{A}$ in eq.(E.31).

• **Field quantization :** The field quantization in electromagnetic interactions can be done only for the vector potential $\boldsymbol{A}$. The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are classical quantities which are defined before the field quantization.

E.6 Gravitational wave

People often discuss the gravitational wave which is supposed to come from the Einstein equation. In this case, one sees that the equation for the metric tensor is all real, and thus the solution of this equation must be also real. Therefore, the gravitational wave, if at all exists, is a real function, and thus it cannot propagate in vacuum unless one believes the aether hypothesis.

• **No quantization of gravity :** In addition, there is no physical meaning to quantize the metric tensor and therefore, there is no chance that the gravitational wave propagates in vacuum.

E.6.1 General relativity

Since we treat the gravitational wave, we should make a comment on the general relativity. Einstein invented the general relativity which is the second order differential equation for the metric tensor $g^{\mu\nu}$. A question may arise as to why the general relativity can be related to the gravitational theory. This reason is simply because Einstein claimed that he had proved the gravitational Poisson equation should be derived from the general relativity at the weak gravitational limit. However, in his proof, he assumed the following strange equation

$$g^{00} \simeq 1 + 2\phi \tag{E.35}$$

where ϕ denotes the gravitational field. Because of this equation (E.35), he could derive the gravitational Poisson equation

$$\boldsymbol{\nabla}^{\mathbf{2}}\phi(\boldsymbol{r}) = 4\pi G\rho(\boldsymbol{r}) \tag{E.36}$$

where G and ρ denote the gravitational constant and the density, respectively.

• **Eq.(E.35) is correct ? :** Here, we show that eq.(E.35) is not only strange but simply incorrect. In order to do so, we should examine the physical meaning of the equation $g^{00} \simeq 1 + 2\phi$. We should notice that 1 (unity) in the right hand side of eq.(E.35) is a simple number. This is clear since the metric tensor is just the coordinate system itself. However, the gravitational field ϕ is a dynamical variable, and therefore this summation of two different categories is simply meaningless.

• **No connection between general relativity and gravity :** By now it should be clear that the general relativity has nothing to do with gravity. It is a theory for the coordinate system (metric tensor), but it is not a theory for nature.

Note :
The new gravitational theory is explained in detail in Chapter 6 in the text book of "Fundamental problems in quantum field theory" .

Reference :

Fundamental Problems in Quantum Field Theory
T. Fujita and N. Kanda, Bentham Publishers, 2013

Bibliography

[1] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, ”Relativistic Quantum Theory”, (Pergamon Press, 1974)

[2] R.P. Feynman, “Space-Time Approach to Quantum Electrodynamics,” Phys. Rev. vol. 76, pp. 769−789, Sep. 1949.

[3] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Fields”, (McGraw-Hill Book Company,1965)

[4] J.J. Sakurai, ”Advanced Quantum Mechanics”, (addison-Wesley,1967)

[5] J. Schwinger, “Quantum Electrodynamics. I. A Covariant Formulation,” Phys. Rev. vol. 74, pp. 1439−1461, Nov. 1948.

[6] S. Tomonaga, “On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields,” Prog. Theor. Phys. vol. 1, pp. 27−42, Aug-Sept. 1946.

[7] G. ’t Hooft and M. Veltman, “Regularization and renormalization of gauge fields,” Nucl. Phys. B. vol. 44, pp. 189−213, 1972.

[8] G. ’t Hooft and M. Veltman, “Combinatorics of gauge fields,” Nucl. Phys. B. vol. 50, pp. 318−353, 1972.

[9] T. Fujita, N. Kanda, H. Kato, H. Kubo, Y. Munakata, S. Oshima and K. Tsuda, “New Renormalization Scheme of Vacuum Polarization in QED”, arXiv:0901.3421

[10] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Mechanics”, (McGraw-Hill Book Company,1964)

[11] W. Pauli and F. Villars, “On the Invariant Regularization in Relativistic Quantum Theory,” Rev. Mod. Phys. vol. 21, pp. 434−444, Sep. 1949.

[12] K. Nishijima, “Fields and Particles”, (W.A. Benjamin, INC, 1969)

[13] J.C. Ward, “An Identity in Quantum Electrodynamics,” Phys. Rev. vol. 78, pp. 182−182, Apr. 1950.

[14] N. Kanda, “Light-Light Scattering”, hep-ph/1106.0592

[15] F. Mandl and G. Shaw, “Quantum Field Theory”, (John Wiley & Sons, 1993)

[16] T. Fujita and N. Kanda, “A Proposal to Measure Photon-Photon Scattering”, hep-ph/1106.0465

[17] H. Euler, “ Über die Streuung von Licht an Licht nach der Diracshen Theorie,” Annalen der Physik 1936; **26**: 398−448.

[18] W. Heisenberg and H. Euler, “Folgerungen aus der Diracschen Theorie des Positrons,” Zeits. f. Physik vol. 98, pp. 714−732, Dezember 1935.

[19] R. Karplus and M. Neuman, “The Scattering of Light by Light,” Phys. Rev. vol. 83, pp. 776−784, Aug. 1951.

[20] T. Fujita and N. Kanda, “Tomonaga’s Conjecture on Photon Self-Energy”, physics.gen-ph/1102.2974

[21] F. Moulin, D. Bernard, and F. Amiranoff, “Photon-photon elastic scattering in the visible domain,” Z. Phys. C. vol. 72, pp. 607−611, 1996.

[22] D. Bernard, F. Moulin, F. Amirano, A. Braun4, J.P. Chambaret, G. Darpentigny, G. Grillon, S. Ranc, and F. Perrone, “Search for stimulated photon-photon scattering in vacuum,” Eur. Phys. J. D. vol. 10, pp. 141−145, 2000.

[23] I. Larin et al. , “New Measurement of the π^0 Radiative Decay Width,” Phys. Rev. Lett. vol. 106, 162303, Apr. 2011.

[24] S.L. Adler, “Axial-Vector Vertex in Spinor Electrodynamics,” Phys. Rev. vol. 177, pp. 2426−2438, Jan. 1969.

[25] C. Itzykson and J.B. Zuber, “Quantum Field Theory”, (McGraw-Hill, 1980)

[26] L. D. Landau, Dokl. Akad. Nawk., USSR vol. 60, pp. 207, 1948.

[27] C. N. Yang, “Selection Rules for the Dematerialization of a Particle into Two Photons,” Phys. Rev. vol. 77, pp. 242−245, Jan. 1950.

[28] N. Kanda, R. Abe, T. Fujita and H. Tsuda, “Z^0 decay into two photons”, hep-ph/1109.0926

[29] C. Amsler et al. (Particle Data Group), Physics Letters B vol. 667, pp. 1, 2008.

[30] J. Schwinger, “Gauge Invariance and Mass. II,” Phys. Rev. vol. 128, pp. 2425−2429, Dec. 1962.

[31] T. Fujita, ”Symmetry and Its Breaking in Quantum Field Theory”, (2nd edition, Nova Science Publishers, 2011)

[32] N. S. Manton, “The Schwinger Model and Its Axial Anomaly,” Ann. Phys. vol. 159, pp. 220−251, 1985.

[33] C.N. Yang and R.L. Mills, “Conservation of Isotopic Spin and Isotopic Gauge Invariance,” Phys. Rev. vol. 96, pp. 191−195, Oct. 1954.

[34] M. Lacombe, B. Loiseau, J.M. Richard, R. Vinh Mau, J. Cote, P. Pires, and R. de Tourreil, “Parametrization of the Paris N-N potential,” Phys. Rev. C. vol. 21, pp. 861−873, Mar. 1980.

[35] M. Lacombe, B. Loiseau, J.M. Richard, R. Vinh Mau, J. Cote, P. Pires, and R. de Tourreil, “Parametrization of the deuteron wave function of the Paris N-N potential,” Phys. Lett. B. vol. 101, pp. 139−140, May. 1981.

[36] R. Machleidt, K. Holinde, and Ch. Elster, “The bonn meson-exchange model for the nucleon?nucleon interaction,” Phys. Rep. vol. 149, pp. 1−89, May. 1987.

[37] R. Machleidt, Adv. Nucl. Phys. 1989; **19**: 189.

[38] A. Bohr and B.R. Mottelson, ”Nuclear Structure” (Vol. 1), (World Scientific,1998)

[39] D. Kiang, M.A. Preston and P. Tip, “One-Boson-Exchange Potential and Nuclear Matter,” Phys. Rev. vol. 170, pp. 907−915, Jun. 1968.

[40] R.D. Haracz and R.D. Sharma, “Two-Boson-Exchange Effects in Nucleon-Nucleon Scattering,” Phys. Rev. vol. 176, pp. 2013−2018, Dec. 1968.

[41] M.H. Partovi and E.L. Lomon, “Field-Theoretical Nucleon-Nucleon Potential,” Phys. Rev. D. vol. 2, pp. 1999−2032, Nov. 1970.

[42] W.R. Wortman, “Two-Pion-Exchange Contributions to Nucleon-Nucleon Scattering,” Phys. Rev. vol. 176, pp. 1762−1768, Dec. 1968.

[43] F. Gross, “Relativistic quantum mechanics and field theory”, (John Wiley & Sons, 1993)

[44] T. Fujita, N. Kanda and S. Oshima, “Nuclear Potential with Two Pion Exchange”, arXiv:1209.3067

[45] T. Fujita, M. Hiramoto and H. Takahashi, “Boson after symmetry breaking in quantum field theory”, in *Focus on Boson Research* (Nova Science Publisher, 2005)

[46] T. Fujita, N. Kanda and H. Tsuda, “Higgs Mechanism and New Propagator of Massive Vector Bosons,” J. Mod. Phys. vol. 3, pp. 619−624, Jul. 2012.

[47] J. Goldstone, Nuovo Cimento vol. 19, pp. 154, 1961.

[48] J. Goldstone, A. Salam and S. Weinberg, “Broken Symmetries,” Phys. Rev. vol. 127, pp. 965−970, Aug. 1962.

[49] Y. Nambu and G. Jona-Lasinio, “Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I,” Phys. Rev. vol. 122, pp. 345−358, Apr. 1961.

[50] T. Fujita, M. Hiramoto, T. Homma and H. Takahashi, “New Vacuum of Bethe Ansatz Solutions in Thirring Model,” J. Phys. Soc. Japan vol. 74, pp. 1143−1149, Jan. 2005.

[51] T. Fujita, T. Kobayashi, M. Hiramoto, H. Takahashi, “Non-equivalence between Heisenberg XXZ spin chain and Thirring model,” Eur. Phys. J. C. vol. 39, pp. 511−518, Jan. 2005.

[52] P.W. Higgs, “Broken symmetries, massless particles and gauge fields,” Phys. Lett. vol. 12, pp. 132−133, Sep. 1964.

[53] S.L. Glashow, “Partial-symmetries of weak interactions,” Nucl. Phys. vol. 22, pp. 579−588, Feb. 1961.

[54] A. Salam, In Elementary particle physics (Nobel Symposium No. 8), Ed. N. Svartholm; Almqvist and Wilsell, Stockholm (1968)

[55] S. Weinberg, “A Model of Leptons,” Phys. Rev. Lett. vol. 19, pp. 1264−1266, Nov. 1967.

[56] T. Fujita and N. Kanda, “No Anomaly and New Renormalization Scheme,” J. Mod. Phys. vol. 3, pp. 665−681, Aug. 2012.

[57] T. Fujita, A. Kusaka, K. Tsuda and S. Oshima, “Unphysical Gauge Fixing in Higgs Mechanism”, arXiv:0806.2957

[58] T. Fujita, “Physical observables in gauge field theory” in *New Fundamentals in Fields and Particles*, Transworld Research Network (2009), p.1−p.20

[59] E. Fermi, ”Elementary Particles”, (Yale University Press, New Haven, 1951)

[60] R. Feynman and M. Gell-Mann, “Theory of the Fermi Interaction,” Phys. Rev. vol. 109, pp. 193−198, Jan. 1958.

[61] S.S Gershtein and J.B. Zeldovich, JETP vol. 2, pp. 576, 1955.

[62] T.D. Lee and C.N. Yang, “Question of Parity Conservation in Weak Interactions,” Phys. Rev. vol. 104, pp. 254−258, Oct. 1956.

[63] C.S. Wu, “Parity Experiments in Beta Decays,” Rev. Mod. Phys. vol. 31, pp. 783−790, 1959.

[64] T. Fujita, “Physical observables in path integral formulation”, in *New Fundamentals in Fields and Particles*, Transworld Research Network (2009), p.31−p.45

[65] T. Fujita, “Quantum Gravity without General Relativity”, arXiv:0804.2518

[66] S. Kanemaki, A. Kusaka, S. Oshima and T. Fujita, “Problems of scalar bosons”, in *New Fundamentals in Fields and Particles*, ed. by T. Fujita, Transworld Research Network (2009), p.47−p.60

[67] C.W. Misner, K.S. Thorne and J.A. Wheeler, “Gravitation” (Freeman, 1973)

[68] T. Fujita and N. Kanda, “Novel Solution of Mercury Perihelion Shift”, physics.gen-ph/0911.2086

[69] T. B. Bahder, “Relativity of GPS measurement,” Phys. Rev. D. vol. 68, 063005, Sep. 2003.

[70] B.W. Parkinson and J.J. Spilker, eds., Global Positioning System, Progress in Astronautics and Aeronautics, **163, 164** (1996)

[71] T. Fujita and N. Kanda, “Physics of Leap Second”, physics.gen-ph/0911.2087

[72] A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie,” Annalen der Physik vol. 49, pp. 769−822, März. 1916.

[73] T. Fujita and S. Oshima, “Electric dipole moments of neutron-odd nuclei ,” J. Phys. G. vol. 39, pp. 095106(6pp), Jul. 2012.

[74] K. Muto, “Time reversal invariance and EDM in atomic system”, in *New Fundamentals in Fields and Particles*,
ed. by T. Fujita, Transworld Research Network (2008), p.119−p.155

[75] S. Oshima, “Electric dipole moments (EDM) of ionic atoms,” Phys. Rev. C. vol. 81, pp. 038501(4pp), Mar. 2010.

[76] P. G. Harris, C. A. Baker, K. Green, P. Iaydjiev, S. Ivanov, D. J. R. May, J. M. Pendlebury, D. Shiers, K. F. Smith, M. van der Grinten and P. Geltenbort, “New Experimental Limit on the Electric Dipole Moment of the Neutron,” Phys. Rev. Lett. vol 82, pp. 904−907, Sep. 1999.

[77] K. F. Smith, N. Crampin, J. M. Pendlebury, D. J. Richardson, D. Shiers, K. Green, A. I. Kilvington, J. Moir, H. B. Prosper and, D. Thompson, N. F. Ramsey, B. R. Heckel and, S. K. Lamoreaux, P. Ageron and, W. Mampe and A. Steyerl, “A search for the electric dipole moment of the neutron,” Phys. Lett. B. vol. 234 pp. 191−196, Jan. 1990.

[78] M. Kobayashi and T. Maskawa, “CP-Violation in the Renormalizable Theory of Weak Interaction,” Prog. Theor. Phys. vol. 49 pp. 652−657, 1973.

[79] E.P. Shabalin, Sov. J. Nucl. Phys. vol. 28, pp. 75, 1978.

[80] E.P. Shabalin, Sov. J. Nucl. Phys. vol. 31, pp. 864, 1980.

[81] T.P. Cheng and L.F. Li, "Guage Theory of Elementary Particle Physics", (Oxford University Press, 1988)

[82] H.A. Bethe, "The Electromagnetic Shift of Energy Levels," Phys. Rev. vol. 72, pp. 339−341, Aug. 1947.

[83] M.I. Eides, H. Grotch and V.A. Shelyuto, "Theory of light hydrogenlike atoms," Phys. Rep. vol. 342, pp. 63−261, Feb. 2001.

[84] W. Heisenberg, "Bemerkungen zur Diracschen Theorie des Positrons," Zeits. f. Physik. vol. 90, pp. 209−231, Juni. 1934.

[85] E.A. Uehling, "Polarization Effects in the Positron Theory," Phys. Rev., vol. 48, pp. 55−63, Jul. 1935.

[86] R. Pohl et al. , "The size of the proton ," Nature Letters vol. 466, pp. 213−217, Jul. 2010.

[87] T. Sakamoto, T. Fujita, N. Kanda, H. Kato and K. Tsuda, "A Puzzle of Lamb Shifts in Muonic Hydrogen", to be published

[88] K.A. Woodle, A. Badertscher, V.W. Hughes, D.C. Lu, M. W. Ritter, M. Gladisch, H. Orth, G. zu Putlitz, M. Eckhause, J. Kane and F.G. Mariam, "Measurement of the Lamb shift in the n = 2 state of muonium," Phys. Rev. A. vol. 41, pp. 93−105, Jan. 1990.

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