# Chapter 5

# Weak Interactions

**Abstract:** In this chapter, we first present a brief review of the weak interaction theory. In particular, we discuss why the conserved vector current model had to be modified to a new theory. After that, we clarify the physics of the spontaneous symmetry breaking and then discuss the intrinsic problem of the Higgs mechanism in the Weinberg-Salam model. In addition, we present the calculation of the vertex correction due to the weak vector bosons and show that there is no logarithmic divergence in this vertex corrections. Therefore, there is no need of the renormalization procedure in the weak interaction models with massive vector bosons.

**Keywords:** CVC theory, vertex corrections of weak vector boson, Weinberg-Salam model, spontaneous symmetry breaking, Higgs mechanism, right propagator of massive vector boson, Lorentz condition

## 5.1 Introduction

The physics of weak interactions started from the Fermi model of the four fermion interaction Hamiltonian. This model pointed to the essentially correct physics picture of the weak decay processes. However, the four fermion interaction model has a quadratic divergence in the second order perturbation calculations even with a very small coupling constant. Therefore, the model cannot be accepted for a correct theory unless one makes some modifications, even though this model is applied to physical processes with the first order perturbation theory and has made a great success.

At the same time, there were several strong experimental evidences that the four fermion interaction model should be mediated by very heavy bosons, and indeed, the experimental discovery of the weak vector bosons ($W^{\pm},\ Z^0$ ) was followed. In the mean time, Weinberg and Salam proposed a weak interaction model which is based on the $SU(2) \otimes U(1)$ non-abelian gauge theory. The reason why they employ the gauge theory is simply because they believed that the gauge theory should be renormalizable, though without any foundations.

However, the problem is that this standard model has two serious mistakes. The first one is related to the non-abelian nature of gauge fields in the model Lagrangian density. As we discuss in the previous chapter, the charges of the non-abelian gauge fields are gauge dependent, and therefore they are not physical observables at all. This means that these gauge fields cannot become free particles unless one makes mistakes somewhere within the theoretical framework. The second mistake in the standard model is connected to their treatment of the Higgs mechanism. There, the local gauge invariance is broken by hand in order to give a finite mass to the gauge field at the Lagrangian density level, and this is a wrong procedure. This is mainly based on the fact that the symmetry breaking physics is completely misunderstood at the time of the construction of the theory, and indeed the symmetry breaking physics is only concerned with the property of the interacting vacuum state, and it cannot induce any change of the gauge field properties in the Hamiltonian since the symmetry breaking has nothing to do with the field operators. In reality, the chiral symmetry is never broken spontaneously, as we see below.

In this chapter, we review what is the basic problem of the standard model of the weak interactions. In short, the problem of the Weinberg-Salam model is concerned with the symmetry nature which should be kept at any time in the Lagrangian density, even though the state (here the vacuum state of the interacting field theory model) can find the symmetry property which is different from the one found in the free field theory model. There is nothing surprising since the true vacuum of the interacting Hamiltonian may well have a non-vanishing charge associated with the symmetry of the Hamiltonian while the free field theory model may have zero charge of the symmetry group. On the other hand, the Weinberg-Salam model had to break the symmetry itself at the Lagrangian density level because it started from the local gauge theory whose fields must be always massless, and this is more than a serious defect of the model Hamiltonian, but it is physically a wrong procedure. This clearly indicates that, instead of the Weinberg-Salam model, one should find a new model Hamiltonian with three massive vector bosons from the beginning, and it turns out that this is indeed renormalizable. In fact, there is no logarithmic divergence in the calculations of any physical observables in the new model, and thus one does not have to worry about the renormalization procedure.

## 5.2 Critical Review of Weinberg-Salam Model

The Weinberg-Salam model has basically two important ingredients. The first one is concerned with the fermion and vector field coupling that leads to the four fermion interaction model in the second order perturbative calculations. This is a very reasonable assumption, and indeed one sees that the model can reproduce almost all of the experimental observations. The second part is the Higgs mechanism which has, in fact, a serious problem in connection with the *unitary gauge* fixing. In this mechanism, the condition of $\phi = \phi^\dagger$ is imposed on the Higgs fields. However, this does not correspond to a proper gauge fixing. Instead, this is simply a procedure for giving a finite but very large mass to a gauge field by breaking the local gauge invariance by hand. This suggests that the starting Lagrangian density of the weak interactions should be reconsidered, and indeed we should start from the three massive vector boson fields from the beginning. The massive vector fields should couple to the fermion currents as the initial ingredients. Here, it is shown that the new renormalization scheme with massive vector bosons has no intrinsic problem, and the massive vector boson fields do not give rise to any divergences for physical observables and therefore we do not need any renormalization procedure.

### 5.2.1 Spontaneous Symmetry Breaking

Before going to the discussion of the Higgs mechanism, we should clarify the physics of the spontaneous symmetry breaking. The whole idea of the symmetry breaking has been critically examined in the recent textbook [31, 45, 46], and the physics of the spontaneous symmetry breaking is, by now, well understood in terms of the standard knowledge of quantum field theory. In particular, if one wishes to understand the vacuum state in a field theory model of fermions, then one has to understand the structure of the negative energy states of the corresponding field theory model.

The terminology of the spontaneous symmetry breaking is misleading, and one should say that it is incorrectly used. It does not express the right physics of the symmetry breaking [47, 48, 49]. This is simply because the breaking of the symmetry cannot, of course, occur in the Hamiltonian of isolated system [50, 51]. If the symmetry breaking is concerned with the comparison of the vacuum states between the free field theory and the interacting field theory models, then we see that the chiral charge associated with the chiral symmetry transformation in the interacting vacuum state may well have a finite but different charge from the vacuum state of the free field theory which indeed has a zero chiral charge. For the total Hamiltonian $H = H_0 + H_I$, we have the vacuum state $|vac\rangle_{exact}$ which is an eigenstate of $H$, and the vacuum state may well have the eigenvalue of the chiral charge operator $\hat{Q}_5$ as [31]

$$e^{i\alpha\hat{Q}_5}|vac\rangle_{exact} = e^{in\alpha}|vac\rangle_{exact}$$

where $n$ is $\pm 1$ for the Thirring model. On the other hand, the free vacuum state $|vac\rangle_{free}$ which is an eigenstate of $H_0$ should have

$$e^{i\alpha\hat{Q}_5}|vac\rangle_{free} = |vac\rangle_{free}.$$

Here, one can see that there is nothing special in this symmetry arguments. The most important of all is that there is no symmetry breaking in the Hamiltonian of $H$. Only the exact vacuum state has a finite chiral charge, in contrast to the zero chiral charge of the free vacuum state.

On the other hand, some people completely misunderstood this physics of symmetry breaking and thought that the vacuum state of the interacting Hamiltonian itself broke the chiral symmetry [49]. This should arise from the two kinds of misunderstanding in their calculations. The first point is that they made use of the approximation scheme of Bogoliubov transformation, and this approximation method happens to induce a deceptive term which looks like a mass term though its mass is infinite [31]. The second misunderstanding is concerned with the concept of the cutoff momentum, and in fact, their result of the mass term is expressed by the cutoff momentum $\Lambda$ which should be set to infinity at the end of the calculation. In this respect, it is clear that one cannot discuss its physics by rewriting the Lagrangian density into a new shape. As one knows, the property of the vacuum state should be determined from the eigenstate of the total Hamiltonian in the corresponding field theory model.

In summary, the symmetry of the Hamiltonian can never be spontaneously broken, and the eigenstate of the Hamiltonian should keep the symmetry property, unless the symmetry breaking terms should be added to the Hamiltonian by hand. As we discuss below, the physics of the Higgs mechanism has nothing to do with the property of the vacuum state, and therefore it is not related to the symmetry breaking physics at all [52].

### 5.2.2 Higgs Mechanism

As we show below, the whole procedure of the Higgs mechanism cannot be justified at all. This is mainly connected to the misunderstanding of the gauge fixing where one degree of freedom of the gauge fields must be reduced in order to solve the equations of motion of the gauge fields. Therefore, one cannot insert the condition of the gauge fixing into the Lagrangian density. This is clear since the Lagrangian density only plays a role for producing the equation of motions. Indeed, the Lagrangian density itself is not directly a physical observable, and the Hamiltonian constructed from the Lagrangian density is most important after the fields are quantized. For the field quantization, one has to make use of the gauge fixing condition which can determine the gauge field $A_\mu$ together with the equation of motions. This means that only the final Hamiltonian density is relevant to the description of physical observables, and thus the success of the Glashow-Weinberg-Salam model [53, 54, 55] is entirely due to the final version of the weak Hamiltonian which is not at all the gauge field theory but is a model field theory of the massive vector fields which couple to the fermion currents. The success of the standard model is, of course, due to the fact that it can be reduced to the theory of conserved vector current (CVC).

In this respect, it is very important to examine the renormalizability of the final version of the weak Hamiltonian. Here, we show that the renormalizability of the model field theory can be indeed justified. This is basically due to the fact that there is no divergence in the vertex corrections of fermions due to the massive vector boson propagations once we employ a proper propagator of the massive vector bosons. Here, we briefly review how we can obtain the new propagator of the massive vector boson, and the correct shape of the propagator of the massive vector bosons should be given as [56]

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}. \tag{5.0}$$

This shape is determined by solving the equations of motion for the massive vector bosons. As long as we employ the above propagator, we find that the anomalous magnetic moment of electron due to weak $Z^0$ bosons does not have any divergences and it is indeed very small number which is consistent with experiment. Thus, one can see that the physical observables with the massive vector boson propagations are all finite and that there are neither conceptual nor technical problems in the renormalization scheme of the massive vector bosons interacting with fermions. Namely, there is no need of the wave function renormalization.

### 5.2.3 Gauge Fixing

Now we discuss the basic problem of the Higgs mechanism [46]. The Lagrangian density of the Higgs mechanism is given as

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}u_0\left(|\phi|^2 - \lambda^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.1}$$

where

$$D^\mu = \partial^\mu + igA^\mu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{5.2}$$

Here, we only consider the U(1) case since it is sufficient for the present discussions. The above Lagrangian density is indeed gauge invariant, and in this respect, the scalar field may interact with gauge fields in eq.(5.1). However, it should be noted that there is no experimental indication that the fundamental scalar field can interact with any gauge fields in terms of the Lagrangian density of eq.(5.1). In this sense, this is only a toy model. Now, the equations of motion for the scalar field $\phi$ become

$$\partial_\mu(\partial^\mu + igA^\mu)\phi = -u_0\phi\left(|\phi|^2 - \lambda^2\right) - igA_\mu(\partial^\mu + igA^\mu)\phi \tag{5.3}$$

$$\partial_\mu(\partial^\mu - igA^\mu)\phi^\dagger = -u_0\phi^\dagger\left(|\phi|^2 - \lambda^2\right) + igA_\mu(\partial^\mu - igA^\mu)\phi^\dagger. \tag{5.4}$$

On the other hand, the equation of motion for the gauge field $A_\mu$ can be written as

$$\partial_\mu F^{\mu\nu} = gJ^\nu \tag{5.5}$$

where

$$J^\mu = \frac{1}{2}i\left\{\phi^\dagger(\partial^\mu + igA^\mu)\phi - \phi(\partial^\mu - igA^\mu)\phi^\dagger\right\}. \tag{5.6}$$

One can also check that the current $J^\mu$ is conserved, that is

$$\partial_\mu J^\mu = 0. \tag{5.7}$$

This Lagrangian density of eq.(5.1) has been employed for the discussion of the Higgs mechanism.

### 5.2.4 Gauge Freedom and Number of Independent Equations

Now, we should count the number of the degrees of freedom and the number of equations. For the scalar field, we have two independent functions $\phi$ and $\phi^\dagger$. Concerning the gauge fields $A^\mu$, we have four since there are $A^0$, $A^1$, $A^2$, $A^3$ fields. Thus, the number of the independent fields is six. On the other hand, the number of equation is five since the equation for the scalar fields is two and the number of the gauge fields is three. This number of three can be easily understood, even though it looks that the independent number of equations in eq.(5.5) is four, but due to the current conservation the number of the independent equations becomes three. This means that the number of the independent functions is six while the number of equations is five, and they are not equal. This is the gauge freedom, and therefore in order to solve the equations of motion, one has to put an additional condition for the gauge field $A_\mu$ like the Coulomb gauge which means $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$. In this respect, the gauge fixing is simply to reduce the redundant functional variable of the gauge field $A_\mu$ to solve the equations of motion, and nothing more than that.

### 5.2.5 Unitary Gauge Fixing

In the Higgs mechanism, the central role is played by the gauge fixing of the unitary gauge. The unitary gauge means that one takes

$$\phi = \phi^\dagger. \tag{5.8}$$

This is the constraint on the scalar field $\phi$ even though there is no gauge freedom in this respect. For the scalar field, the phase can be changed, but this does not mean that one can erase one degree of freedom. One should transform the scalar field in the gauge transformation as

$$\phi' = e^{-ig\chi}\phi$$

but one must keep the number of degree of freedom after the gauge transformation. Whatever one fixes the gauge $\chi$, one cannot change the shape of the scalar field $\phi$ since it is a functional variable and must be determined from the equations of motion. The gauge freedom is, of course, found in the vector potential $A_\mu$ as we discussed above. In this sense, one sees that the unitary gauge fixing is a simple mistake [57]. The basic reason why people overlooked this simple-minded mistake must be due to their obscure presentation of the Higgs mechanism. Also, it should be related to the fact that, at the time of presenting the Higgs mechanism, the spontaneous symmetry breaking physics was not understood properly since the vacuum of the corresponding field theory was far beyond the proper understanding. Indeed, the Goldstone boson after the spontaneous symmetry breaking was taken to be almost a mysterious object since there was no experiment which suggests any existence of the Goldstone boson. Instead, a wrong theory prevailed among physicists. Therefore, they could assume a very unphysical procedure of the Higgs mechanism and people pretended that they could understand it all.

### 5.2.6 Non-abelian Gauge Field

Now, one should be careful for the renormalizability of the non-abelian gauge field theory. As one can easily convince oneself, the non-abelian gauge theory has an intrinsic problem of the perturbation theory [58]. This is connected to the fact that the color charge in the non-abelian gauge field depends on the gauge transformation, and therefore it cannot be physical observables. This means that the free gauge field which has a color charge is gauge dependent, and thus one cannot develop the perturbation theory in a normal way. In QCD, this is exhibited as the experimental fact that both free quarks and free gluons are not observed in nature. The absence of free fields is a kinematical constraint and thus it is beyond any dynamics. Therefore, one cannot discuss the renormalizability of the non-abelian gauge field theory models due to the lack of the perturbation scheme in this model field theory [31, 58]. Therefore, the problem of the renormalizability in the non-abelian field theory model is a meaningless subject since the perturbation theory is not defined in this model field theory.

### 5.2.7 Summary of Higgs Mechanism

The intrinsic problem of the Higgs mechanism is discussed in terms of the gauge fixing condition. This is also related to the misunderstanding of the spontaneous symmetry breaking physics. Here, we have shown that the Higgs mechanism cannot be justified since the gauge invariance of the Lagrangian density is violated by hand. However, we believe that the final version of the weak Hamiltonian should be correct, and therefore we should discuss the renormalization scheme of the massive vector bosons in detail. As we discuss above, the basic reason why the standard model Hamiltonian becomes a reasonable model is due to the fact that they make mistakes twice and thus it gets back to the right Hamiltonian which can describe the nature. The first mistake is related to the non-abelian character of the gauge field theory model while the second mistake is concerned with the breaking of the local gauge invariance in terms of Higgs mechanism, and it is, of course, an incorrect treatment. Therefore, if we remove the Higgs fields and the non-abelian nature of the massive vector bosons from the Weinberg-Salam model, then the final Hamiltonian of the standard model should be physically acceptable.

At this point, we should make a comment on the present status of the Higgs particle search. At present (January, 2013), there is no indication of the existence of the Higgs particle in spite of the fact that the total period of the experimental efforts of the Higgs search must be almost more then three decades. The main difference between the $W$ and $Z$ bosons and Higgs particle searches can be understood in the following way. The Higgs particle search started from the theoretical requirements (though incorrect) without having any firm experimental motivations of its existence, while the W-boson cases had many experimental indications of their existence before they were discovered by the UA1 and UA2 collaborations of the CERN-SPS experiments in 1983.

## 5.3 Theory of Conserved Vector Current

It should be important to construct the Lagrangian density which can describe the weak interaction processes. The basic starting point is, of course, the conserved vector current (CVC) theory which can describe most of the observed weak decay processes quite well. This CVC theory should be derived from the second order perturbation theory by exchanging the weak vector bosons between corresponding fermions.

### 5.3.1 Lagrangian Density of CVC Theory

The theory of the weak interactions is developed in terms of the four fermion interaction model [59] by Fermi and, after some time, Feynman and Gell-Mann extended it to the conserved vector current (CVC) theory, which is quite successful for describing experiments [60, 61, 62, 63]. The Lagrangian density of the CVC theory can be written as

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c.$$

where $G_F$ denotes the weak coupling constant $G_F \simeq 1.2 \times 10^{-5} \frac{1}{M_p^2}$. Also $J^\mu$ is composed of the leptonic and hadronic currents and is written as

$$J^\mu = j_\ell^\mu + j_h^\mu$$

where both of the currents can be expressed as

$$j_\ell^\mu = \bar{\psi}_{\nu_e} \gamma^\mu (1-\gamma^5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma^\mu (1-\gamma^5) \psi_\mu + \cdots$$

$$j_h^\mu = \cos\theta \bar{\psi}_u \gamma^\mu (1-\gamma^5) \psi_d + \sin\theta \bar{\psi}_u \gamma^\mu (1-\gamma^5) \psi_s + \cdots .$$

It should be important to note that the current-current interaction model can describe many experimental data to a very high accuracy, and this is, indeed, a well-known fact before the discovery of the weak vector bosons of $W^\pm, Z^0$.

### 5.3.2 Renormalizability of CVC Theory

However, this model Hamiltonian of CVC theory should have a serious problem related to the divergence in the second order perturbation theory. Since the coupling constant $G_F$ is very small compared to the fine structure constant, one can expect that the second order perturbation must be reliable. On the contrary, however, the second order calculation has a quadratic divergence since the coupling constant $G_F$ has the dimension of the inverse square of the energy. Therefore, it is clear that this theoretical framework should have an intrinsic problem of the divergence, and thus it should be very important to construct a theory which should not have any divergence.

### 5.3.3 Renormalizability of Non-Abelian Gauge Theory

Now, in order to construct a theory which is renormalizable, it was believed that the gauge field theory should be renormalizable at the time when people discovered the CVC theory. Therefore, it is natural that the non-abelian gauge theory of $SU(2) \otimes U(1)$ was proposed by Weinberg-Salam. However, one sees by now that the non-abelian gauge field has a charge associated with its gauge group, but the charge is not a physical observable since it is gauge dependent. Therefore, there is no way to develop any perturbation theory in this non-abelian gauge field theory. This means that the non-abelian gauge theory has an intrinsic problem before going to the renormalization scheme. [64]

## 5.4 Lagrangian Density of Weak Interactions

Even though the Higgs mechanism itself has an intrinsic problem, the final Hamiltonian density may well be physically meaningful. This is clear since, from this Hamiltonian density, one can construct the CVC theory which describes the experimental observables quite well.

### 5.4.1 Massive Vector Field Theory

In this respect, we may write the simplest Lagrangian density for two flavor leptons which couple to the SU(2) vector fields $W_\mu^a$

$$\mathcal{L} = \bar{\Psi}_\ell(i\partial_\mu\gamma^\mu - m)\Psi_\ell - gJ_\mu^a W^{\mu,a} + \frac{1}{2}M^2 W_\mu^a W^{\mu,a} - \frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a} \tag{5.9}$$

where $M$ denotes the mass of the vector boson. Here, we do not write the hadronic part, for simplicity. The lepton wave function $\Psi_\ell$ has two components

$$\Psi_\ell = \begin{pmatrix} \psi_e \\ \psi_\nu \end{pmatrix}. \tag{5.10}$$

Correspondingly, the mass matrix can be written as

$$m = \begin{pmatrix} m_e & 0 \\ 0 & m_\nu \end{pmatrix}. \tag{5.11}$$

The fermion current $J_\mu^a$ and the field strength $G_{\mu\nu}^a$ are defined as

$$J_\mu^a = \bar{\Psi}_\ell\gamma_\mu(1-\gamma_5)\tau^a\Psi_\ell, \qquad G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a. \tag{5.12}$$

This Lagrangian density is almost the same as the standard model Lagrangian density, apart from the Higgs fields and the abelien nature. In fact, there is no experiment in weak process which cannot be described by the Lagrangian density of eq.(5.9). The only thing which, people thought, may be a defect in the above Lagrangian density is concerned with the renormalization of the theory. As we see below, the problem of the renormalization is completely solved by employing the right propagator of the massive vector bosons. This means that we find that there is no logarithmic divergence in the evaluation of the vertex corrections due to the propagations of the massive vector bosons. Therefore, we do not need any renormalization procedure since all the physical observables are calculated to be finite.

## 5.5 Propagator of Massive Vector Boson

Here, we briefly review the derivation of the new propagator of the massive vector boson which has recently been evaluated properly in terms of the polarization vector [56]. The correct shape of the boson propagator is found to be the one given as

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}. \tag{5.13}$$

This is quite important since this does not generate any quadratic divergences in the self-energy diagrams of fermions any more while the old propagator in the textbooks

$$D^{\mu\nu}_{old}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2 - i\varepsilon}$$

gives rise to the quadratic divergence [31, 15]. This old propagator is obtained by making use of the Green's function method. However, the summation of the polarization vectors cannot be connected to the Green's function as we discuss below, and thus the employment of the old propagator is incorrect if one should treat the physical processes which involve the loop integral.

### 5.5.1 Lorentz Conditions of $k_\mu \epsilon^\mu = 0$

Here, we briefly explain how we can obtain eq.(5.13). The free Lagrangian density for the vector field $Z^\mu$ with its mass $M$ is written as

$$\mathcal{L}_Z = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} M^2 Z_\mu Z^\mu$$

with $G^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$. In this case, the equation of motion becomes

$$\partial_\mu (\partial^\mu Z^\nu - \partial^\nu Z^\mu) + M^2 Z^\nu = 0. \tag{5.14}$$

Since the free massive vector boson field should have the following shape of the solution

$$Z^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{3} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \, \epsilon^\mu_{k,\lambda} \left[ c_{\boldsymbol{k},\lambda} e^{ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} \right] \tag{5.15}$$

Here, we can insert this solution into eq.(5.14) and obtain the following equation for the polarization vector $\epsilon^\mu$

$$(k^2 - M^2)\epsilon^\mu - (k_\nu \epsilon^\nu)k^\mu = 0. \tag{5.16}$$

The condition that there should exist a non-zero solution for the $\epsilon^\mu$ requires that the determinant of the matrix should be zero, namely

$$\det\{(k^2 - M^2)g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{5.17}$$

This equation can be easily solved, and we find the following equation

$$k^2 - M^2 = 0 \tag{5.18}$$

which is the only physical solution of eq.(5.17). Therefore we insert this solution into eq.(5.16) and obtain the equation for the polarization vector $\epsilon^\mu$

$$k_\mu \epsilon^\mu = 0 \tag{5.19}$$

which should always hold. Here, we should note that this process of determining the condition on the wave function of $\epsilon^\mu$ is just the same as solving the free Dirac equation. Obviously this is the most important process of determining the wave functions in quantum mechanics, and surprisingly, this has been missing in the treatment of determining not only the massive vector boson propagator but also the photon propagator as well. Also, one can notice that the condition of eq.(5.19) is just the same as the Lorentz gauge fixing condition in quantum electrodynamics (QED), and this is often employed as the gauge fixing. However, one sees by now that the Lorentz condition itself can be obtained from the equation of motion, and therefore it is more fundamental than the gauge fixing, even though the theory of massive bosons has no gauge freedom. This indicates that the Lorentz gauge fixing in QED should not be a proper gauge fixing procedure since the Lorentz gauge fixing cannot give a further constraint on the polarization vector in the perturbation theory of QED. In addition, the number of degrees of freedom for the gauge fields can be understood properly since photon must have the two degrees of freedom due to the two constraint equations (the Lorentz condition and the gauge fixing condition).

### 5.5.2 Right Propagator of Massive Vector Boson

Now, we can evaluate the propagator of the massive vector field in the S-matrix expression. The second order perturbation of the S-matrix for the bosonic part can be written in terms of the T-product of the boson fields and it becomes

$$\langle 0|T\{Z^\mu(x_1)Z^\nu(x_2)\}|0\rangle = i\sum_{\lambda=1}^{3}\int \frac{d^4k}{(2\pi)^4}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}\frac{e^{ik(x_1-x_2)}}{k^2 - M^2 - i\varepsilon}. \tag{5.20}$$

After the summation over the polarization states, we find the following shape for $\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}$ as

$$\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda} = -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{5.21}$$

which satisfies the Lorentz invariance and the condition of the polarization vector $k_\mu \epsilon^\mu = 0$. One sees that this is the only possible solution. From eq.(5.21), one finds that the right propagator of the massive vector boson should be the one given in eq.(5.13)

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}.$$

Here it may be important to note that the polarization vector $\epsilon^\mu_{k,\lambda}$ should depend only on the four momentum $k^\mu$, and it cannot depend on the boson mass at this expression. Later, one may replace the $k^2$ term by $M^2$ in case the vector boson is found at the external line. But in the propagator, the replacement of the $k^2$ term by $M^2$ is forbidden.

### 5.5.3 Renormalization Scheme of Massive Vector Fields

In 1970's, people found that some experiments indicate there might be heavy vector bosons exchanged between leptons and baryons in the weak processes. Therefore, people wanted to start from the massive vector bosons. However, it was somehow believed among educated physicists that only gauge field theories must be renormalizable. We do not know where this belief came from. In fact, there is no strong reason that only the gauge field theory is renormalizable. On the contrary, we know by now that only QED may well have a strange divergence in the vertex corrections.

## 5.6 Vertex Corrections by Weak Vector Bosons

Now we can calculate the vertex correction $\Lambda^\rho(p', p)$ of electromagnetic interaction due to the $Z^0$ boson. The Lagrangian density for the $Z^0$ boson which couples to the electron field $\psi_e$ should be written as

$$\mathcal{L}_{Z^0} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 Z_\mu Z^\mu - g_z \bar{\psi}_e \gamma_\mu (1 - \gamma_5)\psi_e Z^\mu \tag{5.22}$$

where the free Lagrangian density part of electron is not written here for simplicity. This vertex correction is a physical process which can be directly related to the physical observables, and therefore we should be concerned with its divergences. The vertex correction $\Lambda^\rho(p', p)$ can be written by evaluating the corresponding Feynman diagrams as [56]

$$\Lambda^\rho(p', p) = -ig_z^2 e \int \frac{d^4k}{(2\pi)^4} \left( \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon} \right) \gamma_\mu \gamma^5 \frac{1}{\not{p}' - \not{k} - m_e} \gamma^\rho \frac{1}{\not{p} - \not{k} - m_e} \gamma_\nu \gamma^5 \tag{5.23}$$

where only the term corresponding to the $\gamma^5 \gamma_\mu$ is written for simplicity.

### 5.6.1 No Divergences

First, we show that the apparent logarithmic divergent terms in eq.(5.23) vanish to zero, and this can be easily proved since we can find

$$\Lambda^\rho(p,p) = -ieg_z^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 2xdx \frac{\left(\gamma_\mu \not{k}\gamma^\rho \not{k}\gamma^\mu - \frac{\not{k}\not{k}\gamma^\rho \not{k}\not{k}}{k^2}\right)}{(k^2 - s - i\varepsilon)^3} = 0 \tag{5.24}$$

where $s = M^2(1-x) + m_e^2 x^2$. Therefore, there is no logarithmic divergence for the vertex correction from the weak massive vector boson propagations. This is very important in that the physical processes do not have any divergences when we make use of the proper propagator of the massive vector boson.

### 5.6.2 Electron $g-2$ by $Z^0$ Boson

The finite part of the vertex correction due to the $Z^0$ boson can be easily calculated and, therefore, the electron $g-2$ should be modified by the weak interaction to

$$\frac{g-2}{2} \simeq \frac{7\alpha_z}{12\pi}\left(\frac{m_e}{M}\right)^2 \simeq 2 \times 10^{-14} \tag{5.25}$$

where

$$\alpha_z = \frac{g_z^2}{4\pi} \simeq 2.73 \times 10^{-3}.$$

This is a very small effect, and therefore, it is consistent with the electron $g-2$ experiment. We should note that, if we employed the standard propagator of the massive vector boson as given in the field theory textbooks [15], then we would have obtained a very large effect on the electron $g-2$, even if we had successfully treated the problem of the quadratic and logarithmic divergences in some way or the other, by renormalizing them into the fermion self-energy contributions. This strongly suggests from the point of view of the renormalization scheme that the propagator of the massive vector field should be the one given by eq.(5.13).

### 5.6.3 Muon $g-2$ by $Z^0$ Boson

Here, we should also give a calculated value of the muon $g-2$ due to the $Z^0$ boson since it is just the same formula as eq.(5.24) except the mass of lepton. The result becomes

$$\left(\frac{g-2}{2}\right)_\mu \simeq \frac{7\alpha_z}{12\pi}\left(\frac{m_\mu}{M}\right)^2 \simeq 8.6 \times 10^{-10} \tag{5.26}$$

which is much larger than the electron case. This is, however, still too small to be observed by the muon $g-2$ experiments at the present stage.