Cosmology and Field Theory

Takehisa Fujita

(All Physics Institute)

PREFACE

At the present time, the theoretical scheme in modern fundamental physics is almost in the chaotic state, and therefore, it should be quite important for young people to find a physics textbook which can point to a right direction in studying modern physics. To improve the present situation, therefore, I wrote the textbook of Japanese version (Dawn of Universe) which should help young people understand the right modern physics. In the mean time, I started to translate the Japanese version into English. However, in the course of hard working, I got to realize that many parts of the textbook should be newly rewritten since there should be some progress of interpreting the basic problems in field theory and general relativity. In this sense, this book in English version turns out to be not necessarily the translation of "Dawn of Universe", but rather and almost a newly rewritten field theory textbook which is mainly for readers who want to be researchers in physics.

At present, most of the field theory textbooks contain incorrect descriptions such as the general relativity, renormalization scheme, spontaneous symmetry breaking, chiral anomaly and so on, and this is quite unfortunate. In this book, I should try and work hard to explain the basic part of field theory in a correct fashion, and should clarify the basic problems in the general relativity, renormalization scheme, symmetry breaking, anomaly and so on.

At this point, I should make a comment on the way how young people should study physics. In order to carry out something interesting in physics, one should understand physics in depth. But at the same time, one should learn many kinds of skills in numerical computations or mathematical skills in solving many types of differential equations. The reason why I have to stress the importance of leaning skills is mainly because many researchers in recent years tend to be too much knowledge-biased so that they cannot calculate a new type of model calculations by themselves. Therefore, young people who want to be a good physicist should work hard to try to master many types of skills. It should be noted that the learning of skills in physics must be much more important than just accumulating knowledges in science. But obviously, the former is much more difficult and cumbersome than the latter.

This textbook is intended to present a correct knowledge of modern physics and the way of well-trained thinking in physics. This book should be considered to be the first step toward understanding the textbook "Fundamental Problems in Quantum Field Theory" (Bentham Publishers, 2013) [5].

April 2020, Takehisa Fujita

Contents

Chapter 1

Introduction

First, I should write the outline of the story in this book. Even for experts, field theory or general relativity must be difficult to understand. Therefore, I should like to make a brief review in which way this book is designed so that readers may easily find the main plot of the story.

In this textbook, I should describe two theoretical schemes which, however, contradict with each other. The first and old scheme which is based on the genuine theoretical framework such as the general relativity, spontaneous symmetry breaking and renormalization procedure should mainly contain some kind of divergences. On the other hand, the second and new scheme which is based on the phenomenological approach such as the new gravity model or no anomaly framework should have no divergences in the physical observables such as the vertex corrections. These two schemes conflict with each other until the end.

1.1 Ptolemaic and Phenomenology Schools

I should call these two schools "Ptolemaic school" (old scheme) and "Phenomenology School" (new scheme), and therefore, the story should be viewed as the theoretical dispute between the two schools. The former should be based on the theoretical deduction as the first principle, and in particular, its central ideology comes from the general relativity. On the other hand, the latter should start from the experiments and observations as the first principle, and thus, any theoretical models should be constructed so as to understand nature.

In this book, I should try and work hard to explain the theoretical claim of the two schools as clearly as possible, and thus, readers should work hard to make up their own pictures of modern physics. At present, we know that many physical phenomena can be described in terms of simple field theory languages. Up to recent days (or even at present), various strange phenomena which are predicted by the general relativity have been discussed quite intensively. However, Big Bang model and Black Hole are all found to be illusions. Instead, an orthodox and simple gravity model in the field theory terminology is discovered and this gravitational model can describe all the observed data related to the gravity in a correct way.

On the other hand, concerning the symmetry problem in nature, some theoretical models which are based on the spontaneous symmetry breaking have been the leading theoretical framework for a long time. However, if one compares the predictions of these models with experiments, then one finds that they are completely ruled out as a theoretical framework. In addition, the approximations employed by these model calculations cannot be justified at all. Instead, some old and traditional field theory models can describe all the experiments related to the symmetry breaking physics.

1.1.1 What are Ptolemaic and Phenomenology Schools?

Here, I should briefly explain Ptolemaic and Phenomenology Schools. The Ptolemaic school should mainly aim at constructing theoretical frameworks which are based on the Gedanken experiment, and therefore, people in this school should not care so much for reproducing experiments. On the other hand, the Phenomenology School should always aim at describing nature by all means, and therefore, people in this school try to construct some phenomenological theory to understand physical observables. Up to now, almost all of the basic observed quantities that occur in nature can be understood quite successfully in terms of field theory models.

Unfortunately, there is no chance that the two schools might coexist. However, readers should carefully examine the theoretical frameworks of the two schools and find out the future direction of physics. The present author belongs to the latter group, but should try hard to make reliable and fair explanations as to what is now going on in physics world.

The Ptolemaic and Phenomenology Schools should be defined in the following way. But these terminologies are used symbolically and these words themselves are not necessarily connected to their physics.

- **Ptolemaic School :**
They claim that the universe should be created from the point by trusting the general relativity, and space is expanding. Space is taken to be a real object, and light and matter should evolve together with space as if stars should move with the heavens like the geocentric theory.

- **Phenomenology School :**
The basic aim is to understand physical phenomena in nature and the theoretical framework is determined from experiments and observations. The relativity principle and symmetry preservation (Curie principle) is the theoretical basis.

1.1.2 Scenario of Cosmology

Here, I should briefly explain the theoretical claim of the two schools concerning the scenario of cosmology. The cosmology has a unique position in physics since it should be connected with the macroscopic world like stars and galaxies which are composed of elementary particles like protons, electrons and nuclei. Therefore, one must understand the basic interactions between elementary particles including photons together with the large scale physics of formation of stars and galaxies.

• **Scenario of Universe Creation**

(1) **Ptolemaic School** :
This universe is considered to be created as a big bang, and then the universe is expanding. Matters (galaxies) are also expanding together with space expansion. As the basic theory, this school is based on the general relativity which starts from the Gedanken experiment. This universe is taken to be a unique and only one universe in whole space.

(2) **Phenomenology School** :
The universe has been existing from the infinite past, and the universe and space must be infinite. The basic theory is the new gravity model which is incorporated into the Dirac equation. The Newton equation derived from the Dirac equation with the new gravity potential has an additional potential which can describe various phenomena in astrophysics.

1.1.3 Symmetry in Nature

The pictures of the two schools for the symmetry in nature should be quite different from each other. The Ptolemaic School should be based on the spontaneous symmetry breaking idea while the Phenomenology School considers that the symmetry should be preserved unless some external forces act on the system.

• **Symmetry in Nature**

(1) **Ptolemaic School** :
The spontaneous symmetry breaking model is assumed, and using the Higgs mechanism, people make use of the weak interaction model. Since the weak vector boson is discovered, this is considered to be a standard model. Here, the existence of Higgs particle is essential, but it is never found yet in spite of more than 30 years of experimental efforts.

(2) **Phenomenology School** :
Any models that contradict the Curie principle should not be considered since there is no spontaneous symmetry breaking in nature. The model of weak interaction starts from massive weak bosons and this can reproduce the CVC theory. In this theory, there is no divergence for physical observables, and thus, the renormalization procedure is not required.

1.1.4 Theory and History of Ptolemaic School

The reason why the general relativity was believed for such a long time should be related to the fact that the new gravity model is discovered only 10 years ago. If it were constructed 60 years ago before the discovery of the background radiation, then people would have never paid any attention to the general relativity.

• **Solvay Conferences and Controversy of Quantum Mechanics :**
The controversy of quantum mechanics between Bohr and Einstein at the Solvay Conferences on Physics in 1930 indicates that Einstein could not understand the essence of quantum mechanics which is a probabilistic behavior. In fact, this controversy is ended by the complete victory of Bohr against Einstein. It may well be that Einstein continued to keep the deterministic view of the world, and he wanted to defend the general relativity which is the center of this ideology. By now, it is established that the fundamental physics is described in terms of quantum and probabilistic pictures.

• **Spontaneous Symmetry Breaking :**
The similar case occurs in the field theory as well in connection with the spontaneous symmetry breaking model. This is a strange claim and quite a few people must have thought that there is something wrong if the symmetry is broken without any external force. In fact, this contradicts the Curie principle. But surprisingly, the picture of this model is accepted by experts in this field for a long time.

The physics of spontaneous symmetry breaking is not consistent with the common knowledge of theoretical physics, and therefore, the explanation of this model by experts was always unclear. In fact, the approximation scheme employed in this model calculation by Nambu is the most important factor that has led to a wrong conclusion. However, the examination of the approximation scheme was not carried out properly, and thus, the model has lost its aim of understanding nature. In this way, those models which make use of the spontaneous symmetry breaking idea become all incorrect, which is quite unfortunate. Therefore, the concept of spontaneous symmetry breaking has left negative legacy in this field of research.

• **Solution of Puzzle :**
The fact that those strange theories survived for many years must be a puzzling problem of modern physics. Most likely, many accidental things may well have coincided together, but still it is a difficult to solve the puzzle.

There should be a common claim in both Einstein and Nambu. They often emphasized the originality of their model calculations. However, these original and newly developed ideas had nothing to do with physical observables, and this is a whole problem of these models.

1.2 Universe and Dream

Up to now, many people may well have some kind of dream as to what should be strange objects such as Black Hole or Big Bang. Unfortunately, however, all of these strange phenomena which are predicted by the general relativity are completely ruled out. Now, we know that the universe is infinite, and it has existed from eternal past. There is no limit of time and space in this whole universe.

1.2.1 New Dream

Can we find some new interesting phenomena in place of Black Hole? The answer is yes! and therefore, here I should make a brief explanation concerning a new dream.

• Infinite Universe :
What should be then a new dream? This is related to the fact that the whole universe must be truly infinite. Accordingly, the number of galaxies must be infinite. This means that the number of stars should be infinite, and thus, there should exist infinitely many planets which should have some kind of life like human being.

In the childhood, the place I play around was a small limited area. Then, at University day, the place I worked became a whole Japan, and eventually, it became the whole earth around.

The earth is located in the solar system which is, then, revolving a little outer part in the Milky Way. This Milky Way should have presumably a few hundred billion stars. Our galaxy (Milky Way) is located in the universe and the nearest galaxy is Andromeda which is 2.4 million light years from us. Presumably, there are few hundred billion galaxies in our universe whose size may well be around 15 billion light year.

However, by now, it becomes evident that there should be infinite numbers of universes, and therefore, whole universe must be infinite. This suggests that there must be infinite numbers of the earth-like planets in the whole world. This may indicate that there must be many of living objects like human being.

1.2.2 Living Objects in Extra-Our-Universe (EOU)

Some time ago, the story of ET (Extra-Terrestrial) became very popular and it must be a dream to get to know what should be ET. That is, a question as to whether there may be any possibility of finding ET. At present, it is impossible to find any living objects which should come from outside the earth. However, it may well be interesting to discuss some living objects as to in which way there should exist some kind of life in the Extra-Our-Universe. In this case, we have to consider the conditions of their existence beyond space and time.

• Existence Conditions of Living Objects :
Now we should discuss the conditions in which way these living objects can survive, in particular, the size and mass of planets or the role of the sun when stars are twin or something like that.

At least, there must be sufficiently large amount of water in the planets, and therefore, the size of the planets must be larger or equal to the earth. In this case, sea should be made there and there must be sea volcano which seems to be important for the living things to emerge. This volcano should be always made from time to time since the energy source of the volcano eruption must be due to the decay of ^{238}U nuclei. After some time, if chloroplast can be created, then the living objects can get the energy from the sun, and thus, they can develop further.

1.3 Evolution Theory and General Relativity

Here, I should discuss the difference and similarity between Darwin's theory of evolution and general relativity. This description may not be directly related to the main story of this book, but I believe that it should help readers understand the relevant physics in depth.

The two theoretical schemes of Darwin's theory of evolution and general relativity should have had great influence over scientific developments. However, their type of effects is quite different from each other.

Darwin's theory of evolution had powerful effects on people's way of thinking while the general relativity helped people increase their scientific knowledge in modern physics. Unfortunately, however, the physics of Big Bang or Black Hole is all too obscure to understand, and experts in this field did not understand what it means by Black Hole, for example.

1.3.1 Points in Common

The point in common between two schemes may be described in terms of their new and original concept. That is, both theories should have quite a new aspect which is very different from the existing schemes.

• **Evolution :**
In Darwin's theory of evolution, there are many points which contradict common senses at that time, and therefore there should have been many objections from various types of intellectuals even though experts should not hesitate to accept the concept of evolution. In fact, Darwin's theory of evolution is constructed by accumulating many observed data and phenomena, and therefore, the theoretical scheme is reliable and solid. This theoretical scheme should be somewhat similar to the Maxwell equation in physics.

• **General Relativity :**
On the other hand, the general relativity has some implication of extending classical mechanics to relativistic mechanics, and therefore, there should not be so much serious repulsion from experts. Also, the general relativity has some novel aspect in physics. However, this scheme starts from the Gedanken experiment and thus, there is no connection to nature. In this respect, this scheme became a theory which is either accepted or rejected. Because of this problem, there is no serious and critical examination of the theoretical scheme itself.

1.3.2 Points in Difference

The essential difference between two theories must be the scientific approach itself and below I should discuss it in a concrete fashion.

• **Evolution :**
In Darwin's theory of evolution, he constructed his theory by investigating many biological species very carefully. This should be to make a comprehensive survey of observation into one theoretical scheme. This is just the same as the Maxwell equation in physics.

In addition, evolution theory has a probabilistic aspect which did not agree with the main ideology of deterministic view originated from the Christianity. Therefore, the theory should be accepted scientifically, but it should take sometime before the theory became a common sense among people.

Evolution has a probabilistic aspect and is close to quantum mechanics. But this is quite reasonable since the basic structure of living objects should be understood by quantum mechanics.

• **General Relativity :**
The general relativity is based on the Gedanken experiment, and this is a very dangerous starting point since nature is much more complex than expected. Therefore, it is most likely true that the Gedanken experiment itself is incorrect.

Further, this theory is based on the deterministic view, and in fact, the metric tensor should be determined as the function of time. This is in the opposite direction to quantum mechanics. In this sense, the theoretical framework did not make any conflict with the ideology at that time. However, this contradicts quantum mechanical pictures and thus the theory could not become a basic theory in modern physics.

Probably, quite a few people may well believe that the general relativity must be a part of relativity. However, the general relativity has nothing to do with the Lorentz transformation. On the contrary, it violates the transformation property of the relativity. This is because the general relativity fixed the coordinate system in that the metric tensor in the Einstein equation should be affected by the presence of star distributions. In this case, the center of the coordinate system should correspond to the center of gravity of stars.

This choice of the coordinate system determines the distribution function and this contradicts the modern physics since normally we can choose the origin of the coordinate system as we wish. However, the general relativity has lost this important freedom of choosing the origin since the space near stars should be affected and distorted. This is connected to the coordinate system of the metric tensor which is measured from the center of star distribution.

1.4 Special and General Relativity

Einstein invented a new word of special relativity together with the general relativity. Therefore, this naming of special relativity may well have induced some kind of confusion for non-specialists in physics. Some of them might well think that the general relativity is related to the relativity principle. Here I should explain briefly that the expression of special and general is not suitable word in physics since the general relativity has nothing to do with relativity principle. Further, the special relativity is a very general law and consists of the most important transformation property.

1.4.1 Relativity Principle

The starting point of physics is relativity principle. This principle states that all physical observables must be the same for any inertial system. Now the inertial system is defined as the rest system on the earth when the earth is at rest. Another inertial system is just on the train when it is moving linearly with a constant velocity. In this case, all the physical observables must be the same for both systems (rest system and moving system), and indeed, this is confirmed experimentally.

1.4.2 Theory of Relativity

In the two inertial systems, we require that any physical law must be the same. In this case, what should be the transformation rule? In fact, this formulation of the transformation property is just the theory of relativity.

• Galilei Transformation :
Here, the coordinate of the rest system is denoted as $R(t, x, y, z)$ while the coordinate of the moving system is written as $S(t', x', y', z')$. The velocity v of the moving system is considered to be much smaller than the velocity of light c. Now we assume that the system is moving along $x-$axis. In this case, there are the following relations

$$x = x' + vt', \quad y = y', \quad z = z', \quad t = t' \tag{1.1}$$

which is called Galilei transformation. We can easily check that under the above transformation, Newton equation should be invariant. However, this transformation cannot keep the Maxwell equation invariant. Therefore, the Galilei transformation should not be a proper one as the relativity transformation.

1.4.3 Special Relativity

When the velocity v of S−system is close to the velocity of light, then the transformation is given by the Lorentz transformation.

• Lorentz Transformation :
In this case, the Lorentz transformation is given as

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \quad y = y', \quad z = z' \tag{1.2}$$

where γ is defined as

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{1.3}$$

Here it should be noted that the reason why the two system should have different time is related to the fact that the observers should be defined in both systems. The only relevant physical quantity should be observables which must be the same in both systems.

The Lorentz transformation is derived such that the Maxwell equation should have the same shape of the differential equations. In this respect, the relativity equation is not special but is quite general.

• Four Dimensional Space :
In the theory of relativity, we often make use of the terminology "four dimensional space". But we note that this space is a mathematical space and it has nothing to do with real space. In fact, when we write the Lorentz transformation, it is normally convenient to prepare the four dimensional space as the basis state. But real space is, of course, three dimensions, and time should play a very special role in physics and it is essentially different from space.

• $3 \oplus 1-$Dimensional Space :
In particular, time is very important for dynamics. If physical quantities do not depend on time, then they are called "conserved quantities". For example, the angular momentum of a particle trapped in the central potential does not depend on time, and therefore, it is conserved.

Therefore, often in field theory, we write space and time by $3 \oplus 1$ dimensions. However, even now, people may often mix up time together with space since it should not make any serious errors in evaluating physical quantities. This is also related to the fact that the relativity is a kinematics.

1.4.4 Unknown Function in General Relativity

Since the relativity is a kinematics, it is not directly related to physical laws. More precisely, all the basic equations must be invariant under the Lorentz transformation, and thus. the equations of motion in physics should satisfy the transformation property of relativity principle.

On the other hand, the general relativity is completely different from relativity principle. Einstein introduced a metric tensor which depends on time (t) and space ($\boldsymbol{r}$), and constructed the equation for the metric tensor $g^{\mu\nu}(x)$, which is called general relativity. The reason why he had to stick to the word "relativity" is not clear since it has nothing to do with relativity principle. Indeed, his equation should be made for some specific and fixed inertia system, and therefore, the general relativity violates the relativity principle and the naming of general relativity is not properly made in terms of the basic physics point of view.

1.4.5 Metric Tensor

Minkowski introduced a squared infinitesimal displacement in four dimensions by

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \tag{1.4}$$

In this case, the $(ds)^2$ can be written in terms of metric tensor $g^{\mu\nu}$

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = g^{\mu\nu} dx_\mu dx_\nu \tag{1.5}$$

where $dx^\mu = (cdt, dx, dy, dz)$ and $dx_\mu = (cdt, -dx, -dy, -dz)$ are introduced. Here we find that $g^{\mu\nu}$ can be written as

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.6}$$

which is the Minkowski metric tensor.

• Extension of Metric Tensor :
In the Einstein equation, this metric tensor $g^{\mu\nu}$ is assumed to depend on the coordinates t, x, y, z, and therefore, it is written as $g^{\mu\nu} = g^{\mu\nu}(t, x, y, z)$. The Einstein equation is simply expressed as

$$[\textbf{Second Order Derivatives of } g^{\mu\nu}] = [\textbf{Tensor from Star Distribution}] \ .$$

This equation means that if there should be distributions of stars around, then the metric tensor should be affected. But the physical ground why the metric tensor should be influenced by stars is unclear, and indeed there should be no corresponding phenomena found in nature.

1.4.6 Coordinates of Metric Tensor

A question may arise as to what should be the coordinate (t, x, y, z). The coordinate that appears in the metric tensor must be the general and symbolic coordinate which should be different from the coordinate that describes the star distribution. In this respect, there is some confusion in the Einstein equation.

• **Coordinate System of Star Distributions :**
Here, there is a serious problem (inconsistency) in the Einstein equation. This equation states mathematically that the metric tensor should be affected if there should be some distribution of stars. However, the important question is related to the causality problem since the coordinate system describing the star distributions must be written in terms of the metric tensor which is before the effect of the Einstein equation. This corresponds to the violation of the causality.

1.4.7 General Relativity and Gravity

General relativity is considered to be a theory of gravity, but this may be only due to the claim of Einstein. As we see below, it is impossible to prove that the general relativity is connected to gravity.

Here we show a fundamental mistake of the claim that the Einstein equation is connected to the gravity. First, we should write the Poisson equation for the gravitational field of ϕ_g which is written as

$$\boldsymbol{\nabla}^2 \phi_g = 4\pi G_0 \rho. \tag{1.7}$$

Now the above equation can be derived from the Einstein equation if we assume the following equation for the metric tensor of g^{00}

$$g^{00} \simeq 1 + 2\phi_g. \tag{1.8}$$

This ansatz is, of course, incorrect since the metric tensor should be unknown functions which can be determined after we solve the Einstein equation. There is no way to determine the shape of the unknown function before solving equations.

Further, the gravitational field of ϕ_g is a dynamical variable which can be defined only if the concept of gravity is introduced. On the other hand, the metric tensor is defined so as to determine the coordinate variables of inertia system, and thus it has nothing to do with the dynamical variables. This is the very basic point of physics, and physicists should understand this fundamental principle in depth.

1.4.8 Maxwell and Einstein Equations

In order to clarify some basic problems in the Einstein equation, we should compare the physical meaning of the Einstein equation with the Maxwell equation. Therefore, we first write Maxwell and Einstein equations

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = ej^\nu \qquad \text{(Maxwell equation)} \tag{1.9}$$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G_0 T^{\mu\nu} \qquad \text{(Einstein equation).} \tag{1.10}$$

Here the left side of the Maxwell equation is written in terms of vector potential A^μ, and therefore, A^μ should be unknown functions. On the other hand, the left side of the Einstein equation is written in terms of Ricci tensor ($R^{\mu\nu}$) which is expressed in terms of the second order differentials of the metric tensor $g^{\mu\nu}$. Therefore, the metric tensor must be unknown functions of the Einstein equation.

Now we should clarify the physical meaning of quantities (j^μ and $T^{\mu\nu}$) in the right side of both equations. In particular, we should understand how we can evaluate these quantities by making use of fundamental equations of motion such as Schrödinger and Newton equations with many body problems involved.

(1) Maxwell Equation

First, we discuss the current density j^μ which appears in the right side of the Maxwell equation.

- **j^μ in Maxwell Equation :**

The Maxwell equation states that the shape of vector potential A^μ can be determined if the current density j^μ is given. Now a question may arise as to how we can evaluate this current density j^μ. This calculation should involve many body problems of electrons, and this should not be easy at all. However, there is no conceptual difficulty in the evaluation of j^μ even though the calculation must be quite complicated indeed.

(2) Einstein Equation

Now we examine the energy-momentum tensor which appears in the right side of the Einstein equation.

• Energy-Momentum Tensor in Einstein Equation :
The right side of the Einstein equation is written in terms of energy-momentum tensor which is obtained from the distribution of stars. Therefore, if the distribution of stars is known, then the metric tensor $g^{\mu\nu}$ can be determined.

• Strange Equality :
It should be noted that the right side part of the Einstein equation is described in terms of dynamical variables while the left side corresponds to the coordinate system. In this respect, the equality here should not make sense physically, and this is one of the most serious problems of the Einstein equation.

• Determination of Star Distribution by Gravitational Fields :
In spite of the fact that the equality of the Einstein equation cannot be physically justified, we should proceed to go further to the discussion of the Einstein equation.

Here we prove that the metric tensor is not related to the gravitational field ϕ_g by any means. The main point is concerned with the right side of the Einstein equation which is the distribution of stars. It should be noted that the stars and their distribution can be determined by solving the many body Newton equations with the gravitational fields. Namely, the gravitational field should be made use of in constructing the right side of the Einstein equation.

• No Relation between Metric Tensor and Gravity :
Therefore, it is, by now, clear that the metric tensor which should be obtained as the solution of the Einstein equation has nothing to do with the gravitational field ϕ_g. This is related to the causality of the Einstein equation since the determination of the distribution of stars should have required the gravitational potential in advance before solving the Einstein equation. Thus, there is, absolutely, no way to relate the metric tensor to the gravitational field. This is a rigorous proof that eq.(1.8) is invalid.

1.5 Renormalization Scheme

In order to explain quantum field theory in modern physics, we should discuss the renormalization scheme in detail since it is much more important than the general relativity. Unfortunately, however, the theory of renormalization is quite complicated and it is normally very difficult even for experts to understand it in depth. Here, I should briefly explain what should be the essence of the renormalization scheme in quantum electrodynamics (QED).

1.5.1 Why Renormalization Scheme?

Modern physics is written in terms of field theory terminology. In this case, the perturbation theory is the only reliable theoretical tool to evaluate physical observables. This is true for all theoretical schemes of quantum electrodynamics (QED), quantum chromodynamics (QCD), weak interaction and gravity. Among them, QED is best studied, and the renormalization scheme in QED is developed by Tomonaga and Feynman and others.

• **Infinity :**
In the perturbative calculation in QED, some physical observables turned out to be infinite, and therefore, this infinity must be cured or removed in some way or other. In order to do so, there should be basically two ways to solve the problem of infinity in the physical observables. The first method is called renormalization scheme in which the infinity is removed by renormalizing the wave function. In this case, the infinity is absorbed into the wave function since the shape of the infinity is just the same as the infinity that appears in the self-energy of electron. This is called the renormalization scheme, and it is shown that this scheme can reproduce the observed value of electron magnetic moment very accurately.

On the other hand, there should be another approach in which there should be something wrong or inappropriate in the treatment of the calculation of (g-2). This is the claim of Dirac and others, but at that time when the renormalization scheme was developed, the basic defect in the calculation was not found and clarified.

1.5.2 Divergence of Electron Self-Energy

In the perturbative calculation of quantum field theory, the self-energy of electron or photon becomes infinite. This divergence problem, however, is known to occur also in the self-energy of electric field produced by electron in classical electrodynamics. This is, of course, treated in most of the textbooks in electromagnetisms. But people do not care for this infinity because the self-energy of electric field is not observable.

On the other hand, people believe that the same type of divergences in quantum field theory must be removed, and they discuss what should be done for treating this infinity of electron self-energy.

• Mass Renormalization :
In particular, the mass renormalization scheme is often discussed in most of the field theory textbooks. However, this is obviously a meaningless procedure since it has nothing to do with any physical observables. Important quantities in perturbation theory must be physical observables, and this point is missing in the mass renormalization scheme.

• Vertex Corrections of Electron :
As we stressed above, the self-energy cannot be physical observables, and thus, we do not have to care for the divergence of the electron self-energy. However, the vertex correction to the electron magnetic moment is the physical observable and therefore, if it has some divergence, then we have to consider some kind of manipulations to remedy it.

In order to remove the divergence in the vertex corrections, Feynman, Tomonaga and others proposed the renormalization scheme. Here I do not explain the method in detail, but I should briefly describe the renormalization procedure. Intuitively, this method is based on the fact that the divergence shape of self-energy of electron is just the same as that of the vertex corrections. Therefore, this divergence can be removed if we redefine the wave function such that the wave function can absorb this divergence term. However, the divergence is still left in the wave function and this procedure is too artificial to be reliable.

• Dirac's Claim :
Concerning the divergence in the vertex corrections to the electron magnetic moment, Dirac claimed that there must be something wrong in the treatment of the theoretical scheme in the perturbative calculation. In the report published in AIP Conference Proceedings in 1981 [Dirac, AIP Conference Proceedings 74, 129 (1981)], he repeatedly pointed out that there should be some problems in the renormalization scheme. However, people ignored his claim and accepted the theoretical scheme of the renormalization procedure until recently. The main reason why people could not understand Dirac's claim must be partly because the renormalization scheme is simple to evaluate the vertex corrections and partly because it can reproduce the observed value of g-2, even though the agreement between theory and experiment must be accidental.

1.5.3 Divergence of Photon Self-Energy

The self-energy of photon has the same type of divergence, but its property is quite different from the electron self-energy. The divergence of electron self-energy is a logarithmic infinity while the self-energy of photon has a quadratic divergence. Therefore, from the point of renormalization scheme, this divergence cannot be absorbed into the wave function.

• Gauge Condition :
At this point, people introduced somewhat a strange condition which is called "gauge condition" on the self-energy of photon. Physically there is no problem even if there should appear any divergences in the self-energy of photon since it is not observable. However, people employ this unphysical condition and made use of it in the renormalization scheme. It should be noted that one can easily prove that the gauge condition is also mathematically incorrect. In addition, the photon self-energy is not a physical observable and therefore, there is no point that we should worry about this infinity.

• Confusion of Renormalization Scheme :
In reality, the confusion of the renormalization scheme became more and more serious. Usually once a theory is accepted, then it should be very difficult to make the theory into the right direction. In this sense, it should be stressed that we should consider only the physical observables which are meaningful quantities in the perturbative calculations. In this case, we should ask ourselves what should be physical observables in connection to the vacuum polarization. This is, of course, clear that the triangle diagrams must be related to the physical observables, and therefore, it should be most important to evaluate the triangle diagrams as to whether these diagrams may have any divergences or not.

1.5.4 Triangle Diagrams

As we state above, the self-energy of photon is not physical observables, and therefore, we do not have to care for its divergence. However, if we find any divergences in the Feynman diagrams which contain the vacuum polarization process, then we should carefully examine what to do.

• Decay of $\pi^0 \to 2\gamma$:
The first calculation of the triangle diagrams was carried out by Nishijima in 1968. In fact, he calculated the decay of $\pi^0 \to 2\gamma$ process and obtained the finite result which agrees very well with the experiment.

• Chiral Anomaly :
However, right after Nishijima's calculation, Adler published a paper (in Physical Review) in which he carried out the calculation of triangle diagrams of the decay of $Z^0 \to 2\gamma$ process with serious mistakes. Indeed his calculation is completely wrong. Unfortunately, however, nobody at that time could point out the mistake of Adler's paper and soon it was accepted by most experts.

What should be the main reason as to why people believe this wrong work by Adler? It is, of course, difficult to find a good answer for his serious mistake, but it may well be that Adler presented a novel idea of anomaly equation which is mathematically beautiful indeed. But the anomaly equation is obtained by regularizing the linear divergence term which does not exist in the correct calculation of the triangle diagrams. Also, it should be noted that the regularization is simply a meaningless procedure in quantum field theory.

• **Finite Values of Observables in Vacuum Polarization :**
In reality, if we calculate the triangle diagrams properly, then we see that there is neither linear nor logarithmic divergences in any type of diagrams. This shows that the observables of physical processes in connection with vacuum polarizations are all finite, and therefore, renormalization procedure is not needed. This is just consistent with the claim of Dirac.

1.5.5 Is Renormalization Procedure Necessary?

All the physical observables in connection with vacuum polarization turn out to be finite and therefore, we do not have to consider any renormalizations. Up to now, among physical observables, only the vertex correction to the electron magnetic moment becomes divergent.

• **Vertex Correction to Electron Magnetic Moment :**
The vertex corrections to the electron magnetic moment should be the third order perturbative calculation in QED. However, this calculation has a logarithmic divergence even though it is an observable. Therefore, we should treat this divergence in some way or other. In the renormalization scheme, the divergence is removed by renormalizing the wave function, and people calculate the magnetic moment with the new wave function and obtain a finite and correct result. However, people have never examined carefully as to why the infinity appears in the vertex corrections. By now we know that the infinity of the vertex correction should arise from the Feynman propagator which is not a correct propagator with some theoretical inconsistency.

• **Vertex Correction to Lepton Magnetic Moment from Massive Vector Boson :**
Concerning the role of the propagator in the vertex correction, there is a good example. This is the vertex correction from weak interactions. In addition to the photon vertex correction, we should consider physical processes from the weak interactions. In this case, the massive vector boson (Z^0boson) should contribute to the vertex correction at the third order perturbative calculation. Surprisingly, however, there is no divergence in the vertex correction from the massive vector boson, and the calculated result is finite and very small, which is consistent with experiment. Here, we should also give a calculated value of the muon $g-2$ due to the Z^0 boson. The result becomes

$$\left(\frac{g-2}{2}\right)_\mu \simeq \frac{7\alpha_z}{12\pi}\left(\frac{m_\mu}{M}\right)^2 \simeq 8.6\times 10^{-10}.$$

This indicates that there is no need of renormalization in this calculation and it is a very healthy theoretical scheme.

• **Only Vertex Correction from Photon Has Infinity :**
This is very important that there is no divergence if we make use of the proper propagator. People agreed with the renormalization scheme since they thought that the infinity in the vertex correction is inevitable. However, this infinity appears for the vertex correction from photon only if the Feynman propagator is used.

• **Infinity Is Originated from Propagator of Photon! :**
Concerning the divergence of the vertex correction due to photon, we should first look at the property of the photon propagator which is the Feynman propagator. It is well known that the Feynman propagator of photon should have the basic problem since it cannot satisfy the constraint of the polarization vector. This is already pointed out in 1960's textbooks [1, 2]. However, this Feynman propagator of photon can reproduce the right T-matrix of electron-electron scattering, and thus it is used as a standard propagator. This problem is a bit too complicated and thus readers should refer to the explanation of textbook [5].

1.5.6 Important Comments from Nishijima

Some time ago, when I discussed with Professor Nishijima, he told me that I should examine whether the renormalization scheme of gravity theory is important or not. At that time, I was developing the new gravity theory. Therefore, I checked quite a few papers concerning the renormalization scheme. Among many papers in connection with renormalization scheme, several papers written by Heisenberg group in 1930's are very interesting and suggestive. I learned a lot from their papers even though they never discussed the quadratic divergence of the self-energy of photon. Instead, they simply threw away these divergences, keeping only the finite terms, and this is the point I could not understand in their papers.

• **Renormalizability of Non-Abelian Gauge Theory :**
In addition to the papers by Heisenberg group, there was one paper which I could not understand at all. It was too difficult to follow their claims, and I did not know what to do. This is the paper written in 1967 by Russian physicists who insisted that they proved the renormalizability of non-abelian gauge theory. This is quite an important claim and therefore, I worked very hard to understand its physical meaning of this paper. But I could not understand their claim, and thus I asked Professor Nishijima about this paper. However, to my surprise, he simply told me that he could not understand this paper either. This was very shocking since he was, at that time, one of the most famous and eminent physicists in the world. This was very serious since this means that the paper which insisted to have proved the renormalizability of non-abelian gauge theory must be incorrect.

1.5.7 No Perturbation Scheme in Non-Abelian Gauge Theory

Before going to the renormalization scheme, we should discuss the perturbative method itself in the non-abelian gauge theory. Indeed, we can easily prove that there should be no free state in the non-abelian gauge theory, and therefore, we cannot define the perturbative scheme itself before the renormalization. This can be proved since the color charge of the non-abelian gauge theory is gauge dependent and thus the constituents of the gauge model should not be observables. Since the free Lagrangian density is gauge dependent, there is no way to define the perturbation scheme, and thus, we cannot calculate any observables in the non-abelian gauge theory models at all. This means that there is no divergence in this model from the beginning, and thus, it has nothing to do with the renormalization scheme.

• **Constituents in Non-Abelian Gauge Theory (QCD) :**
The quantum chromodynamics is the non-abelian gauge theory and this is the basic field theory model to describe the strong interaction. In this case, quarks and gluons are not physical observables, and indeed they are not observed yet experimentally until now.

• **Constituents in Non-Abelian Gauge Theory (Weak Interactions) :**
Also, the Weinberg-Salam model that is supposed to describe the weak interactions is the non-abelian gauge theory. Therefore, the constituents of the model (vector bosons $W^{\pm}$, Z^0) should not be observables if the model is solved exactly. However, these authors of the model insist that they can break the local gauge symmetry in terms of Higgs mechanism, and therefore, they believe that these weak bosons must become observables after the approximations of Higgs mechanisms. In this way, they finally obtain the weak interaction model which is just the same as the CVC theory. Indeed, the CVC theory is known to reproduce the experiments very well, and thus, the final version of Weinberg-Salam model is quite successful for describing nature. However, from these arguments, it should be very difficult to accept that this model theory must be a standard model for weak interactions.

1.5.8 Renormalization of New Gravity

The new gravity model is described in terms of massless scalar field. It is quite possible that many physicists must have studied the gravity model in terms of scalar field. However, because of the renormalization problem, people may not have considered the scalar field model seriously as a gravity model.

Here I should explain why this gravity model with the scalar field is successful for describing all the observed physical quantities concerning gravity. This is closely related to the renormalization scheme.

• Renormalizability of Gravity

For a long time, people were worried about the renormalizability of any field theory models. This must be due to the fact that the renormalization scheme of QED is considered to be very successful for describing experiments. In addition, QED is a gauge theory, and therefore, people thought that any correct model should be a gauge theory. This is, of course, a ridiculous belief but most physicists took it rather seriously.

In reality, the strange divergence for physical observables must be only for the gauge theory model of QED with the Feynman propagator. Therefore, we should think of any other physical models for gravity so as to reproduce the observed quantities in connection with gravitational observables.

• Other Models than Gauge Theory

As stressed above, many physicists believe that the gravity must be constructed by a gauge theory model. But this has no physical foundation, and indeed the gauge theory cannot make a model which can produce only an attractive interaction. At the same time, people know quite well that the scalar field should produce the interaction which is always attractive.

• No Necessity of Field Quantization

In quantum field theory, it is sometimes necessary to consider the field quantization. This is, of course, required from experiments. For example, the vector potential should be quantized since photon can be understood only after the field quantization.

However, in case there is no requirement from experiments, then we should not make any field quantization. Here, it should be noted that the fermion fields must be always quantized in terms of anti-commutation relations since this is just required from the Pauli principle.

• Gravity as Scalar Field

The gravitational interaction is always attractive, and thus, this is the most important constraint in constructing the field theory model of gravity. Therefore, the gravity must be a scalar field. In addition, the gravity is a long range force, and thus, the field must be massless. Further, there is no graviton found in nature, and therefore, the field should not be quantized. This is all that is necessary to construct the field theory of gravity. In addition, the inertia mass is found to be just the same as the gravitational mass, and thus, the shape of interaction must be the same type as the mass term. The new gravity model should be explained in detail later in chapter 4.

Chapter 2

Symmetry in Physics

In nature, there are many types of symmetry which should play a very important role in physics. The symmetry corresponds to the invariance of Lagrangian density under the space and time transformation. From this, we can learn a lot about the properties of the system without solving the dynamics. Generally speaking, the symmetry property makes theoretical treatments easier and leads us to a simple but deep understanding of physics.

There are two important principles concerning the basic laws in physics. The first one is the relativity principle which is related to the invariance of the system under the Lorentz transformation, and this should hold rigorously. This symmetry must be satisfied by all the theoretical models. The second one is the Curie principle, and any models should satisfy this principle after solving the dynamics. This principle of Curie is related to the causality and should be always kept in nature.

2.1 Transformation and Invariance

What should be transformations in physics? Most of experts should know well the translational symmetry or rotational symmetry in space. Here we should briefly explain the transformation property in connection with the symmetry.

2.1.1 Lagrangian

In modern physics, Lagrangian plays a crucial role for obtaining the equation of motion. Here, we first discuss the Lagrangian in classical mechanics. The simplest Lagrangian must be two body system since one body problem is only for a free particle. The Lagrangian L of the two body system can be written as

$$L \equiv T - V = \frac{1}{2} m_1 \dot{\boldsymbol{r}}_1{}^2 + \frac{1}{2} m_2 \dot{\boldsymbol{r}}_2{}^2 - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{2.1}$$

where T denotes the kinetic energy (the first two terms) while the third term (potential term) corresponds to the two body potential.

2.1.2 Lagrangian Density

On the other hand, field theory can be described by the Lagrangian density which is written in terms of field variable ψ (fermion field) and its derivatives $\partial_\mu\psi$. In fact, QED, QCD, weak interactions and gravity are all written in terms of the Lagrangian density which should be some functions of field variables. In this case, the symmetry is related to the invariance under the transformation of the field variable ψ and its derivatives $\partial_\mu\psi$. In particular, the Lagrangian density in field theory model must be invariant under the Lorentz transformation. Here, we should just write, as an example, the QED Lagrangian density which is composed of fermion field ψ as well as gauge field A^μ as

$$\mathcal{L} = i\bar{\psi}\partial_\mu\gamma^\mu\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $F^{\mu\nu}$ is defined as $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. All the terms in the Lagrangian density should be a scalar under the Lorentz transformation.

2.2 Transformation in Classical Mechanics

First we should discuss the symmetry property in classical mechanics. This can be done in terms of Lagrangian which should depend on the coordinates (t, x, y, z). Therefore the symmetry is related to the transformation of the coordinates.

• **Translation of Space :**
Under the space transformation of $\boldsymbol{r} \Rightarrow \boldsymbol{r} + \boldsymbol{c}$ with $\boldsymbol{c}$ a constant vector, Lagrangian (2.1) is invariant. From this, we can prove that the momentum of the system is conserved. To see it explicitly, we show that, for the infinitesimally small $\boldsymbol{c}$, we find

$$\delta L = L(\boldsymbol{r}+\boldsymbol{c}, \dot{\boldsymbol{r}}) - L(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \boldsymbol{r}} \cdot \boldsymbol{c} = 0. \tag{2.2}$$

Further, from the following Lagrange equation

$$\frac{d}{dt}\frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \dot{\boldsymbol{r}}} = \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \boldsymbol{r}} \tag{2.3}$$

we obtain

$$\frac{d}{dt}\frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \dot{\boldsymbol{r}}} = \frac{d\boldsymbol{p}}{dt} = 0, \quad \Rightarrow \quad \boldsymbol{p} = \text{const.} \tag{2.4}$$

• Translation of Time :
Under the time transformation of $t \Rightarrow t + d$, the Lagrangian of eq.(2.1) is invariant. In this case, the energy of the system is conserved. This can be shown in the following way. First, for the infinitesimally small d, we find

$$\delta L = L(t+d) - L(t) = \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial t} d = 0. \tag{2.5}$$

On the other hand, we can calculate

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \boldsymbol{r}} \dot{\boldsymbol{r}} + \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \dot{\boldsymbol{r}}} \ddot{\boldsymbol{r}} + \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial t} = \frac{d}{dt} \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \dot{\boldsymbol{r}}} \dot{\boldsymbol{r}} + \frac{\partial L(\boldsymbol{r}, \dot{\boldsymbol{r}})}{\partial \dot{\boldsymbol{r}}} \ddot{\boldsymbol{r}} \\ &= \dot{\boldsymbol{p}} \cdot \dot{\boldsymbol{r}} + \boldsymbol{p} \cdot \ddot{\boldsymbol{r}} = \frac{d}{dt} \Big(\boldsymbol{p} \cdot \dot{\boldsymbol{r}} \Big). \end{aligned} \tag{2.6}$$

Thus, we obtain

$$\frac{d}{dt} \Big(\boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L \Big) = 0 \tag{2.7}$$

which means that the Hamiltonian $H \equiv \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L = T + V$ is conserved.

• Space Rotation :
Under the rotation of the finite angle with respect to the coordinate center, the potential is invariant if it is only a function of radial part, that is $V(\boldsymbol{r}) = V(r)$. In this case, the force $\boldsymbol{F}$ becomes

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(r) = -\frac{\boldsymbol{r}}{r} \left(\frac{dU(r)}{dr} \right). \tag{2.8}$$

In this case, the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ should be conserved. This can be shown by doing the following direct calculation as

$$\frac{d\boldsymbol{L}}{dt} = \dot{\boldsymbol{r}} \times m\dot{\boldsymbol{r}} + \boldsymbol{r} \times m\ddot{\boldsymbol{r}} = -\boldsymbol{r} \times \boldsymbol{r} \left(\frac{dU(r)}{r dr} \right) = 0. \tag{2.9}$$

2.3 Transformation Property of Fields

When we consider the transformation of the Lagrangian density or Hamiltonian density under the symmetry operator U, we first evaluate the transformation of the field ψ as

$$\psi' = U\psi. \tag{2.10}$$

In addition, we calculate and see how the Lagrangian density $\mathcal{L}$ should be transformed under the symmetry operator U

$$\mathcal{L}' \equiv \mathcal{L}(\psi', \partial_\mu \psi') = \mathcal{L}(U\psi, \partial_\mu (U\psi)). \tag{2.11}$$

If the Lagrangian density does not change its functional shape under the transformation of U,

$$\mathcal{L}' = \mathcal{L} \tag{2.12}$$

then it is invariant under the transformation of U.

After the field quantization, the transformation procedure becomes somewhat different. The basic physical quantity in quantized field theory becomes the Hamiltonian $\hat{H}$, and the important point is that the Hamiltonian is now an operator and the problem becomes the eigenvalue equation for the field Hamiltonian

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{2.13}$$

where $|\Psi\rangle$ is called *Fock state.* Since an operator $\mathcal{O}$ transforms under the symmetry operator U as

$$\mathcal{O}' = U\mathcal{O}U^{-1} \tag{2.14}$$

the Hamiltonian $\hat{H}$ transforms as

$$\hat{H}' = U\hat{H}U^{-1}. \tag{2.15}$$

In this case, the Fock state $|\Psi\rangle$ should transform as

$$|\Psi'\rangle = U|\Psi\rangle. \tag{2.16}$$

The transformation properties of the Lagrangian density should be kept for the quantized Hamiltonian after the field quantization.

2.3.1 Some Examples of Transformation in Field Theory

In field theory, Lagrangian density depends on the field variable ψ and its derivatives $\partial_\mu \psi$, and therefore, the symmetry is related to the transformation properties of the field variable ψ. The main part of the symmetry in field theory should be found in Appendix B, and therefore, readers should go to this Appendix in order to check these equations in detail.

• **Lorentz Transformation :**
All the Lagrangian densities must be invariant under the Lorentz transformation. Thus, the Lagrangian density must be a Lorentz scalar, which is the absolutely necessary condition.

• **Translation of Time and Space :**
In field theory, the field variable $\psi(t, \boldsymbol{r})$ should depend on time t and space coordinate $\boldsymbol{r}$. Here if we make the infinitesimal translation of quantities $(\delta t, \delta \boldsymbol{r})$, then the field variable should vary accordingly. If the Lagrangian density should be invariant for this translation of space and time, then we can derive the conservation law of energy and momentum tensors.

• **Global Gauge Transformation :**
Suppose that, under the change of the field variable from ψ to $\psi' = e^{i\alpha}\psi$, the Lagrangian density should be invariant. In this case, the charge of the system should be conserved.

• **Global Chiral Gauge Transformation :**
If the Lagrangian density is invariant under the change of the field variable from ψ to $\psi' = e^{i\alpha\gamma_5}\psi$, then this system should have the chiral symmetry, and its chiral charge is a conserved quantity.

• **SU(N) Transformation :**
Suppose the field variable ψ should have N components, and we consider the transformation of $\psi' = U\psi$ where U denotes an element of SU(N) group. If the Lagrangian density is invariant under the SU(N) transformation, this system should have a SU(N) invariance and the state should be specified by the representation of the SU(N) group.

2.3.2 Lorentz Transformation

The most important symmetry in physics must be the Lorentz invariance. The Lorentz invariance should hold in the theory of all the fundamental interactions. This is based on the observation that any physical observables should not depend on the systems one chooses if the systems $\mathcal{S}$ and $\mathcal{S}'$ are related to each other by the

Lorentz transformation,

$$x'^{\mu} = \alpha^{\mu}{}_{\nu} x^{\nu}. \tag{2.17}$$

If the $\mathcal{S}'$ system is moving with its velocity of v along the x_1-axis, then the matrix $\alpha^{\mu}{}_{\nu}$ can be explicitly written as

$$\{\alpha^{\mu}_{\nu}\} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

where

$$\cosh\omega = \frac{1}{\sqrt{1-v^2}} \tag{2.19}$$

is introduced. In this case, the Dirac wave function ψ should transform by the Lorentz transformation as

$$\psi'(x') = S\psi(x), \tag{2.20}$$

where S denotes a 4×4 matrix. Now, the Lagrangian density for free Dirac field is written both in $\mathcal{S}$ and $\mathcal{S}'$ systems

$$\mathcal{L} = \bar{\psi}(x)(i\partial_\mu \gamma^\mu - m)\psi(x) = \bar{\psi}'(x')(i\partial_\mu{}' \gamma^\mu - m)\psi'(x'). \tag{2.21}$$

From the equivalence between $\mathcal{S}$ and $\mathcal{S}'$ systems, one obtains

$$\bar{\psi}'(x') = \bar{\psi}(x) S^{-1}, \tag{2.22}$$

$$S\gamma^\mu S^{-1} \alpha_\mu{}^\nu = \gamma^\nu. \tag{2.23}$$

If one solves eq.(2.23), then one can determine the shape of S explicitly when the $\mathcal{S}'$ system is moving along the x_1-axis

$$S = \exp\left(-\frac{i}{4}\,\omega\sigma_{\mu\nu} I_n^{\mu\nu}\right), \tag{2.24}$$

where $\sigma_{\mu\nu}$ and $I_n^{\mu\nu}$ are defined as

$$\sigma_{\mu\nu} = \frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right), \qquad I_n^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.25}$$

2.4 Principle of Curie

At the end of 19th century, Curie proposed a physical law concerning the symmetry in nature. The principle of Curie states that the asymmetric phenomena should be realized only when there is a force that breaks this symmetry. Otherwise, the symmetry in nature must be preserved for ever. Up to now, there is no incident which contradicts the principle of Curie.

2.4.1 Piezoelectric Effect

There is an interesting phenomenon which is called a piezoelectric effect. When one puts a mechanical pressure on some crystal substance (soft matter), then this substance makes electric polarizations which can induce electric displacements. These phenomena were discovered by Pierre Curie more than 100 years ago.

In these phenomena, the mechanical force on the crystal should break some of its symmetry and thus, the force can induce the polarization of the crystal. This induced polarization is directly related to the electromagnetic effects. Therefore, the piezoelectric effect is to connect the mechanical force to the electric interaction, and therefore, this effect is made for many kinds of applications. In particular, it is, by now, very popular that, by touching liquid crystal display, one can transform the hand signal into the electric signal.

2.4.2 Principle of Curie in Field Theory

In field theory, the principle of Curie should hold, and the symmetry of the system cannot be broken unless there should be some force from outside the system. Unfortunately, however, the claim of the spontaneous symmetry breaking became quite popular at some time. This is the claim by Nambu and other people, but it is, by now, clear that their paper has made some important approximation in their calculation and the apparent breaking of the symmetry is simply a mistake. If they knew and understood the principle of Curie in a correct way, then they would not have made such an easy mistake like the spontaneous symmetry breaking.

2.5 Symmetry in Field Theory

There are many different kinds of symmetry in field theory and the symmetry in connection with Lorentz transformation must be most important. In addition to the Lorentz transformation, the translation symmetry of space and time should be very important, and from this symmetry, we can derive the conservation laws related to the energy and momentum of the corresponding system.

2.5.1 Symmetry and Conservation Law

Here we should write the energy and momentum tensor $T^{\mu\nu}$ which is defined as

$$\mathcal{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\partial^\nu \psi^\dagger - \mathcal{L} g^{\mu\nu}$$

where $\partial_\mu \mathcal{T}^{\mu\nu} = 0$ should hold. This equation of $\partial_\mu \mathcal{T}^{\mu\nu} = 0$ shows the energy and momentum conservation in field theory.

• **Energy-Momentum Tensor in Einstein Equation :**
Here we should make a comment on the energy-momentum tensor which appears in the Einstein equation. This energy-momentum tensor is somewhat strange in physics since classical mechanics should not have any field theory picture, and therefore, the energy-momentum tensor that are some function of fields cannot be made in a proper way. In the general relativity, Einstein made use of the distribution function of stars in order to make the energy-momentum tensor. However, it is clear that the star distribution function should not be any fundamental physical quantities. In this respect, the Einstein equation cannot be taken as the basic equation in physics.

2.5.2 Global Gauge Symmetry

In some of field theory models, there should be a global gauge symmetry. In this case, when we transform the state vector of ψ, then the Lagrangian density is invariant, and this leads to the conservation of charge.

• **Global Gauge Transformation :**
The global gauge transformation is to transform the state vector ψ as

$$\psi' = e^{i\alpha}\psi.$$

If this transformation can keep the Lagrangian density invariant $\mathcal{L}' = \mathcal{L}$, then the vector current j^μ should be conserved from Noether's theorem (Appendix B)

$$\partial_\mu j^\mu = 0, \quad \text{with} \quad j^\mu \equiv -i\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\psi^\dagger\right].$$

Up to now, all the fermion models should have this global gauge symmetry, and therefore, the charge of these models should be conserved.

2.5.3 Chiral Symmetry

In the massless field theory, some models should have the symmetry under the global chiral gauge transformation.

• Global Chiral Gauge Transformation :
The chiral gauge transformation is written as

$$\psi' = e^{i\alpha\gamma_5}\psi.$$

If the Lagrangian density is invariant under this chiral gauge transformation $\mathcal{L}' = \mathcal{L}$, then the chiral current j^5_μ should be conserved

$$\partial_\mu j^5_\mu = 0$$

where α is a constant. The chiral symmetry should be violated if there is a mass term present. This means that the fermion model with the chiral symmetry must be composed of massless fermions. However, if the fermion is massless, then we cannot specify the rest frame in which we should calculate physical quantities. Therefore, massless fermion models are physically meaningless, and therefore, the field theory models with the chiral symmetry cannot physically make sense, and thus, these models should be simply meaningless.

2.5.4 Symmetry and Its Breaking

Nevertheless, Nambu and others presented some special model in which massless fermions are interacting with each other by self-interactions of the following type

$$\mathcal{H}' = g(\bar{\psi}\gamma_5\psi)^2.$$

This is indeed invariant under the chiral transformation of $\psi' = e^{i\alpha\gamma_5}\psi$.

• Spontaneous Symmetry Breaking :
In this model, Nambu and others claimed that the chiral symmetry is spontaneously broken for the vacuum state. This claim turns out to be simply incorrect since the symmetry must be preserved unless some external force is present [4].

• Cause of Mistakes :
What should be the main reason as to why they made such elementary mistakes? There may be two reasons for the mistakes. The first one is connected to the quantum number of the chiral charge. The quantum number of the chiral charge in free field theory should be zero while that of the interacting field theory model turns out to be different from zero. But this has nothing to do with the symmetry breaking, and the quantum number difference of the vacuum states in two different field theory models should be quite natural in physics.

The second reason of their mistakes is related to the approximation scheme they employed. They made use of the Bogoliubov transformation which is, of course, an approximate method, and this method sometimes may induce the apparent mass term for which they thought that the system must have lost the chiral symmetry. However, they simply neglected the higher order terms which certainly recover this symmetry. This is a very low level mistake, and the fact that most people believed this incorrect claim for a long time is disastrous.

2.6 Chiral Anomaly

When the Lagrangian density is invariant under the chiral transformation in field theory models, then the axial vector current should be conserved. This means

$$\partial_\mu j_5^\mu = 0 \qquad (\text{with} \quad j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi)$$

should hold. This conservation law is valid only when the Lagrangian density is written for the massless fermion case. Therefore, as discussed above, this massless fermion model is not realistic since massless fermions cannot make sense in nature due to the lack of its rest system.

On the other hand, the chiral anomaly is not related to the symmetry property of the system. Instead, Adler claimed that the chiral anomaly equation can be derived by regularizing the linear divergence in the triangle diagrams, and therefore, it is the result of the mathematical manipulation. This equation looks interesting and thus attracts many people. However, physically it should not make any sense since the physical law derived from the Noether theorem cannot be broken by the mathematical procedure without any physical reasons. As we discussed in the previous chapter in terms of principle of Curie, the conservation law cannot be broken without any effects of external force.

In fact, it turns out by now that this is a simple mistake at the lowest level since the triangle diagrams do not have any divergences if calculated properly. In addition, it is well known that the triangle diagrams with the axial vector vertex should vanish exactly to zero due to the Landau-Yang theorem from the group theoretical evaluation. Also, we can easily pin down the mistake where it should arise in Adler's paper. Further, the careful calculation of the triangle diagrams with γ_5 vertex was done by Nishijima before Adler's calculations [3]. Indeed, his calculated result can reproduce well the observed value of the decay process $\pi^0 \to \gamma + \gamma$. Therefore, the anomaly equation has nothing to do with physics.

2.6.1 $Z^0 \to \gamma + \gamma$ process

Here, we should explain the physical processes which involve the axial vector coupling with vector bosons. In fact, if we include the weak interactions, then the triangle diagrams with the axial vector coupling should be connected to the physical observables. Namely, there is a possible decay process of a weak vector boson into two photons, that is, $Z^0 \to 2\gamma$. However, this decay process is forbidden due to the Landau-Yang theorem [6, 7]. The physical reason why the decay rate of $Z^0 \to 2\gamma$ process vanishes to zero can be understood from the symmetry arguments. The two photon state $(1^- \otimes 1^-)$ can make a 1^+spin state in terms of spin products. However, the two photon state must be symmetric due to its bosonic nature, while the 1^+spin state which is made of two vector bosons must be anti-symmetric due to the group theoretical arguments. Therefore, the decay of $Z^0 \to 2\gamma$ process is forbidden as a physical process even with the parity non-conserving interaction.

Nevertheless, we should carry out the T-matrix evaluation of the Feynman diagrams corresponding to the Z^0 decay into two photons, and we show that the triangle diagrams with the axial vector coupling have neither linear nor logarithmic divergences. In particular, the fact that there is no linear divergence is proved without any regularizations since it is evaluated before the momentum integrations. The total amplitude of $Z^0 \to 2\gamma$ decay process indeed vanishes to zero.

- **T-matrix Evaluation**

Here, we briefly explain the T-matrix evaluation, and the corresponding T-matrix for the Z^0 boson decaying into two photons can be calculated to be

$$T_{Z^0\to 2\gamma} \simeq e^2 g_z \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\gamma\epsilon_1) \frac{1}{\not{p} - M + i\varepsilon} (\gamma\epsilon_2) \right. \tag{2.26}$$

$$\left. \frac{1}{\not{p} - \not{k}_2 - M + i\varepsilon} \gamma_5 (\gamma\epsilon_v) \frac{1}{\not{p} + \not{k}_1 - M + i\varepsilon} \right] + (1 \leftrightarrow 2). \tag{2.27}$$

- **Linear Divergence**

Here, it is shown that the apparent linear divergence vanishes to zero before the momentum integration. This can be easily proved since the corresponding Trace evaluation of the T-matrix becomes

$$\mathrm{Tr}[\not{p}\gamma_\mu \not{p}\gamma_\nu \not{p}\gamma_\rho \gamma_5]\epsilon_1^\mu \epsilon_2^\nu \epsilon_v^\rho + \mathrm{Tr}[\not{p}\gamma_\mu \not{p}\gamma_\nu \not{p}\gamma_\rho \gamma_5]\epsilon_2^\mu \epsilon_1^\nu \epsilon_v^\rho = 0 \tag{2.28}$$

where we have made use of the following identity equation

$$\mathrm{Tr}[\not{p}\gamma_\mu \not{p}\gamma_\nu \not{p}\gamma_\rho \gamma_5] = -\mathrm{Tr}[\not{p}\gamma_\nu \not{p}\gamma_\mu \not{p}\gamma_\rho \gamma_5]. \tag{2.29}$$

Therefore, the linear divergence disappears in eq.(2.27) before carrying out the momentum integration.

2.7 Problem of Higgs Mechanism

Here, we explain briefly the problem of the Higgs mechanism in terms of simple field theory language. The Lagrangian density of the Higgs mechanism is composed of complex scalar fields which couple with the gauge field. The Higgs mechanism is an approximate scheme in which one fixes the gauge at the Lagrangian density level. In the method of the Higgs model, they claim that the gauge field which is a massless fermion field can acquire its mass in some way or other. But this procedure cannot be justified by any means since incorrect and improper treatments are employed as discussed below.

2.7.1 Higgs Potential

The Lagrangian density of the Higgs mechanism is given as

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^\dagger (D^\mu \phi) - U(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.30}$$

where $U(\phi)$, D^μ, $F^{\mu\nu}$ are defined as

$$U(\phi) = -\frac{1}{4} u_0 \left(|\phi|^2 - \lambda^2 \right)^2 \tag{2.31}$$

$$D^\mu = \partial^\mu + igA^\mu \tag{2.32}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu . \tag{2.33}$$

The field potential $U(\phi)$ is called Higgs potential which is introduced in the spontaneous symmetry breaking physics by hand. This is a self-interacting potential which is quite difficult to understand from the basic physics point of view. At least, this type of self-interacting potential has never been required from experiments, and therefore, it is physically far from realistic.

2.7.2 Higgs Mechanism

Here, we only consider the U(1) case since it is sufficient for the present discussions. The above Lagrangian density is indeed gauge invariant, and in this respect, the scalar field may interact with gauge fields in eq.(2.30). However, it should be noted that there is no experimental indication that the fundamental scalar field can interact with any gauge fields in terms of the Lagrangian density of eq.(2.30). In this sense, this is only a toy model. Now, the equations of motion for the scalar field ϕ become

$$\partial_\mu (\partial^\mu + igA^\mu)\phi = -u_0 \phi \left(|\phi|^2 - \lambda^2 \right) - igA_\mu (\partial^\mu + igA^\mu)\phi \tag{2.34}$$

$$\partial_\mu (\partial^\mu - igA^\mu)\phi^\dagger = -u_0 \phi^\dagger \left(|\phi|^2 - \lambda^2 \right) + igA_\mu (\partial^\mu - igA^\mu)\phi^\dagger . \tag{2.35}$$

On the other hand, the equation of motion for the gauge field A_μ can be written as

$$\partial_\mu F^{\mu\nu} = gJ^\nu \tag{2.36}$$

where

$$J^\mu = \frac{1}{2} i \left\{ \phi^\dagger (\partial^\mu + igA^\mu)\phi - \phi(\partial^\mu - igA^\mu)\phi^\dagger \right\}. \tag{2.37}$$

We can also check that the current J^μ is conserved, that is

$$\partial_\mu J^\mu = 0. \tag{2.38}$$

This Lagrangian density of eq.(2.30) has been employed for the discussion of the Higgs mechanism.

2.7.3 Gauge Freedom and Number of Independent Equations

Now, we should count the number of the degrees of freedom and the number of equations. For the scalar field, we have two independent functions ϕ and $\phi^\dagger$. Concerning the gauge fields A^μ, we have four since there are A^0, A^1, A^2, A^3 fields. Thus, the number of the independent fields is six. On the other hand, the number of equations is five since the equation for the scalar fields is two and the number of the gauge fields is three. This number of three can be easily understood, even though it looks that the independent number of equations is four, but due to the current conservation the number of the independent equations becomes three. This means that the number of the independent functions is six while the number of equations is five, and they are not equal. This is the gauge freedom, and therefore in order to solve the equations of motion, one has to put an additional condition for the gauge field A_μ like the Coulomb gauge which means $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. In this respect, the gauge fixing is simply to reduce the redundant functional variable of the gauge field A_μ to solve the equations of motion, and nothing more than that.

2.7.4 Unitary Gauge Fixing

In the Higgs mechanism, the central role is played by the gauge fixing of the unitary gauge. The unitary gauge means that one takes

$$\phi = \phi^\dagger. \tag{2.39}$$

This is the constraint on the scalar field ϕ even though there is no gauge freedom in this respect. For the scalar field, the phase can be changed, but this does not mean that we can erase one degree of freedom. We should transform the scalar field in the gauge transformation as

$$\phi' = e^{-ig\chi}\phi \tag{2.40}$$

but we must keep the number of degrees of freedom after the gauge transformation. Whatever we fix the gauge χ, we cannot change the shape of the scalar field ϕ since it is a functional variable which must be determined by solving the equations of motion in eq.(2.34).

The gauge freedom is, of course, the property of the vector potential A_μ, and this becomes important only when we want to solve the equations of motion for the A_μ. This is connected to the fact that the degrees of freedom of the vector potential should be four while the number of equations is three. Therefore, we should reduce one degree of freedom of the A_μ by making the gauge fixing so as to solve the equation. In this sense, one sees that the unitary gauge fixing is a simple mistake. The basic reason why people overlooked this simple-minded mistake must be due to their obscure presentation of the Higgs mechanism. Also, it should be related to the fact that, at the time of presenting the Higgs mechanism, the spontaneous symmetry breaking physics was not understood properly since the vacuum of the corresponding field theory was far beyond the proper understanding. Indeed, the Goldstone boson after the spontaneous symmetry breaking was taken to be almost a mysterious object since there was no experiment which suggests any existence of the Goldstone boson. Instead, a wrong theory prevailed among physicists. Therefore, they could assume a very unphysical procedure of the Higgs mechanism and people pretended that they could understand it all.

2.7.5 Non-abelian Gauge Field

Now, one should be careful for the renormalizability of the non-abelian gauge field theory. As one can easily convince oneself, the non-abelian gauge theory has an intrinsic problem of the perturbation theory. This is connected to the fact that the color charge in the non-abelian gauge field depends on the gauge transformation, and therefore it cannot be physical observables. This means that the free gauge field which has a color charge is gauge dependent, and thus, one cannot develop the perturbation theory in a normal way. In QCD, this is exhibited as the experimental fact that both free quarks and free gluons are not observed in nature. The absence of free fields is a kinematical constraint and thus it is beyond any dynamics. Therefore, one cannot discuss the renormalizability of the non-abelian gauge field theory models due to the lack of the perturbation scheme in this model field theory [4]. Therefore, the problem of the renormalizability in the non-abelian field theory model is a meaningless subject since the perturbation theory is not defined in this field theory.

2.7.6 Summary of Higgs Mechanism

The intrinsic problem of the Higgs mechanism is discussed in terms of the gauge fixing condition which is improperly imposed. This is also related to the misunderstanding of the spontaneous symmetry breaking physics. Here, we have shown that the Higgs mechanism cannot be justified since the gauge invariance of the Lagrangian density is violated by hand.

However, we believe that the final version of the weak Hamiltonian should be correct since the standard model is adjusted to reproduce the CVC theory which is a very reliable model of weak interactions. Indeed, the basic reason why the standard model Hamiltonian becomes a reasonable model is due to the fact that the final version of the model Hamiltonian should not possess the two unphysical properties of the starting Lagrangian density, non-abelian character and gauge invariance. In their treatments, Weinberg and Salam have made two serious mistakes and thus the model should get back to the right Hamiltonian which can describe the nature. As one sees by now, the first mistake is related to the non-abelian character of the gauge field theory model which should not allow us any perturbative treatments. But they made the model into the non-gauge theory by making the improper gauge fixing at the Lagrangian level. The second mistake is concerned with the breaking of the local gauge invariance in terms of Higgs mechanism. By employing the Higgs mechanism which is completely a wrong procedure, the gauge field could become massive. Therefore, if we remove the Higgs fields and the non-abelian nature of the massive vector bosons from the Weinberg-Salam model, then the final Hamiltonian of the standard model should be physically acceptable.

At this point, we should make a comment on the present status of the Higgs particle search. Up to now, there is no indication of the existence of the Higgs particle in spite of the fact that the total period of the experimental efforts of the Higgs search must be almost more than three decades. The main difference between the weak vector bosons (W and Z) and Higgs particle searches can be understood in the following way. The Higgs particle search started from the theoretical requirements (though incorrect) without having any firm experimental motivations of its existence, while the W-boson cases had many experimental indications of their existence before they were discovered by the UA1 and UA2 collaborations of the CERN-SPS experiments in 1983.

2.8 Theory of Weak Interaction

It should be important to construct the Lagrangian density which can describe the weak interaction processes. The basic starting point is, of course, the conserved vector current (CVC) theory which can describe most of the observed weak decay processes quite well. This CVC theory should be derived from the second order perturbation theory by exchanging the weak vector bosons between corresponding fermions.

2.8.1 Lagrangian Density of CVC Theory

The theory of the weak interactions is developed in terms of the four fermion interaction model by Fermi and, after some time, Feynman and Gell-Mann extended it to the conserved vector current (CVC) theory, which is quite successful for describing experiments. The Lagrangian density of the CVC theory can be written as

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c. \tag{2.41}$$

where G_F denotes the weak coupling constant $G_F \simeq 1.2 \times 10^{-5} \frac{1}{M_p^2}$. Also J^μ is composed of the leptonic and hadronic currents and is written as

$$J^\mu = j_\ell^\mu + j_h^\mu \tag{2.42}$$

where both of the currents can be expressed as

$$j_\ell^\mu = \bar{\psi}_{\nu_e}\gamma^\mu(1-\gamma^5)\psi_e + \bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma^5)\psi_\mu + \cdots \tag{2.43}$$

$$j_h^\mu = \cos\theta\bar{\psi}_u\gamma^\mu(1-\gamma^5)\psi_d + \sin\theta\bar{\psi}_u\gamma^\mu(1-\gamma^5)\psi_s + \cdots . \tag{2.44}$$

It should be important to note that the current-current interaction model can describe many experimental data to a very high accuracy, and this is, indeed, a well-known fact before the discovery of the weak vector bosons of $W^\pm, Z^0$.

2.8.2 Divergence in CVC Theory

However, this model Hamiltonian of CVC theory should have a serious problem related to the divergence in the second order perturbation theory. Since the coupling constant G_F is very small compared to the fine structure constant, one can expect that the second order perturbation must be reliable. On the contrary, however, the second order calculation has a quadratic divergence since the coupling constant G_F has the dimension of the inverse square of the energy. Therefore, it is clear that this theoretical framework should have an intrinsic problem of the divergence, and thus it should be very important to construct a theory which should not have any divergences in the second order perturbation theory.

2.8.3 Massive Vector Field Theory

Here, we should write the simplest Lagrangian density for two flavor leptons which couple to the SU(2) vector fields W_μ^a

$$\mathcal{L} = \bar{\Psi}_\ell(i\partial_\mu\gamma^\mu - m)\Psi_\ell - gJ_\mu^a W^{\mu,a} + \frac{1}{2}M^2 W_\mu^a W^{\mu,a} - \frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a} \tag{2.45}$$

where M denotes the mass of the vector boson. Here, we do not write the hadronic part, for simplicity. The lepton wave function Ψ_ℓ has two components

$$\Psi_\ell = \begin{pmatrix} \psi_e \\ \psi_\nu \end{pmatrix}. \tag{2.46}$$

Correspondingly, the mass matrix can be written as

$$m = \begin{pmatrix} m_e & 0 \\ 0 & m_\nu \end{pmatrix}. \tag{2.47}$$

The fermion current J_μ^a and the field strength $G_{\mu\nu}^a$ are defined as

$$J_\mu^a = \bar{\Psi}_\ell\gamma_\mu(1-\gamma_5)\tau^a\Psi_\ell, \qquad G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a. \tag{2.48}$$

The Lagrangian density of eq.(2.45) is almost the same as the standard model Lagrangian density, apart from the Higgs fields and the abelian nature. In fact, there is no experiment in weak process which cannot be described by the Lagrangian density of eq.(2.45). The only thing which, people thought, may be a defect in the above Lagrangian density should be concerned with the renormalization of the theory.

As we see below, the problem of the renormalization is completely solved by employing the right propagator of the massive vector bosons. This means that there is no logarithmic divergence in the evaluation of the vertex corrections due to the propagations of the massive vector bosons. Therefore, we do not have to consider any renormalization procedures since all the physical observables in the perturbative calculations are found to be finite.

2.9 Propagator of Massive Vector Boson

Here, we briefly review the derivation of the new propagator of the massive vector boson which has recently been evaluated properly in terms of the polarization vector. The correct shape of vector boson propagator is found to be the one given as

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}. \tag{2.49}$$

This is quite important since this does not generate any divergences in the vertex correction of fermions any more. On the other hand, the old propagator in the textbooks

$$D^{\mu\nu}_{old}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2 - i\varepsilon} \tag{2.50}$$

gives rise to some divergences [4]. This old propagator is obtained by making use of the Green function method. However, the summation of the polarization vectors cannot be connected to the Green function as we discuss below, and thus the employment of the old propagator is incorrect if one should treat the physical processes which involve the loop integral.

2.9.1 Lorentz Conditions of $k_\mu \epsilon^\mu = 0$

Here, we briefly explain how we can obtain eq.(2.49). The free Lagrangian density for the vector field Z^μ with its mass M is written as

$$\mathcal{L}_Z = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} M^2 Z_\mu Z^\mu \tag{2.51}$$

with $G^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$. In this case, the equation of motion becomes

$$\partial_\mu(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + M^2 Z^\nu = 0. \tag{2.52}$$

Since the free massive vector boson field should have the following shape of the solution

$$Z^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{3} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \epsilon^\mu_{k,\lambda} \left[c_{\boldsymbol{k},\lambda} e^{ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} \right] \tag{2.53}$$

we can insert this solution into eq.(2.52) and obtain the following equation for the polarization vector ϵ^μ

$$(k^2 - M^2)\epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{2.54}$$

The condition that there should exist a non-zero solution for the ϵ^μ requires that the determinant of the matrix should be zero, namely

$$\det\{(k^2 - M^2) g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{2.55}$$

This equation can be easily solved, and we find the following equation

$$k^2 - M^2 = 0 \tag{2.56}$$

which is the only physical solution of eq.(2.55). Therefore, we insert this solution into eq.(2.54) and obtain the equation for the polarization vector ϵ^μ

$$k_\mu \epsilon^\mu = 0. \tag{2.57}$$

Here, we should note that this process of determining the condition on the wave function of ϵ^μ is just the same as solving the free Dirac equation. Obviously, this is the most important process of determining the wave functions in quantum mechanics. Surprisingly, however, this has been missing in the treatment of determining not only the massive vector boson propagator but also the photon propagator as well. Also, one should notice that the condition of eq.(2.57) is just the same as the Lorentz gauge fixing condition in QED. However, one sees by now that the Lorentz condition itself can be obtained from the equation of motion, and therefore, it is more fundamental than the gauge fixing, even though the theory of massive bosons has no gauge freedom.

• **Lorentz Condition in QED :**
Since the Lorentz condition

$$k_\mu \epsilon^\mu = 0$$

is also obtained for the QED case, the Lorentz gauge fixing in QED should not be a proper gauge fixing procedure. This is clear since the Lorentz gauge fixing cannot give a further constraint on the polarization vector in the perturbation theory of QED. In addition, the number of degrees of freedom for the gauge fields can be understood properly since photon must have the two degrees of freedom due to the two constraint equations (the Lorentz condition and the gauge fixing condition).

2.9.2 Right Propagator of Massive Vector Boson

Now, we can evaluate the propagator of the massive vector field in the S-matrix expression. The second order perturbation of the S-matrix for the bosonic part can be written in terms of the T-product of the boson fields and it becomes

$$\langle 0|T\{Z^\mu(x_1)Z^\nu(x_2)\}|0\rangle = i\sum_{\lambda=1}^{3}\int \frac{d^4k}{(2\pi)^4}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}\frac{e^{ik(x_1-x_2)}}{k^2-M^2-i\varepsilon}. \tag{2.58}$$

After the summation over the polarization states, we find the following shape for $\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda}$ as

$$\sum_{\lambda=1}^{3}\epsilon^\mu_{k,\lambda}\epsilon^\nu_{k,\lambda} = -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{2.59}$$

which satisfies the Lorentz invariance and the Lorentz condition of the polarization vector. One sees that this is the only possible solution. Now one finds that the right propagator of the massive vector boson should be the one given in eq.(2.49)

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2 - i\varepsilon}.$$

2.9.3 Vertex Corrections by Weak Vector Bosons

Now we can calculate the vertex correction $\Lambda^\rho(p', p)$ of electromagnetic interaction due to the Z^0 boson. The Lagrangian density for the Z^0 boson which couples to the electron field ψ_e should be written as

$$\mathcal{L}_{Z^0} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 Z_\mu Z^\mu - g_z\bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_e Z^\mu. \tag{2.60}$$

The vertex correction $\Lambda^\rho(p', p)$ can be calculated by evaluating the corresponding Feynman diagrams, and we can find $\Lambda^\rho(p, p) = 0$. Therefore, there is no logarithmic divergence for the vertex correction from the weak massive vector boson propagations. This is very important in that the physical processes do not have any divergences when we make use of the proper propagator of the massive vector boson.

2.9.4 Electron $g-2$ by Z^0 Boson

Now the finite part of the vertex correction due to the Z^0 boson can be calculated and, therefore, the electron $g-2$ should be modified by the weak interaction to

$$\frac{g-2}{2} \simeq \frac{7\alpha_z}{12\pi}\left(\frac{m_e}{M}\right)^2 \simeq 2\times 10^{-14} \tag{2.61}$$

where

$$\alpha_z = \frac{g_z^2}{4\pi} \simeq 2.73\times 10^{-3}. \tag{2.62}$$

This is a very small effect, and therefore, it is consistent with the electron $g-2$ experiment. From this evaluation, we see that the right propagator of the massive vector field should be the one given by eq.(2.49).

2.10 Physics with Asymmetry

The symmetry of the system under some external force should be, in general, broken. Here we should briefly discuss some examples of asymmetric physical phenomena in quantum mechanics and field theory. Also, we should discuss some possible violation of time reversal invariance, and this is related to the neutron electric dipole moment (EDM) though it is not observed yet.

2.10.1 Zeeman Effect

The degeneracy of $1s_{\frac{1}{2}}$ state in hydrogen atom can be removed when the uniform external magnetic field $\boldsymbol{B}$ is applied to this system. In this case, the rotational symmetry in space is broken, and thus, the degenerate states is split into two states whose splitting energy ΔE becomes

$$\Delta E = \pm \frac{e\hbar}{2m_e c} B. \tag{2.63}$$

This is called Zeeman splitting, and it is a very important physical phenomenon that shows some asymmetric property in nature.

• **MRI(Nuclear Magnetic Resonance) :**
The Zeeman effect has been extensively applied to various types of apparatus or device. For example, the MRI is quite often used in medical science. There, the distribution of proton in water can be measured by making use of the Zeeman splitting under the strong magnetic field.

• **Zeeman Effect Method with Polarized Light :**
In order to measure quantities of organic mercury in human body, the Zeeman effect method with polarized light is proposed, which measures the absorption rate of photon. This technique should make use of the asymmetric property of spin states under the magnetic field. Due the conservation law of magnetic quantum numbers, this turns out to be very reliable.

2.10.2 Stark Effect

When the electric field $\boldsymbol{E}$ is applied to the hydrogen atom in the $z-$axis, then the interaction of $H' = ezE$ appears. This interaction breaks a rotational symmetry together with a space reflection symmetry. Therefore, the expectation value of the hydrogen ground state should vanish to zero. However, there is a finite matrix element between the two degenerate states of $2s$ and $2p$, and indeed one can check the first order perturbation effect of this interaction.

2.10.3 Spin-Orbit Interaction

The spin orbit interaction of $H' = \xi(r)\, \boldsymbol{\ell} \cdot \boldsymbol{s}$ is derived as the non-relativistic reduction of the Dirac equation with Coulomb interaction. This spin orbit interaction is proportional to the angular momentum $\boldsymbol{\ell}$, and thus, this interaction breaks the rotational symmetry, which is, of course, a result of the approximation. The Dirac equation should have the rotational symmetry in Lorentz space, and the non-relativistic approximation breaks the Lorentz symmetry, and therefore, the spin orbit interaction appears. It should be noted that the non-relativistic approximation is quite good for the hydrogen atom calculation.

2.10.4 Neutron Electric Dipole Moment (EDM)

By now we see that the neutron EDM cannot be caused by the CP violating interaction. However, this does not mean that the neutron EDM should vanish to zero. It should be still quite important to observe the neutron EDM in order to find the T-violating interaction, and this time it should be an interaction which violates the T-invariance at the operator level. The T-violating interaction should have a well-known shape, and it should be given as

$$\mathcal{H}_{edm} = -\frac{i}{2} d_f \bar{\psi} \sigma_{\mu\nu} \gamma^5 \psi F^{\mu\nu} \tag{2.64}$$

where d_f denotes the EDM of the corresponding fermion. The only point that should be considered more carefully is concerned with the dimension of the coupling constant d_f. The d_f has a dimension of length and therefore it may cause some troubles for the renormalization procedure, if necessary. However, it is most likely true that there should be no need of the renormalization procedure in connection with eq.(2.64) as far as physical observables are concerned. At the same time, it seems to be natural that the T-violating interaction is specified by some dimensional constant. This is because a breaking of the fundamental symmetry like T-violation should be determined by some breaking of dimensional property. In this respect, it should be very interesting to understand whether the T-violation can take place in nature or not.

2.11 Gauge Symmetry and Renormalization

Here we should discuss the renormalization scheme from the point of view of symmetry property in nature. This is because the renormalization procedure is treated mainly in the quantum electrodynamics (QED), and QED is invariant under the local gauge transformation which is a very special and powerful symmetry.

2.11.1 Local Gauge Symmetry

The renormalization theory itself is not directly related to the symmetry of the system. But the renormalization scheme is only applied to the quantum electrodynamics which should possess the invariance of local gauge transformation. Here we first write the Lagrangian density of QED

$$\mathcal{L} = \bar{\psi}\Big(i\partial_\mu\gamma^\mu - m - e\gamma_\mu A^\mu\Big)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.65}$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. This Lagrangian density is invariant under the local gauge transformation of

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi, \quad {A'}_0 = A_0 - \frac{\partial\chi}{\partial t} \tag{2.66}$$

$$\psi' = e^{ie\chi}\psi. \tag{2.67}$$

where χ is an arbitrary function that depends on space-time coordinates. The gauge symmetry means that the Lagrangian density must be invariant under the local gauge transformation of eq.(2.66), and this is easily confirmed. However, the transformation of fermion field $\psi' = e^{ie\chi}\psi$ should be necessary only if the invariance of Lagrangian density should be required.

• Gauge Transformation of Fermion $\psi' = e^{ie\chi}\psi$ Necessary? :
In the field theory textbooks, it is always explained that the condition of eq.(2.67) is necessary for the local gauge transformation. This is certainly true when we ask the local gauge invariance of the Lagrangian density. However, the important quantity in obtaining the equations of motion in field theory must be the action S which is defined as

$$S = \int \mathcal{L}\, d^4x = \int \Big(\bar{\psi}(i\partial_\mu\gamma^\mu - m - e\gamma_\mu A^\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big) d^4x. \tag{2.68}$$

In this case, if we ask the invariance of the action under the local gauge transformation, then the gauge transformation of

$$A'^\mu = A^\mu + \partial^\mu\chi \tag{2.69}$$

should be sufficient, and the transformation of eq.(2.67) is not needed.

• **Global Gauge Transformation :**
This is connected to the fact that the Lagrangian density of QED has the invariance under the global gauge transformation of

$$\psi' = e^{i\alpha}\psi \tag{2.70}$$

where α is a constant. Therefore, the vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$ must be conserved due to Noether's theorem

$$\partial_\mu j^\mu = 0. \tag{2.71}$$

In this case, we can prove that the action of S is invariant under the local gauge transformation of eq.(2.66) due to the current conservation of $\partial_\mu j^\mu = 0$, without making use of the fermion field transformation of eq.(2.67). This can be explicitly seen since the additional term of the action after the gauge transformation δS becomes

$$\delta S = \int e j_\mu (\partial^\mu \chi) d^4x = \int e\Big(\partial_\mu (j^\mu \chi) - (\partial_\mu j^\mu)\chi\Big) d^4x = 0. \tag{2.72}$$

Therefore, the action of QED is invariant under the local gauge transformation of eq.(2.69) without making use of eq.(2.67). In this respect, the fermion part of the gauge transformation of eq.(2.67) should not be a necessary condition in terms of fundamental physics law.

2.11.2 Divergence in Physical Observables

In case there is some infinity in the calculation of physical observables, then what should we do? Most likely, we should examine the theoretical scheme as to whether there might be some errors in this framework. In fact, it is well-known that the Feynman propagator of photon has an intrinsic problem since it has a self-inconsistency. Indeed, we can show that only the gauge theory induces the logarithmic divergence in the calculation of the vertex correction. In addition, if we calculate the vertex correction by the massive vector boson with correct propagator, then there is no divergence in the vertex corrections of the weak vector boson [5].

• **Feynman Propagator of Photon :**
As stated above, the main reason why there appears the logarithmic divergence in the vertex correction by photon is because the Feynman propagator of photon has a serious defect. The Feynman propagator can be written as

$$D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 - i\varepsilon}. \tag{2.73}$$

In fact, this problem of the Feynman propagator of photon is discussed long time ago in the field theory textbooks [1, 2]. However, people did not take it seriously, probably because the Feynman propagator can reproduce the correct T-matrix in the electron-electron scattering process, though accidental. However, if we compare the calculated result of the electron-electron scattering, then we realize that the calculation by Feynman propagator gives the right result because the electrons are on the mass shell. However, if electrons are off the mass shell, then Feynman propagator calculation should not give any correct results.

• **Correct Propagator of Photon :**
The correct propagator of photon can be written as

$$\begin{cases} D^{Coul}(k) = \frac{1}{\boldsymbol{k}^2} & A^0 - \text{part} \\ \\ D^{ab}(k) = \frac{1}{k^2 - i\varepsilon}\left(\delta^{ab} - \frac{k^a k^b}{\boldsymbol{k}^2}\right) & \boldsymbol{A} - \text{part.} \end{cases} \tag{2.74}$$

2.11.3 Confusion in Renormalization Scheme

The main reason why the renormalization scheme is not in the right direction is clear by now. This is because people did not work out the important condition on the wave function, which means that they did not solve the equation of motion on the vector potential before fixing the gauge. From the Lagrange equation, we obtain

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0 \tag{2.75}$$

which should be solved for the polarization vector ϵ^μ, and we find $k^2 = 0$. Inserting this dispersion relation, we obtain the following important constrain equation

$$k_\mu \epsilon^\mu = 0$$

which is indeed the Lorentz condition. In other words, the renormalization scheme was developed without this condition, and therefore, people could not find the right direction in the evaluation of vertex corrections. In particular, the number of freedom of free vector potential A^μ should be four, and fixing the gauge should reduce the number to 3. But in reality, the photon has only two degree of freedom. This should be understood only when we solve the equation of motion of eq.(2.75).

2.11.4 Claim of Dirac Is Ignored, Why?

Concerning the renormalization theory, the latest paper of Dirac was published as a AIP Conference report in 1981 [8]. Dirac pointed out repeatedly that the renormalization scheme should be reconsidered since there must be something wrong in this scheme. The AIP Conference report of Dirac is very logical and interesting, and it is quite strange that people have not paid their attention to his theoretical claim which has been completely neglected by experts. This is somewhat unfortunate and in fact, since then, the renormalization scheme became the standard and basic method in field theory until present day.

Chapter 3

General Relativity

The main aim of this text book is to present a clear and simple explanation of new gravity model. But before doing so, we should make a brief description of the general relativity since quite a few readers may still have a belief that the general relativity should be a good candidate for the theoretical model of gravity.

Unfortunately, however, the general relativity is mathematically correct but physically meaningless, and therefore, it should not be so easy to explain what is the general relativity in terms of right physics terminology. In particular, as shown in the Introduction, the metric tensor in the Einstein equation has nothing to do with the gravitational field ϕ_g, and therefore the belief that the general relativity is related to the gravity is completely ruled out.

In this respect, we should mainly explain the mathematical meaning of the Einstein equation and its solution even though the solutions should have no relation to any physical phenomena. Nevertheless, in the last part of this chapter, we should review and discuss the predictions and problems of Mercury perihelion shifts in connection with the Einstein equation since many readers may well be interested in the results of examining the general relativity.

3.1 Introduction

Einstein constructed the general relativity starting from the Gedanken Experiment even though he must have realized that it should have nothing to do with nature. Therefore, the general relativity has no basis on any experiments, and instead, it is based only on the theoretical deduction.

Further, at the time of construction of the general relativity, quantum mechanics was not discovered yet, and Newton and Maxwell equations were the standard theory of physics. In this respect, Einstein might have intended to extend the Newton equation to the relativistic equation, but this is not clear.

3.2 Metric Tensor

The Einstein equation is constructed for the metric tensor as the unknown functions which should be determined by solving the equation. Therefore, the aim of the Einstein equation is to determine the change of the metric tensor when some stars are distributed around. In this respect, we should first understand what is the metric tensor and how it is introduced in physics.

3.2.1 Minkowski Metric

The concept of metric tensor is originated from Minkowski and this is quite simple. The metric tensor is related to the invariant quantity of the Lorentz transformation. The invariant quantity under the Lorentz transformation is the infinitesimal space-time length squared in four dimensions $(ds)^2$, and it is defined as

$$(ds)^2 \equiv (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \tag{3.1}$$

This is indeed invariant under the Lorentz transformation. Now the metric tensor of Minkowski can be seen by rewriting eq.(3.1) in the following way

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \equiv g^{\mu\nu} dx_\mu dx_\nu \tag{3.2}$$

where

$$\begin{aligned} dx^\mu &= (dx^0, dx^1, dx^2, dx^3) \equiv (cdt, dx, dy, dz) \\ dx_\mu &= (dx_0, dx_1, dx_2, dx_3) \equiv (cdt, -dx, -dy, -dz). \end{aligned}$$

Here in eq.(3.2), in case μ appears twice, then the summation of $\mu = 0, 1, 2, 3$ should be taken and therefore,

$$\begin{aligned} g^{\mu\nu} dx_\mu dx_\nu = \quad & g^{00} dx_0 dx_0 + g^{01} dx_0 dx_1 + g^{02} dx_0 dx_2 + g^{03} dx_0 dx_3 \\ + \quad & g^{10} dx_1 dx_0 + g^{11} dx_1 dx_1 + g^{12} dx_1 dx_2 + g^{13} dx_1 dx_3 \\ + \quad & g^{20} dx_2 dx_0 + g^{21} dx_2 dx_1 + g^{22} dx_2 dx_2 + g^{23} dx_2 dx_3 \\ + \quad & g^{30} dx_3 dx_0 + g^{31} dx_3 dx_1 + g^{32} dx_3 dx_2 + g^{33} dx_3 dx_3. \end{aligned} \tag{3.3}$$

Thus, we find the shape of $g^{\mu\nu}$ as

$$\{g^{\mu\nu}\} = \begin{pmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{\mu\nu} \tag{3.4}$$

where $\eta^{\mu\nu}$ is called Minkowski's metric tensor.

3.2.2 Extension of Metric Tensor

In the general relativity, this metric tensor is extended to

$$g^{\mu\nu} = g^{\mu\nu}(x^\sigma), \qquad x^\sigma = (t, x, y, z) \tag{3.5}$$

so that the metric tensor $g^{\mu\nu}$ is now dependent on space and time. However, the physical meaning of this assumption of $g^{\mu\nu}(x^\sigma)$ is unclear. In particular, the Lorentz invariance of $(ds)^2$ is lost by now since the metric tensor depends on space and time. Indeed, this must be very serious in physics.

3.3 Einstein Equation

Now the Einstein equation is written as

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G_0 T^{\mu\nu} \tag{3.6}$$

where $g^{\mu\nu}$ should be unknown functions. $R^{\mu\nu}$ is called Ricci tensor and is written in terms of second order differential of $g^{\mu\nu}$. $T^{\mu\nu}$ denotes the energy-momentum tensor which can be expressed by the distribution function of stars. G_0 is the gravitational constant. This equation states that, if there is a distribution of stars, then the metric tensor $g^{\mu\nu}$ should be deviated from that of Minkowski.

3.3.1 Ricci Tensor

It may not be very helpful even if we write the shape of the Ricci tensor $R^{\mu\nu}$, but we just write its definition for reference

$$R^{\mu\nu} \quad = \quad \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\lambda\mu} + \Gamma^\lambda{}_{\lambda\sigma}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\lambda\mu} \tag{3.7}$$

where

$$\Gamma^\lambda{}_{\mu\nu} \quad \equiv \quad \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{3.8}$$

This is written in terms of differential geometry, but there is no physical meaning in this Ricci tensor.

3.3.2 Poisson Equation for Gravity

Here, we should discuss the relation between the Einstein equation and gravity. Einstein claimed that the Einstein equation should be related to the gravitational field ϕ_g because he postulated the following equation

$$g^{00} \simeq 1 + 2\phi_g. \tag{3.9}$$

In this case, using $T^{00} \simeq \rho$, one can find the following Poisson equation for the gravitational field ϕ_g

$$\nabla^2 \phi_g = 4\pi G_0 \rho \tag{3.10}$$

where ρ is the distribution function of stars. This equation can reproduce the correct shape of the gravitational field, and in fact, if we take $\rho \simeq M\delta(\boldsymbol{r})$ in eq. (3.10), then we find the gravitational field ϕ_g as

$$\phi_g = -\frac{MG_0}{r} \tag{3.11}$$

which has a right shape for the gravity. Unfortunately, however, the equation (3.9) has no physical foundation and cannot be justified by any means.

3.3.3 Relation Between General Relativity and Gravity

The main reason why the general relativity is considered to be a right theory must be due to a belief that the gravitational Poisson equation can be derived from the Einstein equation. This derivation is entirely dependent on the assumption of the equation $g^{00} \simeq 1 + 2\phi_g$. However, as shown in the Introduction, the metric tensor cannot be related to the gravitational field ϕ_g since the gravitational force should have already been employed in the right side of the Einstein equation to make the distribution of stars. Thus, there is no way to obtain the relation [eq.(3.9)] between the metric tensor g^{00} and the gravitational field ϕ_g.

• Dynamical Variables and Coordinate System :
Indeed, this postulated equation (3.9) is physically meaningless. This is clear since the metric tensor is the coordinate system itself while the gravitational field ϕ_g is a dynamical variable, though dimensionless. Therefore, one cannot add the two quantities of different category together.

In addition, the metric tensor should be unknown functions which must be determined as the solutions of the Einstein equation, and therefore, its shape cannot be determined in advance, even though with the weak gravitational approximation.

3.3.4 No Relation Between Metric Tensor and Gravity

Here, we should stress and repeat again that the metric tensor has nothing to do with the gravitational field ϕ_g as we discuss in the Introduction. Here, we again write the Einstein equation to understand the problem more clearly

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G_0 T^{\mu\nu}. \tag{3.12}$$

where the left side of the Einstein equation is written in terms of the metric tensor $g^{\mu\nu}$. Therefore, the metric tensor must be unknown functions which should be determined by solving the Einstein equation. The right side of the Einstein equation is written in terms of energy-momentum tensor which is obtained from the distribution of stars. Therefore, if the distribution of stars is known, then the metric tensor $g^{\mu\nu}$ can be determined. In spite of the fact that the equality of the Einstein equation cannot be physically justified due to the different categories between metric tensor and the distribution of stars, we should proceed to go further to the discussion of the Einstein equation.

• No Relation between Metric Tensor and Gravity :
Here we prove that the metric tensor is not related to the gravitational field ϕ_g by any means. The main point is concerned with the right side of the Einstein equation which is the distribution of stars. It should be noted that the stars and their distribution can be determined by solving the many body Newton equations with the gravitational fields. Namely, the gravitational field should be made use of in constructing the right side of the Einstein equation.

Therefore, it is, by now, clear that the metric tensor which should be obtained as the solution of the Einstein equation has nothing to do with the gravitational field ϕ_g. This is related to the causality of the Einstein equation since the determination of the distribution of stars should have required the gravitational potential in advance before solving the Einstein equation. Thus, there is, absolutely, no way to relate the metric tensor to the gravitational field. This is a rigorous proof that eq.(1.8) is invalid.

3.3.5 Constant Scale in Einstein Equation

In the Einstein equation, what should be a scale that measures all the physical quantities? Here, we should clarify the scale parameter in the Einstein equation.

• Constant Scale in Maxwell Equation :
Before discussing the role of constant scale in the Einstein equation, we should first clarify the constant scale in the Maxwell equation. The Maxwell equation can be written in terms of vector potential A^μ

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = ej^\nu \tag{3.13}$$

In the left side of eqs. (3.13), there is no constant scale. On the other hand, there is a mass scale in the charge density of the right side and therefore, all the observables must be measured by the electron mass.

• Constant Scale in Einstein Equation :
In this respect, the right side of the Einstein equation should have stars and therefore, the mass of stars must be the relevant scale.

3.4 Solution in Einstein Equation

What should be the most important point of the Einstein equation? This must be clear that we should interpret its physical meaning of the solution in eq.(3.6). In fact, the solution of eq.(3.6) should determine the metric tensor $g^{\mu\nu}$, but we should extract some physics out of this metric tensor. Here, we should treat the small deviation of the metric tensor $g^{\mu\nu}$ from Minkowski metric $\eta^{\mu\nu}$

$$g^{00}(x^\sigma) = 1 + C_0(x^\sigma) \tag{3.14}$$

$$g^{11}(x^\sigma) = g^{22}(x^\sigma) = g^{22}(x^\sigma) = -(1 + C_1(x^\sigma)) \tag{3.15}$$

where $g^{\mu\nu} = 0 \ \ (\mu \neq \nu)$ is assumed. In this case, $C_0(x^\sigma), \ C_1(x^\sigma)$ is also dimensionless, and thus $(ds)^2$ in eq.(3.2) should become

$$(ds')^2 = (1 + C_0(x^\sigma))(cdt)^2 - (1 + C_1(x^\sigma))((dx)^2 + (dy)^2 + (dz)^2) \tag{3.16}$$

For the above solutions, there must be two possible approaches. The first choice must be to preserve $(ds)^2$ invariant. The second approach must be to ignore the relativity and consider some new development in some way or other.

• **Preservation of Relativity :**
In order to keep the relativity principle, we should have the shape of $(ds')^2$ which must agree with that of $(ds)^2$ since this is the requirement of invariance under the Lorentz transformation. Therefore, we find

$$C_0(x^\sigma) = 0, \quad C_1(x^\sigma) = 0. \tag{3.17}$$

• **Neglect of Relativity :**
On the other hand, Einstein thought that the relativity should be ignored, and in this case, the metric tensor should depend on the coordinates. This is, however, very difficult to understand what it means by the violation of relativity.

3.4.1 Schwarzschild Solution

Here we should explain briefly the Schwarzschild solution. First, the shape of $(ds)^2$ is written in the polar coordinate

$$(ds)^2 = A(r)(cdt)^2 - B(r)(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta\,(d\phi)^2 \tag{3.18}$$

where $A(r)$ and $B(r)$ should be determined by solving the Einstein equation. The simplest solution can be written as

$$(ds)^2 = \left(1 - \frac{2G_0M}{c^2r}\right)(cdt)^2 - \frac{(dr)^2}{\left(1 - \frac{2G_0M}{c^2r}\right)} - r^2(d\theta)^2 - r^2\sin^2\theta\,(d\phi)^2 \tag{3.19}$$

which is the Schwarzschild solution. We should note that there appears the mass M of star in the $(ds)^2$. This is quite strange since observers should define the coordinate system while the mass is a physical constant which should not have any effects on the coordinate system.

3.4.2 Black Hole

In eq.(3.19), people define the Black Hole radius r such that it satisfies

$$1 - \frac{2G_0M}{c^2r} = 0. \tag{3.20}$$

Thus, the Black Hole radius R_g can be written as

$$R_g = \frac{2G_0M}{c^2}. \tag{3.21}$$

In the interior region which is smaller than R_g, the $(ds)^2$ becomes negative, and thus the coordinate system cannot be defined any more. This means that the Einstein equation is unphysical and meaningless, and this is all that we learn from the Schwarzschild solution. The appearance of the unphysical result should be connected to the fact that eq.(3.19) has nothing to do with the dynamics of star formation.

3.4.3 Friedmann Universe

Friedmann assumes the shape of $(ds)^2$ as

$$(ds)^2 = (cdt)^2 - a(t)^2((dx)^2 + (dy)^2 + (dz)^2) \tag{3.22}$$

where $a(t)$ should be determined from the Einstein equation. Here, we should write one example of the solution

$$a(t) = a_0 t^{\left(\frac{2}{3(1+\kappa)}\right)} \tag{3.23}$$

which is the solution of Friedmann. Here, κ is a constant, and when κ is zero, it is believed that the space should expand.

However, the coordinate system is employed by the observer, and thus the space expansion has no meaning for the observer since he measures the motion of particles in this system. Einstein and his followers misunderstood the expansion of the coordinate system as the expansion of universe, and this has given rise to many confusions in physics.

3.5 Predictions of General Relativity

All the discussions in this section should be based on the assumption that eq.(3.9) should hold, and therefore, readers should take the following stories as some old works in science history. Historically, people believed that the general relativity should be related to the gravity and it should predict the additional potential as the higher order effects. By now we understand that the treatments of this effects should be unphysical. But still as a science history, we should discuss and clarify what people have done on the higher order effects in the general relativity.

3.5.1 Corrections due to General Relativity

The general relativity is believed to generate some correction terms and this is expressed in terms of the additional potential

$$V_c(r) = -\frac{3}{mc^2}\left(\frac{GmM}{r}\right)^2 . \tag{3.24}$$

As will be discussed later, this potential is known to be non-integrable, and therefore, this should give rise to some unphysical effects such as open orbits. Here we first solve the Newton equation to get the orbit solution. In this case, the differential equation for the orbit with the correction potential becomes

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \frac{r^2}{\ell}\sqrt{2m\left(E + \frac{\alpha}{r} - \frac{\ell^2}{2mr^2} + \frac{3}{mc^2}\left(\frac{GmM}{r}\right)^2\right)}. \tag{3.25}$$

This equation can be solved exactly and the effect due to the correction terms appears in $\cos\varphi$ term and is written as

$$r = \frac{A_g}{1 + \varepsilon\cos\left(\frac{L_g}{\ell}\varphi\right)} \tag{3.26}$$

where A_g and L_g are given as

$$A_g = \frac{L_g^2}{GMm^2}, \quad L_g \equiv \sqrt{\ell^2 - \frac{6G^2M^2m^2}{c^2}} \equiv \ell\sqrt{1-\gamma}. \tag{3.27}$$

Here, the γ is defined as

$$\gamma \equiv \frac{6G^2M^2}{c^2R^4\omega^2} \ll 1. \tag{3.28}$$

3.5.2 Prediction of Mercury Perihelion Shifts by General Relativity

People calculated the Mercury perihelion shifts from the orbit solution of eq.(3.26)

$$r = \frac{A_g}{1+\varepsilon\cos\left(\frac{L_g}{\ell}\varphi\right)} \simeq \frac{A_g}{1+\varepsilon\cos\left(\varphi - \frac{1}{2}\gamma\varphi\right)} \tag{3.29}$$

where they thought that the $\frac{1}{2}\varphi\gamma$ term should give rise to the shift. They claim that the shift angle φ can be obtained from

$$\varphi - \frac{1}{2}\gamma\varphi = 2\pi \tag{3.30}$$

and they find

$$\varphi \simeq 2\pi + \pi\gamma. \tag{3.31}$$

Therefore, the perihelion shifts may be determined as

$$\frac{\Delta\varphi}{2\pi} \simeq \frac{\gamma}{2}. \tag{3.32}$$

However, this is completely a wrong procedure since the $\frac{1}{2}\varphi\gamma$ term should come from the non-integrable potential. Thus, this term should be treated in the perturbation theory which does not give rise to any perihelion shifts as we see below.

3.5.3 Non-integrable Potential

Here we discuss the physical effects of the non-integrable potential. The additional potential from the new gravity model has also a similar shape and therefore, we can write the non-integrable potentials into the simple shape in the following way

$$V_a(r) = \frac{q}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{3.33}$$

where

$$q = \begin{cases} -6 & \text{for General Relativity} \\ \ \ 1 & \text{for New Gravity} \end{cases} . \tag{3.34}$$

In this case, the differential equation for the orbit with the additional potential becomes

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2}. \tag{3.35}$$

This equation can be solved exactly and the effect due to the correction appears in $\cos\varphi$ term and is written as

$$r = \frac{A_g}{1+\varepsilon\cos\left(\frac{L_g}{\ell}\varphi\right)} \tag{3.36}$$

where A_g and L_g are given as

$$A_g = \frac{L_g^2}{GMm^2}, \quad L_g \equiv \sqrt{\ell^2 + \frac{qG^2M^2m^2}{c^2}} \equiv \ell\sqrt{1+\eta} \simeq \ell\left(1+\frac{1}{2}\eta\right). \tag{3.37}$$

Here, the η is defined as

$$\eta \equiv \frac{qG^2M^2}{c^2R^4\omega^2} \tag{3.38}$$

which is a very small number. It is around 10^{-8} for the planet motion such as the earth or Mercury.

3.5.4 Effects of Non-integrable Potential on Solution

The solution of eq.(J.4) has a serious problem in that the orbit is not closed. This is quite well known that the potential with the non-integrable shape such as $V_c(r) = \frac{C}{r^2}$ gives rise to the orbit which is not closed. It is, of course, clear that this type of orbits should not happen in nature.

The abnormal behavior of the solution eq.(J.4) can also be seen from $\cos\left(\frac{L_g}{\ell}\varphi\right) \simeq \cos(\varphi+\frac{1}{2}\eta\varphi)$ term. It should be interesting to see that this term cannot be described in terms of the cartesian coordinates of $x = r\cos\varphi,\ y = r\sin\varphi$. In fact, $\cos(\varphi+\frac{1}{2}\eta\varphi)$ term becomes

$$\cos(\varphi+\frac{1}{2}\eta\varphi) = \frac{x}{r}\cos\frac{1}{2}\eta\varphi - \frac{y}{r}\sin\frac{1}{2}\eta\varphi \tag{3.39}$$

and there is no way to transform the $\cos\frac{1}{2}\eta\varphi$ term into $x,\ y$ coordinates even though we started from this cartesian coordinate. This is very serious since the solution expressed by polar coordinates cannot be written any more in the cartesian coordinates. This is related to the fact that the orbit is not closed due to the non-integrable potential effects.

3.5.5 Discontinuity of Orbit

The effect of the non-integral potential can be further seen as the discontinuity of the orbit trajectory since the orbit is not closed. In order to see this discontinuity of the orbit, we first start from the orbit solution with the non-integral potential, which is eq.(J.4)

$$r = \frac{A_g}{1+\varepsilon\cos\left(1+\frac{1}{2}\eta\right)\varphi}.$$

In this case, we find the radius r at $\varphi = 0$ and $\varphi = 2\pi$ as

$$r = \frac{A_g}{1+\varepsilon}, \qquad \varphi = 0 \tag{3.40}$$

$$r = \frac{A_g}{1+\varepsilon\cos\pi\eta}, \qquad \varphi = 2\pi. \tag{3.41}$$

Therefore the difference Δr becomes

$$\Delta r \equiv r_{(\varphi=2\pi)} - r_{(\varphi=0)} \simeq \frac{1}{2}A_g \pi^2 \eta^2 \varepsilon \simeq 0.15 \text{ cm} \tag{3.42}$$

for the Mercury orbit case of the general relativity as an example. This means that the orbit is discontinuous when φ becomes 2π. This is not acceptable for the classical mechanics, and indeed it disagrees with the observation. In addition, eq.(J.4) cannot generate the perihelion shift, and this can be easily seen from the orbit trajectory of eq.(J.4).

3.6 Perturbative Treatment of Non-integrable Potential

When the non-integrable potential appears as the small perturbation on the Newton equation, what should be the best way to take into account this small potential effect? Here we present a perturbative treatment of the non-integrable potential, and in this case, the equation for the orbit determination becomes

$$\begin{aligned}\frac{dr}{d\varphi} &= \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2} \\ &= r^2\sqrt{1+\eta}\sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}}\end{aligned} \tag{3.43}$$

Therefore, we can rewrite the above equation as

$$\sqrt{1+\eta}\,d\varphi = \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}}} \tag{3.44}$$

Here we note that $\eta = \frac{q}{\ell^2 c^2}(GmM)^2$ is a very small number which is of the order $\eta \sim 10^{-8}$. Now in order to keep the effect of the non-integrable potential in terms of integrable expression, we should make an approximation as

$$\sqrt{1+\eta}\,d\varphi \simeq d\varphi. \tag{3.45}$$

The reason why we should make this approximation is because we should consider the dynamical effect as the perturbation while the η in the right side of eq.(4.28) should only change the value of constants such as E or α in the differential equation. In this way, the equation to determine the orbit becomes

$$\frac{dr}{d\varphi} = r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r} - \frac{1}{r^2}} \tag{3.46}$$

which gives the right orbit solution. Now the orbit is closed, and the solution can be written as

$$r = \frac{A_g}{1+\varepsilon\cos\varphi} \tag{3.47}$$

where A_g is given as

$$A_g = \frac{\ell^2}{GMm^2}(1+\eta). \tag{3.48}$$

Note that the ε is also changed due to the η term, but here we can safely neglect this effect since it does not play any role for physical observables. Therefore, the effect of the additional potential is to change the radius A_g of the orbit even though this change is very small indeed. Now eq.(J.16) clearly shows that there is no perihelion shift, and this is very reasonable since the additional potential cannot shift the main axis of the orbit.

3.6.1 Higher Order Effect of Perturbation

Here we should estimate the higher order effect of the perturbation in eq.(4.28). Denoting the solution of eq.(J.16) by $r^{(0)}$

$$r^{(0)} = \frac{A_g}{1+\varepsilon\cos\varphi}$$

and the perturbative part of the radius by r' ($r = r^{(0)}+r'$), we can write the equation for r' as

$$\frac{dr'}{d\varphi} = \frac{1}{2}\eta(r^{(0)})^2\sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r^{(0)}} - \frac{1}{(r^{(0)})^2}} \tag{3.49}$$

where the right side depends only on φ. Here, we should make a rough estimation and only consider the case in which the eccentricity ε is zero. In this case, the right side does not depend on the variable ε, and thus we can prove that the right side is zero. Therefore, the higher order correction of r' should be proportional to the eccentricity ε and can be written as

$$r' \simeq C_0\eta\varepsilon A_g \tag{3.50}$$

where C_0 should be some numerical constant. For the earth revolution, the value of ε is very small ($\varepsilon \simeq 0.0167$) and thus we can safely ignore this higher order perturbative effect.

3.7 Period Corrections from General Relativity

Here we discuss briefly the period corrections generated by the additional potential of the general relativity. The gravitational potential together with the additional potential from the general relativity is given as

$$V(r) = -\frac{GMm}{r} - \frac{3}{mc^2}\left(\frac{GmM}{r}\right)^2. \tag{3.51}$$

Therefore the Newton equation becomes

$$m\ddot{r} = -\frac{GmM}{r^2} + \frac{L_g^2}{mr^3} \tag{3.52}$$

where L_g^2 is defined as

$$L_g^2 \equiv \ell^2 - \frac{6G^2M^2m^2}{c^2}. \tag{3.53}$$

The solution of the differential equation is given by taking into account the perturbative treatment of the non-integrable potential

$$r = \frac{A_g}{1 + \varepsilon \cos\varphi} \tag{3.54}$$

where A_g is given as

$$A_g = \frac{L_g^2}{GMm^2}. \tag{3.55}$$

Therefore, the period T can be determined when we integrate $\dot{\varphi} = \frac{\ell}{mr^2}$ over the orbit period as

$$\frac{\ell}{m}\int_0^T dt = \int_0^{2\pi} r^2 d\varphi = A_g^2 \int_0^{2\pi} \frac{1}{(1+\varepsilon\cos\varphi)^2} d\varphi \tag{3.56}$$

which can be calculated to be

$$\omega T = 2\pi(1 - 2\gamma). \tag{3.57}$$

In this case, the correction ΔT to the period can be written as

$$\left(\frac{\Delta T}{T}\right)_{GR} \simeq -2\gamma. \tag{3.58}$$

3.7.1 Earth Revolution Period

For the earth revolution around the sun, the correction to the period T due to the general relativity becomes

$$\Delta T_{GR} = -3.8 \quad \text{[s/year]} \tag{3.59}$$

which is in the wrong direction as compared to the observation in terms of the leap second delay. In addition, this value is, by far, too large compared to the leap second, and in fact, the observed value of the leap second is around 0.62 [s/year]. Therefore, the correction to the earth period from the general relativity should be completely ruled out from the observation.

3.8 Gravitational Wave

It is really a shame as a theoretical physicist that we have to make a brief explanation about the gravitational wave. It is beyond imagination that some group of people insisted that they observed a signal of the gravitational wave. Those people who claimed a "discovery" of the gravitational wave should be far from physicists, and their standard of understanding physics must be lower than the fourth grade student of university. The physical observation can be done only if the object should have any interactions with matters whatever it can be. However, the gravitational wave which is a classical wave has no interaction with any physical objects. This means that its observation of their claim does not make sense.

When a physical object can propagate in vacuum, then it must be a particle like photon whatever it may be, even though massless. This is confirmed from the vast amount of experiments, and by now, "the ether hypothesis" is completely excluded. In fact, all modern physics is based on the relativity principle, and there is no experiment which contradicts the relativity.

Therefore, unless the "graviton" were measured, any gravitational wave cannot be observed. However, the strength of gravity is 10^{-35} times smaller than the strength of electromagnetic force, and therefore, it is simply impossible to measure the graviton. This is well known to physicists and therefore, generally, any physics text books do not make the explanation of the gravitational wave.

Chapter 4

New Gravity Model

In physics, there are two fundamental theoretical frameworks, namely, kinematics and dynamics. They are both quite important, but their roles are different from each other. The kinematics is related to the properties of particle motions such as energy, momentum and various symmetries. In particular, the energy and momentum of a particle should be very important to understand the physics of particle motions. On the other hand, the dynamics consists of equation of motion which determines the behaviors of particle motions. In this case, depending on the phenomena in physics, we should choose the interactions of particles and employ the equation of motion accordingly. If we can solve the equation of motion properly, then we may understand corresponding physical phenomena that should occur in nature.

4.1 Introduction

At present, there are four fundamental interactions in modern physics, electromagnetic, weak, strong and gravitational interactions. In terms of coupling constant, the electromagnetic interaction must be a standard, and the strength of the coupling constant which is dimensionless is found to be

$$\alpha = \frac{1}{137}. \tag{4.1}$$

On the other hand, the strong interaction should be stronger by two orders of magnitude than the electromagnetic interaction while weak interaction must be weaker by a few orders of magnitude than the electromagnetic interaction. However, the gravity is, by far, the weakest force among the four interactions. In fact, the gravity is by the order of $\sim 10^{-30}$ smaller than the electromagnetic interaction. Nevertheless, the gravity is very important in the universe for the formation of stars and galaxies since the force range is very long and it is always attractive.

• **Why Gravity Has Large Effects on Star Formation? :**
The gravity is crucially important for the formation of stars even though the interaction strength is quite weak. There are two important aspects in the gravity when the stars should be formed. The first point is connected to the interaction range which is very long since it has the shape of $\frac{1}{r}$. The other point is that the gravity is always attractive and, as long as the corresponding body is massive, there should exist the attractive interactions from all other massive objects even though they are far away from each other. Because of the attractive nature, there should be no shielding in contrast to the electromagnetic cases.

• **Dirac Equation with Gravitational Potential :**
When the energy of a particle becomes as high as its mass, then we have to consider the relativistic equation of motion under the gravitational potential. In this case, the Newton equation is not appropriate for describing a relativistic motion, and thus, we have to find a new equation of motion. Since we know that the classical mechanics is derived from the Schrödinger equation, we should start from the relativistic equation in quantum mechanics. This is the Dirac equation, and therefore, we have to consider the Dirac equation with the gravitational interaction.

However, the Dirac equation with the gravitational potential has not been determined properly for a long time. This problem is connected to the ambiguity as to whether the gravitational potential should be taken as the fourth component of the vector type interaction or the mass term of scalar type interaction. This problem was not settled until recently, and thus, we should consider the gravitational field theory in some way or other. As will be discussed later, the new gravity model is, indeed, constructed in terms of a massless scalar field theory. Therefore the corresponding Dirac equation with the gravitational potential is well established by now [4, 5].

4.2 Dirac Equation and Gravity

The Newton equation works very well under the gravitational potential, and indeed, the Kepler problem is best understood by solving the Newton equation.

• **Ehrenfest Theorem :**
This Newton equation itself is obtained from the Schrödinger equation by making some approximation such as Ehrenfest theorem. In this case, the expectation values of $\boldsymbol{r}$ and $\boldsymbol{p}$ in quantum mechanics lead to the Newton equation.

• **Foldy-Wouthuysen Transformation :**
The Schrödinger equation can be derived from the Dirac equation by making the Foldy-Wouthuysen transformation. Therefore, the Dirac equation must be the starting point from which the Newton equation can be derived.

4.2.1 Dirac Equation and Gravitational Potential

As can be seen from the present discussion, it should be crucially important to have the Dirac equation with the gravitational potential properly taken into account. Otherwise, we cannot obtain the Newton equation with the gravitational potential. In other words, we should not start from the Newton equation with the gravitational potential since it is obtained only after some series of approximations should be properly made.

• **Dirac Equation with Coulomb Potential :**
Before going to the discussion of the Dirac equation with the gravity, we should first discuss the Dirac equation with the Coulomb potential of $V_c(r) = -\frac{Ze^2}{r}$. This is well-known and can be written as

$$\left(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\beta - \frac{Ze^2}{r}\right)\Psi = E\Psi. \tag{4.2}$$

On the other hand, we should be careful in which way we put the gravitational potential of $V(r) = -\frac{G_0 mM}{r}$ into the Dirac equation since there are two different ways, either the same way as the Coulomb case which should be written as

$$\left(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\beta - \frac{G_0 mM}{r}\right)\Psi = E\Psi \tag{4.3}$$

or the scalar type potential like the mass term which we are now treating below.

• **Dirac Equation with Gravitational Potential :**
In fact, the right Dirac equation with the gravitational potential of $V(r) = -\frac{G_0 mM}{r}$ can be written by putting it into the scalar term as

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m - \frac{G_0 mM}{r}\right)\beta\right]\Psi = E\Psi. \tag{4.4}$$

This is obtained from the field theoretical construction of the gravity model. By now, we see that the scalar type potential of gravity must be the right gravitational potential, and we should discuss it in detail below.

4.3 New Gravity Model

When we wish to construct the theory of gravity, the first thing we should work out should be to find the framework in which the gravitational potential can be properly taken into account in the Dirac equation. Without doing this procedure, there should be no way to consider the theory of gravity. In fact, the Dirac equation for a particle with its mass m in the gravitational potential can be written as

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta\right]\Psi=E\Psi \tag{4.5}$$

where M denotes the mass of the gravity center. In addition, if we make the non-relativistic reduction using the Foldy-Wouthuysen transformation, then we find the gravitational potential in classical mechanics

$$V(r)=-\frac{GmM}{r}+\frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{4.6}$$

where the second term should be the additional potential which appears as the relativistic effect. This additional potential of gravity is a new gravitational potential and this must be a new discovery since nineteenth century. It turns out that this new potential can explain the problem of leap second of the earth revolution period which will be discussed later.

• **Rough Estimation of Relativistic Effect :**
Historically, the first check of the relativistic effect was done by Michelson-Morley by using the velocity of the earth revolution which should be the fastest object relevant to the observed speed on the earth. The result of Michelson-Morley experiment showed that the speed of light is not affected by the earth revolution, and this leads to the concept of the relativity principle. The relativistic effect in this case is

$$\left(\frac{v}{c}\right)^2\sim 1.0\times 10^{-8} \tag{4.7}$$

where c and v denote the velocities of light and the earth revolution, respectively. It should be interesting to note that the leap second of the earth revolution period is found to be ($\Delta T/T\sim 2\times 10^{-8}$) which is just the same order of magnitude as the relativistic effect.

4.3.1 Lagrangian Density

When we consider the theory of gravity, we should start from the scalar field theory since it gives always attractive interactions.

• **Lagrangian Density of Gravity :**
Here, we should write the Lagrangian density of a fermion field ψ interacting with the electromagnetic field A_μ and the gravitational field $\mathcal{G}$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m(1+g\mathcal{G})\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\upsilon\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\ \partial^\mu\mathcal{G} \tag{4.8}$$

where m denotes the mass of fermion. The gravitational field $\mathcal{G}$ is a massless scalar field. The reason why people did not consider the scalar field for the gravity should be mainly because they thought that the scalar field should not be renormalizable. However, as we show below, there is no necessity of the field quantization of the gravitational field, and thus, there is no divergence at all.

• **Gravity Cannot Be Gauge Theory :**
For a long time, people believed that the basic field theory must be a gauge theory, even though there is no foundation for this belief. Indeed, the gauge theory has both attractive and repulsive interactions, and therefore, it is clear that this model of gauge field theory should not be suitable for the gravity.

By now, it is known that only the gauge theory of quantum electrodynamics should give rise to some divergences in the calculation of physical observables such as vertex corrections. In fact, there is no divergence for the vertex corrections which are calculated from the massive vector field theory [5].

4.3.2 Equation for Gravitational Field

From the Lagrangian density, we can obtain the equation for the gravitational field from the Lagrange equation. Here, we can safely make the static approximation for the equation of motion, and obtain the equation for the gravitational field $\mathcal{G}_0$ as

$$\boldsymbol{\nabla}^2\mathcal{G}_0 = mg\rho_g \tag{4.9}$$

where $m\rho_g$ corresponds to the matter density. The coupling constant g is related to the gravitational constant G as

$$G_0 = \frac{g^2}{4\pi}.$$

This equation eq.(4.9) is indeed the Poisson equation for gravity.

4.3.3 Dirac Equation with Gravitational Potential

From the Lagrangian density with gravity and electromagnetic interactions, we can derive the Dirac equation

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+m\beta\left(1+g\mathcal{G}\right)-\frac{Ze^2}{r}\right]\Psi=E\Psi. \tag{4.10}$$

Further, in case the gravitational force is produced by nucleus with its mass of M, the Dirac equation becomes

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta-\frac{Ze^2}{r}\right]\Psi=E\Psi \tag{4.11}$$

which is just the equation discussed in the previous section.

4.3.4 Foldy-Wouthuysen Transformation of Dirac Hamiltonian

The Dirac equation with the gravitational interaction

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta\right]\Psi=E\Psi \tag{4.12}$$

can be reduced to the non-relativistic equation in quantum mechanics. This can be done in terms of Foldy-Wouthuysen transformation which is a unitary transformation. Therefore, the transformation procedure is very reliable indeed.

• **Foldy-Wouthuysen Transformation :**
Here, we start from the Hamiltonian with the gravitational potential

$$H=-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+\left(m-\frac{GmM}{r}\right)\beta. \tag{4.13}$$

This Hamiltonian can be rewritten in terms of the Foldy-Wouthuysen transformation which is somewhat a complicated and tedious procedure involved, though it can be done in a straightforward way [1]. In this case, the non-relativistic Hamiltonian should be obtained as

$$H=m+\frac{\boldsymbol{p}^2}{2m}-\frac{GmM}{r}+\frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2-\frac{1}{2m^2}\frac{GMm}{r^3}(\boldsymbol{s}\cdot\boldsymbol{L}) \tag{4.14}$$

which is kept only up to the order of $\left(\frac{\boldsymbol{p}}{m}\right)^2\frac{GM}{r}$.

4.3.5 Classical Limit of Hamiltonian with Gravity

Here, we should calculate the classical equation of motion from the non-relativistic Hamiltonian in quantum mechanics. In this case, the Hamiltonian which is only relevant to the present discussion can be written as

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{GmM}{r} + \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2. \tag{4.15}$$

This can be reduced to the Newton equation by making the expectation values of operators in quantum theory in terms of the Ehrenfest theorem. In this case, we approximate the products by the factorization in the following way

$$\left\langle \frac{1}{2m^2}\frac{GmM}{r}\boldsymbol{p}^2 \right\rangle = \left\langle \frac{1}{2m^2}\frac{GmM}{r} \right\rangle \left\langle \boldsymbol{p}^2 \right\rangle \tag{4.16}$$

which must be a good approximation in the classical mechanics application. In addition, we make use of the Virial theorem

$$\left\langle \frac{\boldsymbol{p}^2}{m} \right\rangle = -\left\langle V \right\rangle . \tag{4.17}$$

Therefore, we finally obtain the following additional potential

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{4.18}$$

which is a new gravitational potential in classical mechanics. The derivation of the additional potential is similar to the Zeeman effects in that both interactions appear in the non-relativistic reduction as the higher order terms of coupling constant.

4.4 Predictions of New Gravity Model

By now, a new gravity model is constructed, and as a byproduct, there appears the additional gravitational potential. This is a very small term, but its effect can be measurable. Indeed, this is the relativistic effect which becomes

$$\left(\frac{v}{c}\right)^2 \sim 1.0 \times 10^{-8} \tag{4.19}$$

for the earth revolution around the sun. On the other hand, the leap second of the earth revolution is found to be

$$\left(\frac{\Delta T}{T}\right) \sim 2 \times 10^{-8} \tag{4.20}$$

which is just the same order of magnitude as the relativistic effect. Therefore, as we see later, it is natural that the leap second value can be understood by the additional potential of the new gravity model.

4.4.1 Period Shifts in Additional Potential

In the new gravity model, there appears the additional potential in addition to the normal gravitational potential. In the case of the earth revolution around the sun, this potential is written as

$$V(r) = -\frac{GmM}{r} + \frac{1}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{4.21}$$

where the second term is the additional potential [5]. Here, G and c denote the gravitational constant and the velocity of light, respectively. m and M correspond to the masses of the earth and the sun, respectively.

• Non-integrable Potential :
As discussed in the previous chapter, the additional potential should be a non-integrable, and therefore, the treatment should be done in terms of the perturbation theory. In this case, the Newton equation with the perturbative procedure of the additional potential can be solved and the period T of the revolution is written as

$$\omega T \simeq 2\pi(1 + 2\eta) \tag{4.22}$$

where η is given as

$$\eta = \frac{G^2 M^2}{c^2 R^4 \omega^2}. \tag{4.23}$$

Here, R is the average radius of the earth orbit. The angular velocity ω is related to the period T by

$$\omega = \frac{2\pi}{T}. \tag{4.24}$$

The period shift due to the additional potential becomes

$$\frac{\Delta T}{T} = 2\eta \tag{4.25}$$

which is the delay of the period of the revolution [4, 5].

4.4.2 Period Shifts of Earth Revolution (Leap Second)

In the earth revolution, the orbit radius, the mass of the sun and the angular velocity can be written as

$$R = 1.496 \times 10^{11} \text{ m}, \quad M = 1.989 \times 10^{30} \text{ kg}, \quad \omega = 1.991 \times 10^{-7}. \tag{4.26}$$

In this case, the period shift becomes

$$\frac{\Delta T}{T} = 2\eta \simeq 1.981 \times 10^{-8} \tag{4.27}$$

Therefore, the period of the earth revolution per year amounts to

$$\Delta T_{N.G.} = 0.621 \quad \text{[s/year]} \tag{4.28}$$

which is a delay. This suggests that the corrections must be necessary in terms of the leap second.

• Leap Second :
In fact, the leap second corrections have been made for more than 40 years. The first leap second correction started from June 1972, and for 40 years, people made corrections of 25 second. Therefore, the average leap second per year becomes

$$\Delta T^{Obs}_{N.G.} \simeq 0.625 \pm 0.013 \quad \text{[s/year]} \tag{4.29}$$

which agrees perfectly with the prediction of eq.(4.28).

• Definition of Newcomb Time :
Newcomb defined the time series of second in terms of the earth revolution period. However, the recent measurement of time in terms of atomic clock turns out to deviate from the Newcomb time [9]. This deviation should be due to the relativistic effects, and indeed this deviation can be understood by the additional potential of gravity.

4.4.3 Mercury Perihelion Shifts

For a long time, people believed that the Mercury perihelion shifts can be understood by the higher order effects of general relativity. However, as seen in the previous chapter, there should be no perihelion shifts for one period of revolution.

Instead, there should be the Mercury perihelion shifts which may arise from the effects of other planets such as Jupiter if we can measure the perihelion shifts for some long period of revolutions. Concerning the Mercury perihelion shifts, however, the measurements as well as the calculations of the effects from other planets should be carried out more carefully. After the calculation of Newcomb in the 19 century, no careful calculation on the perihelion shifts has been done until now.

4.4.4 Retreat of Moon

The moon is also affected by the additional potential of gravity from the earth. The shifts of the moon orbit can be expressed just in the same way as the earth revolution. In this case, η can be written as

$$\eta = \frac{G^2M^2}{c^2R^4\omega^2}. \tag{4.30}$$

Here, R is the radius of the moon orbit. M and ω denote the mass of the earth and the angular velocity, respectively. They are written as

$$R = 3.844 \times 10^8 \ \text{ m}, \quad M = 5.974 \times 10^{24} \ \text{ kg}, \quad \omega = 2.725 \times 10^{-6} \tag{4.31}$$

Therefore, the period shift becomes

$$\frac{\Delta T}{T} = 2.14 \times 10^{-11}. \tag{4.32}$$

Now, we should carry out the calculation as to how the orbit can be shifted, and the shift of the angle can be written as

$$\Delta\theta = 4\pi\eta. \tag{4.33}$$

Thus, the orbit shift $\Delta\ell_m$ can be written as

$$\Delta\ell_m = R\Delta\theta \simeq 0.052 \ \text{ m} \tag{4.34}$$

and therefore, the shift per year becomes

$$\Delta\ell_{m\ (one\ year)} = \Delta\ell_m \times \frac{3.156 \times 10^7}{2.36 \times 10^6} \simeq 69.5 \ \text{ cm}. \tag{4.35}$$

• Calculated Results of Retreat of Moon :
Since the orbit of the moon is ellipse, the orbit shift can be seen as if it were retreated [10]. The orbit is described by

$$r = \frac{R}{1 + \varepsilon \cos\theta}. \tag{4.36}$$

In addition, the eccentricity is quite small ($\varepsilon = 0.055$) and therefore, we can rewrite the above equation as

$$r \simeq R(1 - \varepsilon \cos\theta). \tag{4.37}$$

Thus, the orbit shift Δr at $\theta \simeq \frac{\pi}{2}$ becomes per year

$$\Delta r \simeq R\Delta\theta\ \varepsilon \simeq \Delta\ell_{m\ (one\ year)}\ \varepsilon\ \simeq\ 3.8\ \ \text{cm} \tag{4.38}$$

On the other hand, the observed value of the retreat shift of the moon orbit is

$$\Delta r_m^{obs} \simeq 3.8\ \ \text{cm} \tag{4.39}$$

which agrees very well with the prediction.

• Retreat Shift is not Real! :
It should be noted that this observation is only possible by making use of the Doppler shift measurement. This is not a direct measurement of the moon orbit distance which is not possible due to the uncertainty of the accuracy of light velocity

$$c = (2.99792458\ \pm 0.000000012) \times 10^8\ \ \text{cm/s}. \tag{4.40}$$

The accuracy of the orbit shift $\Delta r_m^{obs} \simeq 3.8$ cm is at the order of 10^{-10} while the light velocity is measured only up to 10^{-8} accuracy. This means that the shift of the orbit radius is just the instantaneous and apparent effect.

4.5 Summary

The new gravity theory of eq.(4.8)) can naturally derive the Dirac equation of eq.(4.4). This is very important in modern physics since we have now the Dirac equation with the gravitational potential properly taken into account. This Dirac equation can be reduced to the non-relativistic Hamiltonian which then gives rise to the Newton equation with the gravitational potential, and this new equation should contain a new gravitational potential as the additional potential.

• **Massless Scalar Field :**
The fact that the gravity is described by the massless scalar field can give rise to some important effects on the non-relativistic reduction. This is in contrast to the Coulomb case, but rather similar to the non-relativistic reduction of the vector potential case. In the non-relativistic reduction of the vector potential term in the Hamiltonian, we find new terms such as Zeeman effects or spin-orbit interactions. In the same way, in the non-relativistic reduction of the scalar potential term in the Hamiltonian, we find the new additional potential. In fact, this new additional potential can reproduce the leap second of the earth revolution.

• **Inertial Mass and Gravitational Mass :**
From experiments, it is known that the inertial mass and gravitational mass are just the same. This equivalence of two masses is one of the grounds in constructing the general relativity. On the other hand, this equivalence is derived as a natural consequence in the new gravity model. This is one of the strong reasons why this new gravity model is reliable indeed.

• **Renormalization Theory :**
For a long time, people believed that any models of field theory must be renormalizable. This is, of course, due to the infinity appeared in the calculation of vertex corrections in QED. The vertex corrections should be physical observables since they are just the electron magnetic moments, and therefore, they must be finite. This infinity, however, should appear only when the Feynman propagator of photon is employed in calculating the vertex corrections as recent investigations show. In fact, the vertex corrections due to the massive vector boson are shown to have no divergence at all. In this respect, there should be no divergence of vertex correction due to photon if we employ the right propagator of photon, instead of Feynman propagator [5]. In this respect, the massless scalar field of gravitation should not be quantized, and thus, there is no divergence in this field theory model of gravity.

Chapter 5

Cosmology

Before twentieth century, the main Weltanschauung should be based on the assumption that God should exist. Therefore, everything should be, in principle, determined at the final stage. This deterministic Weltanschauung should be the main stream of scientific world. This view should be consistent with classical mechanics since all the mechanical trajectories must be determined if we can solve the whole dynamics in an exact fashion like God.

Unfortunately, however, the classical mechanics is an approximate scheme in physics, and thus we cannot say that it can produce correct answers. In this respect, the idea of deterministic Weltanschauung should not be accepted from the modern physics point of view. Physical phenomena occur basically in a probabilistic way, and the evolution of living objects is controlled by quantum mechanical behaviors.

5.1 Introduction

For the discussion of cosmology, we should understand the behavior of elementary particles in depth in terms of field theory. The basic interactions are quantum electrodynamics, quantum chromodynamics, weak interactions and gravity. Among them, the formation of stars can be described by the gravitational interaction, and unfortunately, the theory of gravity has not been constructed until recently. The main reason of this delay of constructing the theory of gravity should be partly due to the misunderstanding of the renormalization scheme in the gauge theory, and partly due to the presence of the general relativity. In particular, Einstein claimed that the general relativity must be the theory of gravity. But in reality, it has nothing to do with the gravity since it is a theory for the coordinate system. In the Introduction, we show in a clear way that the metric tensor of the Einstein equation is not related to the gravitational field by any means.

5.1.1 Big Bang and Creation

The big bang model has been quite popular for a half century until recently. In the big bang model, they assume that the universe must have been created about 15 billion years ago. Why is this strange model accepted by many physicists?

• Similarity with Creation :
This may be partly because the idea of the universe creation is somewhat similar to the Creation that appeared in Bible. However, as a story, the creation of universe is certainly interesting, but as a natural and scientific phenomenon, the Creation cannot be accepted as a physical incident.

• Background Radiations and Expansion of Universe :
There should be another reason why the big bang model is accepted by some experts. This should be related to the observation of the background radiation which is, of course, a very important discovery in physics. Right after the background radiation was discovered, some people proposed a model in which the background radiation must be understood in terms of some relics of the big bang explosion. Since then, the big bang model was taken to be a physically interesting target even though, in the beginning of the proposal of the big bang model, it must have been taken simply as a toy model.

In addition, the big bang model is supported by the solution of the Einstein equation in which space must be expanding after the big bang. In fact, people believe that the space expansion should be identified as the expansion of universe with galaxies. This means that people thought that stars should be expanding together with the space expansion.

• Particle Motion with Space or in Space? :
This is, however, very difficult to understand how particles should move together with space, not in space. The expanding universe should be an observed fact, but this means that galaxies are expanding relatively with each other. Obviously, this does not indicate that space is expanding. This is the basic but serious mistake of the big bang model, and it cannot be considered to be a scientific model.

Here it should be noted that photons created at the beginning of the big bang cannot become a thermal equilibrium. This can be easily seen since the mean free path of photon should be around a few orders of magnitude longer than the present radius of the universe (around 15 billion light year). Therefore, there is no chance that photons can become a thermal equilibrium in the finite and open universe. Therefore, there should be no chance to observe high density radiations in the present universe. The origin of the background radiations should be the result of the Mugen universe, and this point should be discussed later in detail.

5.1.2 Cosmology in General Relativity

We should make a brief description of the cosmology which is based on the general relativity. The general relativity is a theory for the metric tensor, and this coordinate system is supposed to be affected by the presence of stars. Further, people believed that the change of the metric tensor should affect on the cosmology, but it is quite difficult to understand why the cosmology is related to the metric tensor. Up to now, people have discussed the expansion of space in terms of metric tensor, but this is again very difficult to understand what it means by the space expansion.

• **Universe Creation and Expansion :**
The cosmology based on the general relativity should start from the assumption that the universe is created from one point. The reason why it must start from the point is related to the lack of the scale in the general relativity. If it started from a finite object, then they would have to assume some scale which should have a dimension of length. In this case, there must be a physical reason as to what should be this length scale in space expansion, even though there is no scale parameter in the metric tensor.

• **Scenario of Universe Creation :**
The scenario of universe creation is based on the big bang model even though it was proposed as a toy model. The important point of this cosmology should be that, right after the creation of the universe, space expands as a solution of the Einstein equation. However, it is very difficult to understand what it means physically by space expansion. Therefore, it is beyond imagination in which way the point can expand without any length scale. In addition, if we multiply any large constant to the point, we cannot get a finite scale of space since the point has no finite length,

5.1.3 Expansion of Universe

In reality, the claim that the universe is expanding should mean that many galaxies are expanding with each other. In this case, space is not expanding, and this is the whole problem of the big bang model. If galaxies and space are expanding together, then this indicates that galaxies are attached to space in some way or other. But this cannot be understood in terms of physics terminology.

• **Expansion of galaxies :**
In the big bang model, particles are assumed to expand very quickly at very fast speed after particle creations. However, the creation must accompany anti-particles, and this was a serious problem of the big bang cosmology. In fact, it is, by now, proved experimentally that protons are stable particles, and therefore, the world of particle-anti-particle states cannot be reduced to the particle states. This clearly shows that the big bang model is completely ruled out.

5.2 New Cosmology

By now, it is clear that quantum field theory should be the most reliable theoretical framework in all physics. Indeed, this basic field theory models are composed of four interactions which are quantum electrodynamics, weak interaction, strong interaction and gravity. Therefore, the new cosmology must start from the field theory models of four interactions. In particular, we have now the new gravity model at hand, and this enables us to discuss the problem of the whole universe.

The scenario of cosmology we develop here should be based on the modern field theory. This must be still far from elaborate treatments, but it should be in the right direction. The most important point of the new cosmology must be the experimental fact that the basic constituents of this universe which are protons and electrons should be stable. This is the essential difference from the big bang model. This new model of cosmology indicates that the universe must have existed from the infinite past. There should be no possibility that the universe was created suddenly at some point of time.

5.2.1 Stable Elementary Particles and Infinite Past

The starting point of the new cosmology should be the observed fact that the constituents (proton and electron) of the universe is stable. This means that the universe must have existed from the infinite past. This is the most important point when we wish to construct the cosmology. However, we cannot understand the infinite past, and we should notice that we have to take it granted that the universe exists from the infinite past even though we cannot understand the infinity itself. By keeping this point in mind, we develop the new cosmology in a scientifically correct way.

• Introduction of Finite Universe :
From now on, we should introduce the finite universe as for the present universe which has the diameter of about 15 billion light years. The whole universe must be composed of infinite numbers of the same type of finite universes. This finite universe may well have about several hundred billions of galaxies.

5.2.2 Size of Finite Universe and Equation of Motion

Up to now, the size of finite universe is determined by making use of Hubble's law of ($v = Hd$). In this case, the largest value of d can be obtained when v becomes the velocity of light $v = c$. This gives the size of the finite universe as $d \simeq \frac{c}{H} \simeq 1.35 \times 10^{10}$ light years. However, this method of calculating the size is too much simplified, and in reality, the size of the universe should be determined by solving the expansion dynamics of the universe which should not be very easy to do. Therefore, the number $d \simeq 1.35 \times 10^{10}$ light years should be taken as the zeroth order estimation of the size of the finite universe.

• **Equation to Determine Size of Finite Universe :**
In order to determine the size of the finite universe, we should set up the mechanical equation of motion for dynamics of the finite universe. As the simplest model, we should start from calculating the dynamics at the explosion point of the finite universe (Cosmic Fireball).

Suppose the mass of the finite universe is taken to be M_U, and we should solve the dynamics of the mass m_s of star which should spring out of the finite universe. The Newton equation for this object must be

$$m_s \ddot{x} = -\frac{G m_s M_U}{x^2} \tag{5.1}$$

where the dynamics can be well approximated by one dimensional motion. Form this equation, we find the energy conservation law

$$E = \frac{1}{2} m_s \dot{x}^2 - \frac{G m_s M_U}{x}. \tag{5.2}$$

The size of the finite universe must be roughly determined from this dynamics. However, the choice of the initial values should not be so easy to be determined. In any case, we assume that the escape velocity from the fireball of the finite universe should be v_0. If the radius of fireball is taken to be R_f, then we find

$$E = \frac{1}{2} m_s v_0^2 - \frac{G m_s M_U}{R_f}. \tag{5.3}$$

From these dynamics, we may determine the radius of the finite universe at present. But this should be a homework problem for readers.

5.3 Model of Mugen Universe

When we wish to construct the universe model which should be infinite, there must exist some observed facts as a physical foundation. The most important observation is related to the stability of proton and electron which are the constituents of the universe. It is clear that the life time of electron should be infinite since there is no particle which is lighter than electron. On the other hand, proton is quite heavy, and thus, in principle it may decay into lighter particles. However, this decay is ruled out both experimentally and theoretically, and therefore, it is indeed stable.

In addition to the stability of the constituents, there should be another important observation that should support the infinite universe. This is the background radiation which should play a very important role for the infinite universe as we discuss later.

5.3.1 Cosmic Fireball and Explosion of Finite Universe

The assumption that this finite universe with several hundred billions of galaxies must have exploded about 15 billion years ago should be rather reasonable. However, these numbers of 15 billion years should not have any special physical meaning.

The important point should be the observed fact that galaxies of our finite universe have been expanding with each other, which is Hubble's law. A question may arise as to how these galaxies should be expanding with each other. The observations show that, the galaxy is more distant from us, it is moving away faster. In fact, this should be just what Hubble's law indicates, and suggests that, at the initial stage of cosmic fireball explosion, galaxies must be expanding with higher speed.

• **Stop of Expansion in Finite Universe :**
The expansion of finite universe should eventually stop at some point of time due to the attractive force of gravity among galaxies. After that, galaxies should be going to fuse into larger group of galaxies. Finally, two or three gigantic galaxy clusters should be formed, and then, they should be going to collide into the formation of big cosmic fireball.

• **Cosmic Fireball :**
We may call this big explosion "cosmic fireball". This cosmic fireball may well be somewhat similar to the baryonic states of big bang model. In this sense, at the early stage of the cosmic fireball, nuclei up to He may be made up by nuclear fusion. After some time, the cosmic fireball should lose its thermal energy and cool down. However, these procedures of expansion should be carefully studied, and at present, it is too difficult to make any concrete pictures of expansion.

5.3.2 Debris of Previous Finite Universe

According to the new cosmology model, the formation of galaxies and finite universe must be repeatedly made for eternal time. The formation of this present finite universe is considered to have occurred some 15 billion years ago. In this case, what should be before the present finite universe formation? If there should be any relics at the explosion, it should be easier to understand, but it may well be very difficult to find any relics of the explosions.

• **Debris of Explosion of Finite Universe :**
What should be some relics after the explosion of the previous finite universe? This may be the large scale structure of galaxy cluster. Readers should refer to some astrophysics textbook for the galaxy cluster. The fact that the space distribution of galaxy cluster is found to be quite unbalanced should suggest that this special and ununiform space distribution might well be due to the collision of two gigantic galaxy clusters at the final stage.

5.3.3 Infinite Past and Future and Infinite Space

According to the new cosmology, the galaxy formation and its explosion in the finite universe must have repeatedly occurred from the infinite past to future. Unfortunately, however, we cannot understand any infinity objects, and even though we use the terminology of infinity, this does not mean that we understand it. Mathematically we see that we can only count some finite numbers, and according to the brain science, we know that the number of brain cell should be about trillion and therefore, there is no chance that we understand infinity.

• **Paradox of Olbers :**
Space should be infinite. However, there is Olbers paradox which means that the sum of light should become infinite if the universe is infinite and the distribution of stars is uniform in the universe. This is quite a reasonable constraint, and therefore, it is believed that universe must be finite. However, there is a serious misunderstanding in the discussion of Olbers paradox. This is related to the assumption of light velocity, and since the velocity of light is finite, it takes infinite time when it should come to the earth from infinite distance. Therefore, even if we sum up all the lights from infinite universe, the sum of light cannot become infinite. In this respect, we see that Olbers paradox cannot be applied to the infinite universe due to the finite velocity of light.

Therefore, it is not very surprising that infinite numbers of similar finite universe to the present one should exist, even though we cannot understand its meaning of infinity.

5.3.4 New Picture of Universe

The basic picture of the new cosmology is connected to the repetition of expanding finite universe at the early stage and coalescing galaxies at the final stage. Eventually this finite universe should result in the cosmic fireball and then it should expand again and so on.

• **Center of Finite Universe :**
Naturally, a question may arise as to whether there is a center in this finite universe or not. Solar systems or galaxies have a center which is quite a heavy object, sun or super neutron star. However, the ensemble of galaxies should look like a nucleus which has no center. A finite universe is composed of many galaxies which may distributed like nucleons in nucleus. Each galaxy should play the same and equal role for making up the finite universe. In this respect, there is no special center in the finite universe, and if we can average over the galaxies of the finite universe, then we may find roughly the center position of finite universe.

5.3.5 Background Radiation and Mugen Universe

In this finite universe, there is the background radiation with 2.7 K. This low energy photon is uniformly distributed, and therefore, the total energy of the background radiation in the finite universe must be quite large. In fact, this photon energy should amount to several percent of the total gravitational energy in this finite universe.

If this energy of background radiation should have stemmed from the cosmic fireball explosion of the finite universe, then this photon energy should eventually escape away from this finite universe. Therefore, the infinite times of explosion should have already lost the total energy of the gravitational energy completely. This means that the origin of the background radiation should be somewhere else.

In order to understand the origin of background radiation energy, we may consider the following scenario for the radiation. That is, there must be infinite number of the finite universe of the similar type, and these finite universes repeat the explosion and expansions. Thus, from these explosions, radiations which correspond to the background radiation should be all over the whole universe, and this is just the origin of the 2.7 K background radiation.

5.3.6 Mugen Universe

Here we define the structure of the whole universe.

$$\textbf{Structure of universe} \ : \ \begin{cases} 10^{57} \times \textbf{protons} & \Rightarrow \textbf{star} \\ 10^{12} \times \textbf{stars} & \Rightarrow \textbf{galaxy} \\ 10^{12} \times \textbf{galaxies} & \Rightarrow \textbf{finite universe} \\ \infty \quad \times \textbf{finite universe} & \Rightarrow \textbf{mugen universe}. \end{cases}$$

5.3.7 Infinite Number of Finite Universes

In the Mugen universe, there must be infinite numbers of finite universes, and in this case, there is no problem that the vast amount of energy can be lost in terms of photon and neutrino during the explosion. In fact, the same level of energy due to photon and neutrino must be supplied from Mugen universe.

In this way, the problem of the gravitational energy loss by the explosion and expansion of finite universe can be solved. In addition, the origin of the background radiation should be from Mugen universe, and therefore, it is not surprising that the background radiation is in the thermal equilibrium.

• **Gravitational Stability of Mugen Universe :**
Now we should discuss why Mugen universe should be gravitationally stable even though the interaction is always attractive. Here we should consider a simplest case in which one finite universe is located at the origin $x = 0$ in one dimensional space. Then, at $x = \pm a$, there are two finite universes with the same total mass M, and next at $x = 2 \pm a$, there are two finite universes with the same total mass M and so on until at $x = n \pm a$. In this case, as long as n is finite, it is clear that, Mugen universe should eventually collapse into one big object with mass of $2nM$. However, if n is infinity, then Mugen universe should not collapse. and it should be stable, since the gravitational force from the left side of all finite universes should be equal to the one from the right side of all finite universes.

5.4 Can We Measure Neighboring $\beta-$Finite Universe?

We should call our finite universe α. This $\alpha-$finite universe has a size of about 15 billion light years and about 100 billion galaxies. These numbers are from the observations even though the number itself should not be very important.

In addition, we should call the neighboring finite universe β and possibly the $\beta-$finite universe should have similar size and numbers of galaxies. Now a basic question is as to whether there is any possibility of observing some light from $\beta-$finite universe or not.

5.4.1 No Red Shifts from $\beta-$Finite Universe?

$\beta-$finite universe may not be moving away from us. But at the same time, it should not be approaching to us either. This is because Mugen universe must be rather stable among finite universes. In this case, lights from $\beta-$finite universe must be neither red shifts nor blue shifts. Therefore, even if we could observe lights from $\beta-$finite universe, there should be no way to extract any information from the lights.

5.4.2 Lights of Blue Shifts

All the lights from $\alpha-$finite universe must be red shifts since the universe is expanding yet. Therefore, if one could observe lights with strong blue shifts, then they might well have come from the $\beta-$finite universe. This may happen if the $\beta-$finite universe should have been in the state of explosion.

Appendix A

Notations in Field Theory

In field theory, we often employ special notations which are by now commonly used. In this Appendix, we explain some of the notations which are particularly useful in field theory.

A.1 Natural Units

We employ the natural units because of its simplicity

$$\bullet\ c = 1, \quad \hbar = 1. \tag{A.1}$$

To get the right dimensions out, we should make use of

$$\bullet\ \hbar c = 197.33 \ \text{ MeV} \cdot \text{fm}. \tag{A.2}$$

For example, pion mass and its Compton wave length become

$$\begin{cases} \bullet\ m_\pi \simeq 140 \text{ MeV/c}^2 \\ \bullet\ \frac{1}{m_\pi} = \frac{\hbar c}{m_\pi c^2} = \frac{197 \text{ MeV}\cdot \text{ fm}}{140 \text{ MeV}} \simeq 1.4 \text{ fm}. \end{cases} \tag{A.3}$$

The fine structure constant α is expressed by the coupling constant e which is defined in some different ways :

$$\bullet\ \alpha = e^2 = \frac{e^2}{\hbar c} = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.036}\,.$$

- Masses of electron, muon and proton :

$$\begin{cases} \text{Electron mass :} & m_e = 0.511 \quad MeV/c^2 \\ \text{Muon mass :} & m_\mu = 105.66 \quad MeV/c^2 \\ \text{Proton mass :} & M_p = 938.28 \quad MeV/c^2 \end{cases} .$$

- Bohr radius: $a_0 = \dfrac{1}{m_e e^2} = 0.529 \times 10^{-8}$ cm

- Magnetic moments: $\begin{cases} \text{Electron :} & \mu_e = 1.00115965219 \quad \dfrac{e\hbar}{2m_e c} \\ \text{Muon :} & \mu_\mu = 1.001165920 \quad \dfrac{e\hbar}{2m_\mu c} \\ \text{Proton :} & \mu_P = 2.7928473446 \quad \dfrac{e\hbar}{2M_p c} \end{cases} .$

A.2 Hermite Conjugate and Complex Conjugate

For a complex c-number A

$$A = a + bi \quad (a, b: \text{ real}). \tag{A.4}$$

- Complex conjugate A^* :

$$A^* = a - bi. \tag{A.5}$$

- **Matrix** A

If A is a matrix, we define the hermite conjugate $A^\dagger$

$$(A^\dagger)_{ij} = A^*_{ji}. \tag{A.6}$$

- **Differential Operator** $\hat{A}$

If $\hat{A}$ is a differential operator, then the hermite conjugate can be defined only when the Hilbert space and its scalar product are defined. For example, suppose $\hat{A}$ is written as

$$\hat{A} = i\frac{\partial}{\partial x}. \tag{A.7}$$

In this case, its hermite conjugate $\hat{A}^\dagger$ becomes

$$\hat{A}^\dagger = -i\left(\frac{\partial}{\partial x}\right)^T = i\frac{\partial}{\partial x} = \hat{A} \tag{A.8}$$

which means $\hat{A}$ is Hermitian. This can be easily seen in a concrete fashion since

$$\langle\psi|\hat{A}\psi\rangle = \int_{-\infty}^{\infty} \psi^\dagger(x) i \frac{\partial}{\partial x} \psi(x)\, dx = -i \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} \psi^\dagger(x) \right) \psi(x)\, dx = \langle\hat{A}\psi|\psi\rangle, \quad \text{(A.9)}$$

where $\psi(\pm\infty) = 0$ is assumed. The complex conjugate of $\hat{A}$ is simply

$$\hat{A}^* = -i \frac{\partial}{\partial x} \neq \hat{A}. \quad \text{(A.10)}$$

- **Field ψ :**

If the $\psi(x)$ is a c-number field, then the hermite conjugate $\psi^\dagger(x)$ is just the same as the complex conjugate $\psi^*(x)$. However, when the field $\psi(x)$ is quantized, then one should always take the hermite conjugate $\psi^\dagger(x)$. For the complex conjugate of the field $\psi^*(x)$, we may examine the time reversal invariance later.

A.3 Scalar and Vector Products (Three Dimensions) :

- **Scalar Product**

For two vectors in three dimensions

$$\boldsymbol{r} = (x, y, z) \equiv (x_1, x_2, x_3), \quad \boldsymbol{p} = (p_x, p_y, p_z) \equiv (p_1, p_2, p_3) \quad \text{(A.11)}$$

the scalar product is defined

$$\boldsymbol{r} \cdot \boldsymbol{p} = \sum_{k=1}^{3} x_k p_k \equiv x_k p_k, \quad \text{(A.12)}$$

where, in the last step, we omit the summation notation if k is repeated twice.

- **Vector Product**

The vector product is defined as

$$\boldsymbol{r} \times \boldsymbol{p} \equiv (x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3, x_1 p_2 - x_2 p_1). \quad \text{(A.13)}$$

This can be rewritten in terms of components,

$$(\boldsymbol{r} \times \boldsymbol{p})_i = \epsilon_{ijk} x_j p_k, \quad \text{(A.14)}$$

where ϵ_{ijk} denotes anti-symmetric symbol with

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \quad \text{otherwise} = 0. \quad \text{(A.15)}$$

A.4 Scalar Product (Four Dimensions)

For two vectors in four dimensions,

$$x^\mu \equiv (t, x, y, z) = (x_0, \boldsymbol{r}), \quad p^\mu \equiv (E, p_x, p_y, p_z) = (p_0, \boldsymbol{p}) \tag{A.16}$$

the scalar product is defined

$$x \cdot p \equiv Et - \boldsymbol{r} \cdot \boldsymbol{p} = x_0 p_0 - x_k p_k. \tag{A.17}$$

This can be also written as

$$x_\mu p^\mu \equiv x_0 p^0 + x_1 p^1 + x_2 p^2 + x_3 p^3 = Et - \boldsymbol{r} \cdot \boldsymbol{p} = x \cdot p, \tag{A.18}$$

where x_μ and p_μ are defined as

$$x_\mu \equiv (x_0, -\boldsymbol{r}), \quad p_\mu \equiv (p_0, -\boldsymbol{p}). \tag{A.19}$$

Here, the repeated indices of the Greek letters mean the four dimensional summation $\mu = 0, 1, 2, 3$. The repeated indices of the roman letters always denote the three dimensional summation throughout the text.

A.4.1 Metric Tensor

It is convenient to introduce the metric tensor $g^{\mu\nu}$ which has the following properties

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.20}$$

In this case, the scalar product can be rewritten as

$$x \cdot p = x^\mu p^\nu g_{\mu\nu} = Et - \boldsymbol{r} \cdot \boldsymbol{p}. \tag{A.21}$$

A.5 Four Dimensional Derivatives ∂_μ

The derivative ∂_μ is introduced for convenience

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla} \right), \tag{A.22}$$

where the lower index has the positive space part. Therefore, the derivative ∂^μ becomes

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla} \right). \tag{A.23}$$

A.5.1 $\hat{p}^\mu$ and Differential Operator

Since the operator $\hat{p}^\mu$ becomes a differential operator as

$$\hat{p}^\mu = (\hat{E}, \hat{\boldsymbol{p}}) = \left(i\frac{\partial}{\partial t}, \ -i\boldsymbol{\nabla} \right) = i\partial^\mu \tag{A.24}$$

the negative sign, therefore, appears in the space part. For example, if we define the current j^μ in four dimension as

$$j^\mu = (\rho, \boldsymbol{j}), \tag{A.25}$$

then the current conservation is written as

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \frac{1}{i}\,\hat{p}_\mu j^\mu = 0. \tag{A.26}$$

A.5.2 Laplacian and d'Alembertian Operators

The Laplacian and d'Alembertian operators, Δ and $\Box$ are defined as

$$\Delta \ \equiv \ \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{A.27}$$

$$\Box \ \equiv \ \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta. \tag{A.28}$$

A.6 γ-Matrices

Here, we present explicit expressions of the γ-matrices in two and four dimensions. Before presenting the representation of the γ-matrices, we first give the explicit representation of Pauli matrices.

A.6.1 Pauli Matrices

Pauli matrices are given as

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1. \end{pmatrix} \tag{A.29}$$

Below we write some properties of the Pauli matrices.

A.6.2 Representation of γ-matrices

(a) Two dimensional representations of γ-matrices

- Dirac :

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.30}$$

- Chiral :

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.31}$$

(b) Four dimensional representations of gamma matrices

- Dirac :

$$\begin{aligned} \gamma_0 &= \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \\ \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \end{aligned} \tag{A.32}$$

- Chiral :

$$\begin{aligned} \gamma_0 &= \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \\ \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix} \end{aligned} \tag{A.33}$$

where

$$\mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.34}$$

- **Hermiticity**

$$\sigma_1^\dagger = \sigma_1, \quad \sigma_2^\dagger = \sigma_2, \quad \sigma_3^\dagger = \sigma_3. \tag{A.35}$$

- **Complex Conjugate**

$$\sigma_1^* = \sigma_1, \quad \sigma_2^* = -\sigma_2, \quad \sigma_3^* = \sigma_3. \tag{A.36}$$

- **Transposed**

$$\sigma_1^T = \sigma_1, \quad \sigma_2^T = -\sigma_2, \quad \sigma_3^T = \sigma_3 \quad (\sigma_k^T = \sigma_k^*). \tag{A.37}$$

• **Useful Relations**

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{A.38}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{A.39}$$

A.6.3 Useful Relations of γ-Matrices

Here, we summarize some useful relations of the γ-matrices.

• **Anti-commutation relations**

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^5, \gamma^\nu\} = 0. \tag{A.40}$$

• **Hermiticity**

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \ \ (\gamma_0^\dagger = \gamma_0, \ \ \gamma_k^\dagger = -\gamma_k), \quad \gamma_5^\dagger = \gamma_5. \tag{A.41}$$

• **Complex Conjugate**

$$\gamma_0^* = \gamma_0, \quad \gamma_1^* = \gamma_1, \quad \gamma_2^* = -\gamma_2, \quad \gamma_3^* = \gamma_3, \quad \gamma_5^* = \gamma_5. \tag{A.42}$$

• **Transposed**

$$\gamma_\mu^T = \gamma_0 \gamma_\mu^* \gamma_0, \quad \gamma_5^T = \gamma_5. \tag{A.43}$$

A.7 Transformation of State and Operator

When we transform a quantum state $|\psi\rangle$ by a unitary transformation U which satisfies

$$U^\dagger U = 1 \tag{A.44}$$

we write the transformed state as

$$|\psi'\rangle = U|\psi\rangle. \tag{A.45}$$

The unitarity is important since the norm must be conserved, that is,

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1. \tag{A.46}$$

In this case, an arbitrary operator $\mathcal{O}$ is transformed as

$$\mathcal{O}' = U\mathcal{O}U^{-1}. \tag{A.47}$$

This can be obtained since the expectation value of the operator $\mathcal{O}$ must be the same between two systems, that is,

$$\langle\psi|\mathcal{O}|\psi\rangle = \langle\psi'|\mathcal{O}'|\psi'\rangle. \tag{A.48}$$

Since

$$\langle\psi'|\mathcal{O}'|\psi'\rangle = \langle\psi|U^\dagger\mathcal{O}'U|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle \tag{A.49}$$

we find

$$U^\dagger\mathcal{O}'U = \mathcal{O}. \tag{A.50}$$

A.8 Fermion Current

We summarize the fermion currents and their properties of the Lorentz transformation. We also give their nonrelativistic expressions since the basic behaviors must be kept in the nonrelativistic expressions. Here, the approximate expressions are obtained by making use of the plane wave solutions for the Dirac wave function.

$$\bullet\ \text{Fermion currents :} \quad \left(\begin{array}{ll} \text{Scalar :} & \bar{\psi}\psi \simeq 1 \\ \text{Pseudoscalar :} & \bar{\psi}\gamma_5\psi \simeq \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} \\ \text{Vector :} & \bar{\psi}\gamma_\mu\psi \simeq \left(1, \dfrac{\boldsymbol{p}}{m}\right) \\ \text{Axialvector :} & \bar{\psi}\gamma_\mu\gamma_5\psi \simeq \left(\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}, \boldsymbol{\sigma}\right) \end{array} \right. \tag{A.51}$$

Therefore, under the parity $\hat{P}$ and time reversal $\hat{T}$ transformation, the above currents behave as

$$\bullet\ \text{Parity } \hat{P} : \quad \left(\begin{array}{l} \bar{\psi}'\psi' = \bar{\psi}\hat{P}^{-1}\hat{P}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_5\hat{P}\psi = -\bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\hat{P}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi \end{array} \right. \tag{A.52}$$

$$\bullet\ \text{Time reversal } \hat{T} : \quad \left(\begin{array}{l} \bar{\psi}'\psi' = \bar{\psi}\hat{T}^{-1}\hat{T}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_5\hat{T}\psi = \bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\hat{T}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\gamma_5\hat{T}\psi = -\bar{\psi}\gamma_k\gamma_5\psi \end{array} \right. \tag{A.53}$$

A.9 Trace in Physics

A.9.1 Definition

The trace of $N \times N$ matrix A is defined as

$$\mathrm{Tr}\{A\} = \sum_{i=1}^{N} A_{ii}. \tag{A.54}$$

This is simply the summation of the diagonal elements of the matrix A. It is easy to prove

$$\mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}. \tag{A.55}$$

A.9.2 Trace in Quantum Mechanics

In quantum mechanics, the trace of the Hamiltonian H becomes

$$\mathrm{Tr}\{H\} = \mathrm{Tr}\{UHU^{-1}\} = \sum_{n=1} E_n, \tag{A.56}$$

where U is a unitary operator that diagonalizes the Hamiltonian, and E_n denotes the energy eigenvalue of the Hamiltonian. Therefore, the trace of the Hamiltonian has the meaning of the sum of all the eigenvalues of the Hamiltonian.

A.9.3 Trace in $SU(N)$

In the special unitary group $SU(N)$, we often describe the element U^a in terms of the generator T^a as

$$U^a = e^{iT^a}. \tag{A.57}$$

In this case, the generator must be hermitian and traceless since

$$\det U^a = \exp\left(\mathrm{Tr}\left\{\ln U^a\right\}\right) = \exp\left(i\,\mathrm{Tr}\left\{T^a\right\}\right) = 1 \tag{A.58}$$

and thus

$$\mathrm{Tr}\left\{T^a\right\} = 0. \tag{A.59}$$

The generators of $SU(N)$ group satisfy the following commutation relations

$$[T^a, T^b] = iC^{abc}T^c, \tag{A.60}$$

where C^{abc} denotes a structure constant in the Lie algebra. The generators are normalized in this textbook such that

$$\mathrm{Tr}\left\{T^a T^b\right\} = \frac{1}{2}\,\delta^{ab}. \tag{A.61}$$

A.9.4 Trace of γ-Matrices and $\not{p}$

The Trace of the γ-matrices is also important. First, we have

$$\mathrm{Tr}\,\{1\} = 4, \quad \mathrm{Tr}\,\{\gamma_\mu\} = 0, \quad \mathrm{Tr}\,\{\gamma_5\} = 0. \tag{A.62}$$

In field theory, we often define a symbol of $\not{p}$ just for convenience

$$\bullet\ \not{p} \equiv p_\mu \gamma^\mu. \tag{A.63}$$

In this case, the following relation holds :

$$\bullet\ \not{p}\not{q} = pq - i\sigma_{\mu\nu} p^\mu q^\nu. \tag{A.64}$$

The following relations may also be useful :

$$\bullet\ \mathrm{Tr}\,\{\not{p}\not{q}\} = 4pq, \tag{A.65}$$

$$\bullet\ \mathrm{Tr}\,\{\gamma_5 \not{p}\not{q}\} = 0, \tag{A.66}$$

$$\bullet\ \mathrm{Tr}\,[\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = 4\Big\{(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\Big\} \tag{A.67}$$

$$\bullet\ \mathrm{Tr}\,[\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = -4i\varepsilon_{\alpha\beta\gamma\delta}\ p_1^\alpha\ p_2^\beta\ p_3^\gamma\ p_4^\delta \tag{A.68}$$

$$\bullet\ \mathrm{Tr}\,[\gamma^5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}] = -4i\,[g_{\mu_1\mu_2}\varepsilon_{\mu_3\mu_4\mu_5\mu_6} - g_{\mu_1\mu_3}\varepsilon_{\mu_2\mu_4\mu_5\mu_6}$$

$$+g_{\mu_2\mu_3}\varepsilon_{\mu_1\mu_4\mu_5\mu_6} + g_{\mu_4\mu_5}\varepsilon_{\mu_1\mu_2\mu_3\mu_6} - g_{\mu_4\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_5} + g_{\mu_5\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}] \tag{A.69}$$

$$\bullet\ \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'} = -\begin{vmatrix} \delta^\mu{}_{\mu'} & \delta^\mu{}_{\nu'} & \delta^\mu{}_{\alpha'} & \delta^\mu{}_{\beta'} \\ \delta^\nu{}_{\mu'} & \delta^\nu{}_{\nu'} & \delta^\nu{}_{\alpha'} & \delta^\nu{}_{\beta'} \\ \delta^\alpha{}_{\mu'} & \delta^\alpha{}_{\nu'} & \delta^\alpha{}_{\alpha'} & \delta^\alpha{}_{\beta'} \\ \delta^\beta{}_{\mu'} & \delta^\beta{}_{\nu'} & \delta^\beta{}_{\alpha'} & \delta^\beta{}_{\beta'} \end{vmatrix} \tag{A.70}$$

$$\bullet\ \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu'\alpha'\beta'} = -\begin{vmatrix} \delta^\nu{}_{\nu'} & \delta^\nu{}_{\alpha'} & \delta^\nu{}_{\beta'} \\ \delta^\alpha{}_{\nu'} & \delta^\alpha{}_{\alpha'} & \delta^\alpha{}_{\beta'} \\ \delta^\beta{}_{\nu'} & \delta^\beta{}_{\alpha'} & \delta^\beta{}_{\beta'} \end{vmatrix} \tag{A.71}$$

$$\bullet\ \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha'\beta'} = -2\begin{vmatrix} \delta^\alpha{}_{\alpha'} & \delta^\alpha{}_{\beta'} \\ \delta^\beta{}_{\alpha'} & \delta^\beta{}_{\beta'} \end{vmatrix} \tag{A.72}$$

$$\bullet\ \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta'} = -6\delta^\beta{}_{\beta'} \tag{A.73}$$

$$\bullet\ \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} = -24 \tag{A.74}$$

Appendix B

Basic Equations and Principles

B.1 Lagrange Equation

In classical field theory, the equation of motion is derived from the Lagrange equation. Here, we briefly review how we can obtain the equation of motion from the Lagrangian density.

B.1.1 Lagrange Equation in Classical Mechanics

Before going to the field theory, we first discuss the Lagrange equation (Newton equation) in classical mechanics. In order to obtain the Lagrange equation by the variational principle in classical mechanics, we start from the action S

$$S = \int L(q, \dot{q})\, dt, \tag{B.1}$$

where the Lagrangian $L(q, \dot{q})$ depends on the general coordinate q and its velocity $\dot{q}$. At the time of deriving equation of motion by the variational principle, q and $\dot{q}$ are independent as the function of t. This is clear since, in the action S, the functional dependence of $q(t)$ is unknown and therefore we cannot make any derivative of $q(t)$ with respect to time t. Once the equation of motion is established, then we can obtain $\dot{q}$ by time differentiation of $q(t)$ which is a solution of the equation of motion.

The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of q and $\dot{q}$.

$$\begin{aligned} \delta S &= \int \delta L(q, \dot{q})\, dt = \int \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q} \right) dt & \text{(B.2)} \\ &= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt = 0, & \text{(B.3)} \end{aligned}$$

where the surface terms are assumed to vanish. Therefore, we obtain the Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{B.4}$$

B.1.2 Hamiltonian in Classical Mechanics

The Lagrangian $L(q, \dot{q})$ must be invariant under the infinitesimal time displacement ϵ of $q(t)$ as

$$q(t+\epsilon) \to q(t) + \dot{q}\epsilon, \quad \dot{q}(t+\epsilon) \to \dot{q}(t) + \ddot{q}\epsilon + \dot{q}\,\frac{d\epsilon}{dt}\,. \tag{B.5}$$

Therefore, we find

$$\delta L(q,\dot{q}) = L(q(t+\epsilon), \dot{q}(t+\epsilon)) - L(q,\dot{q}) = \frac{\partial L}{\partial q}\,\dot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\,\ddot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\,\dot{q}\frac{d\epsilon}{dt} = 0. \tag{B.6}$$

Neglecting the surface term, we obtain

$$\delta L(q,\dot{q}) = \left[\frac{\partial L}{\partial q}\,\dot{q} + \frac{\partial L}{\partial \dot{q}}\,\ddot{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right)\right]\epsilon = \left[\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{q}}\,\dot{q}\right)\right]\epsilon = 0. \tag{B.7}$$

Thus, if we define the Hamiltonian H as

$$H \equiv \frac{\partial L}{\partial \dot{q}}\,\dot{q} - L \tag{B.8}$$

then it is a conserved quantity.

B.1.3 Lagrange Equation for Fields

The Lagrange equation for fields can be obtained almost in the same way as the particle case. For fields, we should start from the Lagrangian density $\mathcal{L}$ and the action is written as

$$S = \int \mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt, \tag{B.9}$$

where $\psi(x)$, $\dot{\psi}(x)$ and $\frac{\partial \psi}{\partial x_k}$ are independent functional variables.

The Lagrange equation can be obtained by requiring that the action S should be a minimum with respect to the variation of ψ, $\dot{\psi}$ and $\frac{\partial \psi}{\partial x_k}$,

$$\begin{aligned}
\delta S &= \int \delta\mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k}\right) d^3r\, dt = \int \left(\frac{\partial \mathcal{L}}{\partial \psi}\,\delta\psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\delta\dot{\psi} + \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,\delta\left(\frac{\partial \psi}{\partial x_k}\right)\right) d^3r\, dt \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\right)\delta\psi\, d^3r\, dt = 0,
\end{aligned} \tag{B.10}$$

where the surface terms are assumed to vanish. Therefore, we obtain

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})}\,, \tag{B.11}$$

which can be expressed in the relativistic covariant way as

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right). \tag{B.12}$$

This is the Lagrange equation for field ψ, which should hold for any independent field ψ.

B.2 Noether Current

If the Lagrangian density is invariant under the transformation of the field with a continuous variable, then there is always a conserved current associated with this symmetry. This is called *Noether current* and can be derived from the invariance of the Lagrangian density and the Lagrange equation.

B.2.1 Global Gauge Symmetry

The Lagrangian density which is discussed in this textbook should have the following functional dependence in general

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi + \mathcal{L}_I \left[\bar{\psi}\psi, \bar{\psi}\gamma_5\psi, \bar{\psi}\gamma_\mu\psi\right]. \tag{B.13}$$

This Lagrangian density is obviously invariant under the global gauge transformation

$$\psi' = e^{i\alpha}\psi, \quad \psi'^{\dagger} = e^{-i\alpha}\psi^{\dagger}, \tag{B.14}$$

where α ia a real constant. Therefore, the Noether current is conserved in this system. To derive the Noether current conservation for the global gauge transformation, we can consider the infinitesimal global transformation, that is, $|\alpha| \ll 1$. In this case, the transformation becomes

$$\psi' = \psi + \delta\psi, \quad \delta\psi = i\alpha\psi. \tag{B.15}$$

$$\psi'^{\dagger} = \psi^{\dagger} + \delta\psi^{\dagger}, \quad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}. \tag{B.16}$$

• Invariance of Lagrangian Density

Now, it is easy to find

$$\delta\mathcal{L} = \mathcal{L}(\psi', \psi'^{\dagger}, \partial_\mu\psi', \partial_\mu\psi'^{\dagger}) - \mathcal{L}(\psi, \psi^{\dagger}, \partial_\mu\psi, \partial_\mu\psi^{\dagger}) = 0. \tag{B.17}$$

At the same time, we can easily evaluate $\delta\mathcal{L}$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\delta\left(\partial_\mu\psi\right) + \frac{\partial \mathcal{L}}{\partial \psi^{\dagger}}\delta\psi^{\dagger} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\delta\left(\partial_\mu\psi^{\dagger}\right)$$

$$
\begin{aligned}
&= i\alpha \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\mu \psi - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \right) \psi^\dagger - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \partial_\mu \psi^\dagger \right] \\
&= i\alpha \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right] = 0,
\end{aligned}
\tag{B.18}
$$

where the equation of motion for ψ is employed.

• Current Conservation

Therefore, if we define the current j_μ as

$$
j^\mu \equiv -i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right] \tag{B.19}
$$

then we have

$$
\partial_\mu j^\mu = 0. \tag{B.20}
$$

For Dirac fields with electromagnetic interactions or self-interactions, we can obtain as a conserved current

$$
j^\mu = \bar{\psi} \gamma^\mu \psi. \tag{B.21}
$$

B.2.2 Chiral Symmetry

When the Lagrangian density is invariant under the chiral transformation,

$$
\psi' = e^{i\alpha\gamma_5} \psi \tag{B.22}
$$

then there is another Noether current. Here, $\delta\psi$ becomes

$$
\delta\psi = i\alpha\gamma_5\psi. \tag{B.23}
$$

Therefore, a corresponding conserved current for massless Dirac fields with electromagnetic interactions or self-interactions can be obtained

$$
j_5^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \gamma_5 \psi = \bar{\psi} \gamma^\mu \gamma_5 \psi. \tag{B.24}
$$

In this case, we have

$$
\partial_\mu j_5^\mu = 0 \tag{B.25}
$$

which is the conservation of the axial vector current. The conservation of the axial vector current is realized for field theory models with massless fermions.

B.3 Hamiltonian Density

The Hamiltonian density $\mathcal{H}$ is constructed from the Lagrangian density $\mathcal{L}$. The field theory models which we consider should possess the translational invariance. If the Lagrangian density is invariant under the translation a^μ, then there is a conserved quantity which is the energy momentum tensor $\mathcal{T}^{\mu\nu}$. The Hamiltonian density is constructed from the energy momentum tensor of $\mathcal{T}^{00}$.

B.3.1 Hamiltonian Density from Energy Momentum Tensor

Now, the Lagrangian density is given as $\mathcal{L}\left(\psi_i, \dot{\psi}_i, \frac{\partial\psi_i}{\partial x_k}\right)$. If we consider the following infinitesimal translation a^μ of the field ψ_i and $\psi_i^\dagger$

$$\begin{aligned} \psi_i' &= \psi_i + \delta\psi_i, \quad \delta\psi_i = (\partial_\nu\psi_i)a^\nu, \\ {\psi_i^\dagger}' &= \psi_i^\dagger + \delta\psi_i^\dagger, \quad \delta\psi_i^\dagger = (\partial_\nu\psi_i^\dagger)a^\nu, \end{aligned} \tag{B.26}$$

then the Lagrangian density should be invariant

$$\begin{aligned} \delta\mathcal{L} &\equiv \mathcal{L}(\psi_i', \partial_\mu\psi_i') - \mathcal{L}(\psi_i, \partial_\mu\psi_i) \\ &= \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) + \frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,\delta\psi_i^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\delta(\partial_\mu\psi_i^\dagger)\right] = 0. \end{aligned}$$

Making use of the Lagrange equation, we obtain

$$\begin{aligned} \delta\mathcal{L} &= \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,(\partial_\nu\psi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,(\partial_\mu\partial_\nu\psi_i) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial_\nu\psi_i\right)\right] a^\nu \\ &\quad + \sum_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,(\partial_\nu\psi_i^\dagger) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,(\partial_\mu\partial_\nu\psi_i^\dagger) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial_\nu\psi_i^\dagger\right)\right] a^\nu \\ &= \partial_\mu\left[\mathcal{L}g^{\mu\nu} - \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger\right)\right] a_\nu = 0. \end{aligned} \tag{B.27}$$

• **Energy Momentum Tensor $\mathcal{T}^{\mu\nu}$**

Therefore, if we define the energy momentum tensor $\mathcal{T}^{\mu\nu}$ by

$$\mathcal{T}^{\mu\nu} \equiv \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger\right) - \mathcal{L}g^{\mu\nu} \tag{B.28}$$

then $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0. \tag{B.29}$$

This leads to the definition of the Hamltonian density $\mathcal{H}$ in terms of $\mathcal{T}^{00}$

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i)}\,\partial^0\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i^\dagger)}\,\partial^0\psi_i^\dagger\right) - \mathcal{L}. \tag{B.30}$$

B.3.2 Hamiltonian Density from Conjugate Fields

When the Lagrangian density is given as $\mathcal{L}(\psi_i, \dot{\psi}_i, \frac{\partial \psi_i}{\partial x_k})$, we can define the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$ as

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}, \quad \Pi_{\psi_i^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger}. \tag{B.31}$$

In this case, the Hamiltonian density can be written as

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L}. \tag{B.32}$$

It should be noted that this way of making the Hamiltonian density is indeed easier to remember than the construction starting from the energy momentum tensor.

- **Hamiltonian**

The Hamiltonian is defined by integrating the Hamiltoian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \left[\sum_i (\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger) - \mathcal{L} \right] d^3r. \tag{B.33}$$

B.3.3 Hamiltonian Density for Free Dirac Fields

For a free Dirac field with its mass m, the Lagrangian density becomes

$$\mathcal{L} = \psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j. \tag{B.34}$$

Therefore, the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$ are obtained

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = \psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0. \tag{B.35}$$

Thus, the Hamiltonian density becomes

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L} = \bar{\psi}_i \left[-i\gamma_k \partial_k + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \tag{B.36}$$

B.3.4 Hamiltonian for Free Dirac Fields

The Hamiltonian H is obtained by integrating the Hamiltonian density over all space and thus can be written as

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi\, d^3r. \tag{B.37}$$

In classical field theory, this Hamiltonian is not an operator but is just the field energy itself. However, this field energy cannot be evaluated unless we know the shape of the field $\psi(x)$ itself. Therefore, we should determine the shape of the field $\psi(x)$ by the equation of motion in the classical field theory.

B.3.5 Role of Hamiltonian

We should comment on the usefulness of the classical field Hamiltonian itself for field theory models. In fact, the Hamiltonian alone is not useful. This is similar to the classical mechanics case in which the Hamiltonian of a point particle itself does not tell a lot. Instead, we have to derive the Hamilton equations in order to calculate some physical properties of the system and the Hamilton equations are equivalent to the Lagrange equations in classical mechanics.

- **Classical Field Theory**

In classical field theory, the situation is just the same as the classical mechanics case. If we stay in the classical field theory, then we should derive the field equation from the Hamiltonian by the functional variational principle as will be discussed in the next section.

- **Quantized Field Theory**

The Hamiltonian of the field theory becomes important when the fields are quantized. In this case, the Hamiltonian becomes an operator, and thus we have to solve the eigenvalue problem for the quantized Hamiltonian $\hat{H}$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{B.38}$$

where $|\Psi\rangle$ is called *Fock state* and should be written in terms of the creation and annihilation operators of fermion and anti-fermion. The space spanned by the Fock states is called *Fock space*.

In normal circumstances of the field theory models such as QED and QCD, it is practically impossible to find the eigenstate of the quantized Hamiltonian. The difficulty of the quantized field theory comes mainly from two reasons. Firstly, we have to construct the vacuum state which is composed of infinite many negative energy particles interacting with each other. The vacuum state should be the eigenstate of the Hamiltonian

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle, \tag{B.39}$$

where E_Ω denotes the energy of the vacuum and it is in general infinity with the negative sign. The vacuum state $|\Omega\rangle$ is composed of infinitely many negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger(s)}_{\boldsymbol{p}}|0\rangle\rangle, \tag{B.40}$$

where $|0\rangle\rangle$ denotes the null vacuum state. In the realistic calculations, the number of the negative energy particles must be set to a finite value, and this should be reasonable since physical observables should not depend on the properties of the deep negative energy particles. However, it is most likely that the number of the negative energy particles should be, at least, larger than a few thousand for two dimensional field theory models.

The second difficulty arises from the operators in the Hamiltonian which can change the fermion and anti-fermion numbers and therefore can induce infinite series of the transitions among the Fock states. Since the spectrum of bosons and baryons can be obtained by operating the fermion and anti-fermion creation operators on the vacuum state, the Fock space which is spanned by the creation and annihilation operators becomes infinite. In the realistic calculations, the truncation of the Fock space becomes most important, even though it is difficult to find any reasonable truncation scheme.

In this respect, the Thirring model is an exceptional case where the exact eigenstate of the quantized Hamiltonian is found. This is, however, understandable since the Thirring model Hamiltonian does not contain the operators which can change the fermion and anti-fermion numbers.

B.4 Variational Principle in Hamiltonian

When we have the Hamiltonian, then we can derive the equation of motion by requiring that the Hamiltonian should be minimized with respect to the functional variation of the state $\psi(\boldsymbol{r})$.

B.4.1 Schrödinger Field

When we minimize the Hamiltonian

$$H = \int \left[-\frac{1}{2m}\psi^\dagger \boldsymbol{\nabla}^2 \psi + \psi^\dagger U \psi \right] d^3r \tag{B.41}$$

with respect to $\psi(\boldsymbol{r})$, then we can obtain the static Schrödinger equation.

• Functional Derivative

First, we define the functional derivative for an arbitrary function $\psi_i(\boldsymbol{r})$ by

$$\frac{\delta \psi_i(\boldsymbol{r}')}{\delta \psi_j(\boldsymbol{r})} = \delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{B.42}$$

This is the most important equation for the functional derivative, and once we accept this definition of the functional derivative, then we can evaluate the functional variation just in the same way as normal derivative of the function $\psi_i(\boldsymbol{r})$.

• **Functional Variation of Hamiltonian**

For the condition on $\psi(\boldsymbol{r})$, we require that it should be normalized according to

$$\int \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, d^3r = 1. \tag{B.43}$$

In order to minimize the Hamiltonian with the above condition, we can make use of the Lagrange multiplier and make a functional derivative of the following quantity with respect to $\psi^\dagger(\boldsymbol{r})$

$$\begin{aligned} H[\psi] &= \int \left[-\frac{1}{2m}\,\psi^\dagger(\boldsymbol{r}')\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + \psi^\dagger(\boldsymbol{r}')U\psi(\boldsymbol{r}') \right] d^3r' \\ &\quad -E\left(\int \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}')\, d^3r' - 1 \right), \end{aligned} \tag{B.44}$$

where E denotes a Lagrange multiplier and just a constant. In this case, we obtain

$$\frac{\delta H[\psi]}{\delta\psi^\dagger(\boldsymbol{r})} = \int \delta(\boldsymbol{r}-\boldsymbol{r}') \left[-\frac{1}{2m}\,\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + U\psi(\boldsymbol{r}') - E\psi(\boldsymbol{r}') \right] d^3r' = 0. \tag{B.45}$$

Therefore, we find

$$-\frac{1}{2m}\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) + U\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{B.46}$$

which is just the static Schrödinger equation.

B.4.2 Dirac Field

The Dirac equation for free field can be obtained by the variational principle of the Hamiltonian. Below, we derive the static Dirac equation in a concrete fashion by the functional variation of the Hamiltonian.

• **Functional Variation of Hamiltonian**

For the condition on $\psi_i(\boldsymbol{r})$, we require that it should be normalized according to

$$\int \psi_i^\dagger(\boldsymbol{r})(\gamma^0)_{ij}\psi_j(\boldsymbol{r})\, d^3r = 1. \tag{B.47}$$

Now, the Hamiltonian should be minimized with the condition of eq.(B.47)

$$\begin{aligned} H[\psi_i] &= \int \psi_i^\dagger(\boldsymbol{r}) \left[-i(\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})_{ij} + m(\gamma^0)_{ij} \right] \psi_j(\boldsymbol{r})\, d^3r \\ &\quad -E\left(\int \psi_i^\dagger(\boldsymbol{r})(\gamma^0)_{ij}\psi_j(\boldsymbol{r})\, d^3r - 1 \right), \end{aligned} \tag{B.48}$$

where E is just a constant of the Lagrange multiplier. By minimizing the Hamiltonian with respect to $\psi_i^\dagger(\boldsymbol{r})$, we obtain

$$(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi(\boldsymbol{r}) - E\psi(\boldsymbol{r}) = 0 \tag{B.49}$$

which is just the static Dirac equation for free field.

Appendix C

Maxwell Equation

The most fundamental equation in physics is the Maxwell equation. This equation is constructed by extracting physical law from experiments, and therefore the equation is basically related to describing nature itself.

C.1 Maxwell Equation

The Maxwell equation is written for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = e\rho, \qquad \text{(Gauss law)} \tag{C.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \text{(No magnetic monopole)} \tag{C.2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \text{(Faraday law)} \tag{C.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = e\boldsymbol{j}, \qquad \text{(Ampere} - \text{Maxwell law)} \tag{C.4}$$

where ρ and $\boldsymbol{j}$ denote the charge and current densities, respectively, and we explicitly write the charge e. The behavior of the charge density ρ and the current density $\boldsymbol{j}$ should be understood by solving the equations of motion for fermions. In this sense, it is important to realize that the Maxwell equation cannot tell us anything about the charge and current densities. In reality, the behavior of the charge and current density in the metal is very complicated, and it is mostly impossible to produce and understand the physics of the charge and current density in the metal in a proper manner. This is, of course, related to the fact that many body problems cannot be solved even for the non-relativistic equations of motion.

It may be important to note that the Maxwell equation does not contain any $\hbar$ even though it is a field theory equation of motion. However, if one considers the energy of photon, then one should introduce the $\hbar$ to express the photon energy like $\hbar\omega$. In this respect, one may say that the free photon is the result of the quantization of the vector field, and the classical field equation which is derived for the vector field in the absence of the matter fields does not prove the existence of photon. It only says that the wave equation for the vector field $\boldsymbol{A}$ indicates that it should behave like a free massless particle.

In this sense, the Maxwell equation itself does not know about the quantization of fields, and the basic theoretical reason why one should quantize the fields is one of the most important problems. At present, we believe that the quantization of vector field $\boldsymbol{A}$ is required from experiment.

On the other hand, the quantization of the Dirac field should be originated from the negative energy states which should require the field quantization with the anti-commutation relation for the creation and annihilation operators within the theoretical framework.

C.2 Vector Potential

In order to describe the Maxwell equation in a different way, one normally introduces the vector potential $(A_0, \boldsymbol{A})$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{C.5}$$

In this case, the Faraday law ($\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$) and no magnetic monopole ($\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$) can be automatically satisfied. In this case, the Maxwell equation can be written in terms of the vector potential $(A_0, \boldsymbol{A})$ as

$$\boldsymbol{\nabla}^2 A_0 = -e\rho, \qquad \text{(Poisson equation)} \tag{C.6}$$

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) \boldsymbol{A} + \frac{\partial}{\partial t} \boldsymbol{\nabla} A_0 = e\boldsymbol{j}, \qquad \text{with} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{C.7}$$

In this expression, we take the Coulomb gauge fixing since this is simple and best.

C.2.1 Static Fields

If the field does not depend on time, then the electric field $\boldsymbol{E}$ can be written as $\boldsymbol{E} = -\boldsymbol{\nabla} A_0$ because $\frac{\partial \boldsymbol{A}}{\partial t} = 0$. By making use of the identity equation for the $\delta-$function,

$$\boldsymbol{\nabla^2} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{C.8}$$

we can obtain the solution for the Poisson equation as

$$A_0(\boldsymbol{r}) = \frac{e}{4\pi}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{C.9}$$

and thus we obtain the electric field by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{e}{4\pi}\int \frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d^3r'. \tag{C.10}$$

On the other hand, the Ampere law becomes

$$\boldsymbol{\nabla}^2\boldsymbol{A} = -e\boldsymbol{j}$$

which can be easily solved, and we can obtain the solution of the above equation as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{e}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r'. \tag{C.11}$$

In this case, the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ can be expressed as

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{e}{4\pi}\int \frac{Jd\boldsymbol{r}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \quad \text{with} \quad Jd\boldsymbol{r}' \equiv \boldsymbol{j}(\boldsymbol{r}')d^3r' \tag{C.12}$$

which is Biot-Savart law.

C.2.2 Free Vector Field and Its Quantization

When there exist neither charge nor current densities, that is, the vacuum state, then the Maxwell equation becomes

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\boldsymbol{A}(t,\boldsymbol{r}) = 0 \tag{C.13}$$

which is the wave equation. However, it is clear that the vector field is a real field, and therefore there is no free field solution at the present condition for the vector field. More explicitly, the solution of the free field should be an eigenstate of the momentum operator $\boldsymbol{p} = -i\boldsymbol{\nabla}$. This means the solution of the vector field with its momentum $\boldsymbol{k}$ should have the following shape

$$\boldsymbol{A}(t,\boldsymbol{r}) = \frac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}, \quad \text{or} \quad \frac{1}{\sqrt{V}}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t} \tag{C.14}$$

which are, however, complex functions. Therefore, we should have another condition on the vector field if we wish to have a free field solution, corresponding to a photon state. This is indeed connected to the quantization of the vector field and we write

$$\hat{\boldsymbol{A}}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}\left[c^\dagger_{\boldsymbol{k},\lambda}e^{-i\omega_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{r}} + c_{\boldsymbol{k},\lambda}e^{i\omega_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \tag{C.15}$$

where $c_{\boldsymbol{k},\lambda}$, $c^\dagger_{\boldsymbol{k},\lambda}$ denote the creation and annihilation operators, and $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. Here, $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ denotes the polarization vector which should satisfy the following condition from the Coulomb gauge fixing

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0 \tag{C.16}$$

which is the most reasonable gauge fixing condition, and up to now, it does not give rise to any problems concerning the evaluation of all the physical observables in quantum electrodynamics.

C.2.3 Commutation Relations

Since the gauge fields are bosons, the quantization procedure must be done in the commutation relations, instead of anti-commutation relations. Therefore, the quantization can be done by requiring that $c_{\boldsymbol{k},\lambda}$, $c^\dagger_{\boldsymbol{k},\lambda}$ should satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} \tag{C.17}$$

and all other commutation relations vanish.

C.3 Photon

For this quantized vector field, we can define one-photon state with $(\boldsymbol{k}, \lambda)$, and it can be written as

$$\boldsymbol{A}_{\boldsymbol{k},\lambda}(x) = \langle \boldsymbol{k}, \lambda | \hat{\boldsymbol{A}}(x) | 0 \rangle = \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega_{\boldsymbol{k}} t} \tag{C.18}$$

which is indeed the eigenstate of the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$. Here, we see that photon is the result of the field quantization. In this respect, photon cannot survive in the classical field theory of the Maxwell equation even though the wave equation suggests that there must be some wave that can propagate like a free particle. Indeed, eq.(C.13) indicates that there should be a free wave solution. However, the vector potential $\boldsymbol{A}$ itself is a real field, and therefore it cannot behave like a free particle which should be a complex function ($e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$). In this respect, the existence of photon should be understood only after the vector field $\boldsymbol{A}$ is quantized. After the field quantization, the energy of photon is measured in units of $\hbar$, that is,

$$E_{\text{photon}} = \hbar\omega. \tag{C.19}$$

The fact that the Maxwell equation does not contain any $\hbar$ may be a good reason why it could not lead us to the concept of the first quantization even though it is indeed a field theory equation.

C.3.1 Field Energy of Photon

The energy of the gauge field can be calculated from the energy momentum tensor $\mathcal{T}^{\mu\nu}$ of the electromagnetic fields and it becomes

$$H_0 = \int \mathcal{T}^{00} d^3r = \frac{1}{2}\int \left[\left(\frac{\partial \boldsymbol{A}}{\partial t}\right)^2 + (\boldsymbol{\nabla} \times \boldsymbol{A})^2 \right] d^3r = \sum_{\boldsymbol{k},\lambda} \omega_{\boldsymbol{k}} \left(c^\dagger_{\boldsymbol{k},\lambda} c_{\boldsymbol{k},\lambda} + \frac{1}{2} \right). \quad \text{(C.20)}$$

This represents the energy of photons, and it is written in terms of the field quantized expression.

C.3.2 Static Field Energy per Time

The energy increase per second can be written as

$$W_0 = e\int \boldsymbol{j} \cdot \boldsymbol{E} d^3r. \quad \text{(C.21)}$$

This equation can be rewritten by making use of the Maxwell equation as

$$W_0 = -\frac{d}{dt}\int \left(\frac{1}{2}|\boldsymbol{B}|^2 + \frac{1}{2}|\boldsymbol{E}|^2 \right) d^3r - \int \boldsymbol{\nabla} \cdot \boldsymbol{S} d^3r \quad \text{(C.22)}$$

where the Poynting vector $\boldsymbol{S}$ is defined as

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}. \quad \text{(C.23)}$$

This first term in this equation corresponds to the normal field energy increase of the static fields $\boldsymbol{E}$ and $\boldsymbol{B}$. The second term is the energy flow from the Poynting vector, but we should note that the energy should flow into the inner part of the system and should be accumulated into the condenser thorough the Poynting vector. But it never flows out into the air. That means that the emission of photons has nothing to do with the Poynting vector. This is, of course, clear since the emission of photon should be only possible through electrons (fermions) as we see below.

C.3.3 Oscillator of Electromagnetic Wave

Photon can be emitted from the oscillator when the electromagnetic field is oscillating. A question is as to how it can emit photons. Now the electromagnetic interaction H_I with electrons can be written as

$$H_I = -e\int \boldsymbol{j} \cdot \boldsymbol{A} d^3r \quad \text{(C.24)}$$

and thus we should start from this expression. The interaction energy increase per time can be written as

$$W \equiv \frac{dH_I}{dt} = -e\int \left[\frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} + \boldsymbol{j} \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right] d^3r \quad \text{(C.25)}$$

where we consider the case without A_0 term, and thus the electric field can be written as

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}. \tag{C.26}$$

Thus, W becomes

$$W = -e\int \frac{\partial \boldsymbol{j}}{\partial t}\cdot \boldsymbol{A} d^3r + e\int \boldsymbol{j}\cdot \boldsymbol{E} d^3r. \tag{C.27}$$

From the above equation, we see that the second term is just W_0, and thus there is no need to discuss it further. Therefore, defining the first term by W_1, we obtain

$$W_1 = -e\int \frac{\partial \boldsymbol{j}}{\partial t}\cdot \boldsymbol{A} d^3r = -\frac{e}{m}\int \left\{\frac{\partial}{\partial t}(\psi^\dagger \hat{\boldsymbol{p}}\psi)\right\}\cdot \boldsymbol{A} d^3r \tag{C.28}$$

where we take the non-relativistic current $\boldsymbol{j}$ as

$$\boldsymbol{j} = \frac{1}{m}\psi^\dagger \hat{\boldsymbol{p}}\psi, \qquad \text{with} \quad \hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}. \tag{C.29}$$

Since the Zeeman Hamiltonian H_Z is written as

$$H_Z = -\frac{e}{2m}\boldsymbol{\sigma}\cdot \boldsymbol{B}_0 \tag{C.30}$$

we can evaluate the current variation with respect to time as

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{1}{m}\left[\frac{\partial \psi^\dagger}{\partial t}\hat{\boldsymbol{p}}\psi + \psi^\dagger \hat{\boldsymbol{p}}\frac{\partial \psi}{\partial t}\right] = \frac{e}{2m^2}\boldsymbol{\nabla} B_0(\boldsymbol{r}) \tag{C.31}$$

where we assume that the $\boldsymbol{B}_0$ is in the $z-$direction $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. Thus, we find

$$W_1 = -\frac{e^2}{2m^2}\int (\boldsymbol{\nabla} B_0(\boldsymbol{r}))\cdot \boldsymbol{A} d^3r \tag{C.32}$$

where we note that $\boldsymbol{A}$ is associated with current electrons while $\boldsymbol{B}_0$ is an external magnetic field. This is the basic mechanism for the production of the electromagnetic waves (low energy photons) through the oscillators. This clearly shows that the electromagnetic wave can be produced only when there are, at least, two coils where one coil should produce the change of the magnetic field which can affect on electrons in another coil.

Appendix D

Dirac Equations

The fundamental equation for fermions is the Dirac equation, and it can describe the energy spectrum of the hydrogen atom to a very high accuracy.

D.1 Dirac Equations

The Dirac equation can naturally describe the spin part of the wave function and this is essentially connected to the relativistic wave equations. In addition to the spin degree of freedom, the Dirac equation contains the negative energy states which are quite new to the non-relativistic wave equations. The existence of the negative energy states requires the Pauli principle which enables us to build the vacuum state, and it should be defined as the state in which all the negative energy states are filled. In this case, this vacuum state becomes stable since no particle can be decayed into the vacuum state due to the Pauli principle.

It should be noted that the Pauli principle can be derived if we ask the quantization of the Dirac field in terms of the anti-commutation relations. In this respect, the quantization of the Dirac field is essential because of the Pauli principle, and the field quantization with anti-commutation relations is absolutely necessary within the theoretical framework.

D.1.1 Free Field Solutions

The Dirac equation for free fermion with its mass m is written as

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta\right)\psi(\boldsymbol{r},t) = 0 \tag{D.1}$$

where ψ has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{D.2}$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{D.3}$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix. The derivation of the Dirac equation and its application to hydrogen atom can be found in the standard textbooks. One can learn from the procedure of deriving the Dirac equation that the number of components of the electron fields is important, and it is properly obtained in the Dirac equation. That is, among the four components of the field ψ, two degrees of freedom should correspond to the positive and negative energy solutions and another two degrees should correspond to the spin with $s = \frac{1}{2}$.

Eq.(D.1) can be rewritten in terms of the wave function components by multiplying β from the left side

$$(i\partial_\mu \gamma^\mu - m)_{ij}\psi_j = 0, \quad \text{for} \quad i = 1, 2, 3, 4 \tag{D.4}$$

where the repeated indices of j indicate the summation of $j = 1, 2, 3, 4$. Here, γ matrices

$$\gamma^\mu = (\gamma_0, \boldsymbol{\gamma}) \equiv (\beta, \beta\boldsymbol{\alpha}) \tag{D.5}$$

are introduced, and the repeated indices of Greek letters μ indicate the summation of $\mu = 0, 1, 2, 3$. The expression of eq.(D.4) is called *covariant* since its Lorentz invariance is manifest. It is indeed written in terms of the Lorentz scalars, but, of course there is no deep physical meaning in covariance.

• Lagrangian Density for Free Dirac Fields

The Lagrangian density for free Dirac fermions can be constructed as

$$\mathcal{L} = \psi_i^\dagger [\gamma_0 (i\partial_\mu \gamma^\mu - m)]_{ij}\psi_j = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi \tag{D.6}$$

where $\bar{\psi}$ is defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0. \tag{D.7}$$

This Lagrangian density is just constructed so as to reproduce the Dirac equation of (D.4) from the Lagrange equation. It should be important to realize that the Lagrangian density of eq.(D.6) is invariant under the Lorentz transformation since it is a Lorentz scalar. This is clear since the Lagrangian density should not depend on the system one chooses.

• Lagrange Equation for Free Dirac Fields

The Lagrange equation for $\psi_i^\dagger$ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} \tag{D.8}$$

and we can easily calculate the following equations

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} = 0, \qquad \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = 0 \tag{D.9}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} = [\gamma_0(i\partial_\mu\gamma^\mu - m)]_{ij}\psi_j \tag{D.10}$$

and thus, this leads to the following equation

$$[\gamma_0(i\partial_\mu\gamma^\mu - m)]_{ij}\psi_j = 0 \tag{D.11}$$

which is just the free Dirac equation.

• Plane Wave Solutions of Free Dirac Equation

The free Dirac equation of eq.(D.11) can be solved exactly, and it has plane wave solutions. A simple way to solve eq.(D.11) can be shown as follows. First, one writes the wave function ψ in the following shape

$$\psi_s(\boldsymbol{r}, t) = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{D.12}$$

where φ and ϕ are two component spinors

$$\varphi = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \qquad \phi = \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}. \tag{D.13}$$

In this case, eq.(D.11) becomes

$$\begin{pmatrix} m - E & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m - E \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = 0 \tag{D.14}$$

which leads to

$$E^2 = m^2 + \boldsymbol{p}^2. \tag{D.15}$$

This equation has the following two solutions.

• **Positive Energy Solution** ($E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(+)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} u_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{D.16}$$

$$u_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_{\boldsymbol{p}}+m}\chi_s \end{pmatrix}, \quad \text{with} \quad s = \pm\frac{1}{2} \tag{D.17}$$

where χ_s denotes the spin wave function and is written as

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{D.18}$$

• **Negative Energy Solution** ($E_{\boldsymbol{p}} = -\sqrt{\boldsymbol{p}^2 + m^2}$)

In this case, the wave function becomes

$$\psi_s^{(-)}(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} v_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{D.19}$$

$$v_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{|E_{\boldsymbol{p}}| + m}{2|E_{\boldsymbol{p}}|}} \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E_{\boldsymbol{p}}|+m}\chi_s \\ \chi_s \end{pmatrix}. \tag{D.20}$$

• **Some Properties of Spinor**

The spinor wave function $u_{\boldsymbol{p}}^{(s)}$ and $v_{\boldsymbol{p}}^{(s)}$ are normalized according to

$$u_{\boldsymbol{p}}^{(s)\dagger} u_{\boldsymbol{p}}^{(s)} = 1 \tag{D.21}$$

$$v_{\boldsymbol{p}}^{(s)\dagger} v_{\boldsymbol{p}}^{(s)} = 1. \tag{D.22}$$

Further, they satisfy the following equations when the spin is summed over

$$\sum_{s=1}^{2} u_{\boldsymbol{p}}^{(s)} \bar{u}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}} \tag{D.23}$$

$$\sum_{s=1}^{2} v_{\boldsymbol{p}}^{(s)} \bar{v}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}. \tag{D.24}$$

D.1.2 Quantization of Dirac Fields

Here, we discuss the quantization of free Dirac fields with its solutions

$$\psi(\boldsymbol{r},t)=\sum_{\boldsymbol{n},s}\frac{1}{\sqrt{L^3}}\left(a_{\boldsymbol{n}}^{(s)}u_{\boldsymbol{n}}^{(s)}e^{i\boldsymbol{p}_{\boldsymbol{n}}\cdot\boldsymbol{r}-iE_{\boldsymbol{n}}t}+b_{\boldsymbol{n}}^{(s)}v_{\boldsymbol{n}}^{(s)}e^{i\boldsymbol{p}_{\boldsymbol{n}}\cdot\boldsymbol{r}+iE_{\boldsymbol{n}}t}\right), \tag{D.25}$$

where $u_{\boldsymbol{n}}^{(s)}$ and $v_{\boldsymbol{n}}^{(s)}$ denote the spinor part of the plane wave solutions. Here, the basic method to quantize the fields is to require that the annihilation and creation operators $a_{\boldsymbol{n}}^{(s)}$ and $a^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for positive energy states and $b_{\boldsymbol{n}}^{(s)}$ and $b^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for negative energy states become operators which should satisfy the anti-commutation relations.

- **Anti-commutation Relations**

The creation and annihilation operators for positive and negative energy states should satisfy the following anti-commutation relations,

$$\{a_{\boldsymbol{n}}^{(s)},a^{\dagger}{}_{\boldsymbol{n}'}^{(s')}\}=\delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'},\quad \{b_{\boldsymbol{n}}^{(s)},b^{\dagger}{}_{\boldsymbol{n}'}^{(s')}\}=\delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}. \tag{D.26}$$

All the other cases of the anti-commutations vanish, for examples,

$$\{a_{\boldsymbol{n}}^{(s)},a_{\boldsymbol{n}'}^{(s')}\}=0,\quad \{b_{\boldsymbol{n}}^{(s)},b_{\boldsymbol{n}'}^{(s')}\}=0,\quad \{a_{\boldsymbol{n}}^{(s)},b_{\boldsymbol{n}'}^{(s')}\}=0. \tag{D.27}$$

D.1.3 Quantization in Box with Periodic Boundary Conditions

In field theory, we should put the theory into the box with its volume $V=L^3$ and require that the wave function must satisfy the periodic boundary conditions (PBC). This is mainly because the free field solutions are taken as the basis states, and in this case, we can only calculate physical observables if we work in the box. Since the free wave function $\psi_s(\boldsymbol{r},t)$ in the box should be proportional to

$$\psi_s(\boldsymbol{r},t)\simeq\begin{pmatrix}\varphi\\ \phi\end{pmatrix}\frac{1}{\sqrt{V}}e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{D.28}$$

the PBC equations become

$$e^{ip_xx}=e^{ip_x(x+L)},\qquad e^{ip_yy}=e^{ip_y(y+L)},\qquad e^{ip_zz}=e^{ip_z(z+L)}. \tag{D.29}$$

Therefore, one obtains the constraints on the momentum p_k as

$$p_x=\frac{2\pi}{L}n_x,\quad p_y=\frac{2\pi}{L}n_y,\quad p_z=\frac{2\pi}{L}n_z,\qquad n_k=0,\pm1,\pm2,\cdots. \tag{D.30}$$

In this case, the number of states N in the large L limit becomes

$$N=\sum_{n_x,n_y,n_z}\sum_s=2\frac{L^3}{(2\pi)^3}\int d^3p \tag{D.31}$$

where a factor of two comes from the spin degree of freedom.

D.1.4 Hamiltonian Density for Free Dirac Fermion

The Hamiltonian density for free fermion can be constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \partial^\nu \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \partial^\nu \psi_i^\dagger \right) - \mathcal{L} g^{\mu\nu}. \tag{D.32}$$

- **Hamiltonian Density from Energy Momentum Tensor**

Now, one defines the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i)} \partial_0 \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i^\dagger)} \partial_0 \psi_i^\dagger \right) - \mathcal{L}. \tag{D.33}$$

Since the Lagrangian density of free fermion is given in eq.(D.6) and is rewritten as

$$\mathcal{L} = i\psi_i^\dagger \partial_0 \psi_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j. \tag{D.34}$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = \mathcal{T}^{00} = \bar{\psi}_i \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \tag{D.35}$$

- **Hamiltonian for Free Dirac Fermion**

The Hamiltonian for free fermion fields is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H} d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi d^3r. \tag{D.36}$$

As we discussed in the Schrödinger field, the Hamiltonian itself cannot give us many information on the dynamics. One can learn some properties of the system described by the Hamiltonian, but one cannot obtain any dynamical information of the system from the Hamiltonian. In order to calculate the dynamics of the system in the classical field theory model, one has to solve the equation of motions which are obtained from the Lagrange equations for fields.

When one wishes to consider the quantum effects of the fields or, in other words, creations of particles and anti-particles, then one should quantize the fields. In this case, the Hamiltonian becomes an operator. Therefore, one has to prepare the Fock states on which the Hamiltonian can operate. Most of the difficulties of the field theory models should be to find the correct vacuum of the interacting system. In four dimensional field theory models, only the free field theory can be solved exactly, and therefore we are all based on the perturbation theory to obtain physical observables.

D.1.5 Fermion Current and its Conservation Law

The Dirac equation has a very important equation of current conservation. This is, in fact, related to the global gauge symmetry which should be always satisfied in Dirac as well as Schrödinger equations. If the Lagrangian density should have the following shape

$$\mathcal{L} = F(\psi^\dagger \psi) \tag{D.37}$$

then it is invariant under the global gauge transformation of

$$\psi' = e^{i\alpha}\psi. \tag{D.38}$$

In this case, if one defines the Noether current j^μ as

$$j^\mu \equiv -i\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\psi^\dagger\right] \tag{D.39}$$

then one has the conservation of current

$$\partial_\mu j^\mu = 0. \tag{D.40}$$

For Dirac fields, one can obtain as a conserved current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{D.41}$$

while the conserved current $j^\mu = (\rho, \boldsymbol{j})$ for the Schrödinger field is written as

$$\rho = \psi^\dagger\psi, \qquad \boldsymbol{j} = \frac{1}{2im}\left(\psi^\dagger \boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^\dagger)\psi\right) \tag{D.42}$$

D.1.6 Dirac Equation for Coulomb Potential

For a hydrogen-like atomic system, one can write the Dirac equation as

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta + \frac{Ze^2}{r}\right)\psi(\boldsymbol{r}, t) = 0 \tag{D.43}$$

where ψ has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{D.44}$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{D.45}$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix. In this case, one can easily prove that the quantities that can commute with the Dirac Hamiltonian must be $\boldsymbol{J}$ and K as defined below

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{s}, \qquad K = \beta(2\boldsymbol{s} \cdot \boldsymbol{L} + 1) \tag{D.46}$$

where $\boldsymbol{L}$ and $\boldsymbol{s}$ are defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \hat{\boldsymbol{p}}, \qquad \boldsymbol{s} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma} \end{pmatrix}. \tag{D.47}$$

Therefore, the energy eigenvalue of the Dirac field can be specified by the quantum numbers of $J,\ J_z,\ K$.

• Energy Eigenvalue with Coulomb in Hydrogen-like Atom

The energy eigenvalue of the Dirac equation can be obtained for the hydrogen-like atomic system. The Dirac equation can be written as

$$\left(-i\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + m_e\beta - \frac{Ze^2}{r}\right)\psi(\boldsymbol{r}, t) = E\psi(\boldsymbol{r}, t) \tag{D.48}$$

where m_e denotes the electron mass. This can be solved exactly, and the energy eigenvalue is given as

$$E_{n,j} = m_e\left[1 - \frac{(Z\alpha)^2}{n^2 + 2\left(n - (j + \frac{1}{2})\right)\left[\sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2} - (j + \frac{1}{2})\right]}\right]^{\frac{1}{2}} \tag{D.49}$$

where α denotes the fine structure constant with $\alpha = \frac{1}{137}$. The quantum number n runs as $n = 1, 2, \ldots$. The energy $E_{n,j}$ can be expanded up to the order α^4 as

$$E_{n,j} - m_e = -\frac{m_e(Z\alpha)^2}{2n^2} - \frac{m_e(Z\alpha)^4}{2n^4}\left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\right) + \mathcal{O}\left((Z\alpha)^6\right). \tag{D.50}$$

The first term in the energy eigenvalue is the familiar energy spectrum of the hydrogen-like atom in the non-relativistic quantum mechanics.

It should be noted that this result is mathematically exact, but the Dirac equation eq.(D.43) itself is simply obtained within one body problem, and it is, of course, an approximation. A question may arise as to how much the reduction of the one body problem can be justified. Namely, the hydrogen atom should be, at least, a two body problem since it involves electron and proton in the hydrogen atom. In fact, the relativistic two body Dirac equation cannot be solved or cannot be reduced to one body problem in a proper manner. The Dirac equation of eq.(D.43) is indeed one body equation, but the mass m should be replaced by the reduced mass, and this is indeed a very artificial procedure. In reality, it may well be even more complicated than the tow body problems, and once the fields are quantized, then the hydrogen atom should become many body problems. This means that one electron state could be mixed up by the two electron-one positron states in the electron wave function after the field quantization. At present, however, there is no reliable calculation with this additional configuration, and therefore we do not know how large these contributions to the energy should be for the hydrogen atom.

D.1.7 Dirac Equation for Coulomb and Gravity Potential

Even when one considers the hydrogen-like atom, there is a gravitational interaction between electron and proton. Here, we write a full Dirac equation in the hydrogen-like atom when the gravitational interaction is included

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \left(m_e - \frac{Gm_eM_pZ}{r}\right)\beta - \frac{Ze^2}{r}\right]\Psi = E\Psi \tag{D.51}$$

where M_p and G denote the proton mass and the gravitational constant, respectively. The gravity is too weak to make any influence on the spectrum in the hydrogen-like atom, but theoretically it should be important that all the interactions in the hydrogen-like atom are now included in the Dirac equation. This equation can be solved exactly, and the energy eigenvalue is found in the textbook [5].

• Classical Limits

As we see in the later chapter, the gravitational force becomes important when we discuss the motion of the planets in the Newton equation. When we make the non-relativistic reduction of the Dirac Hamiltonian, then we find

$$H = \frac{\boldsymbol{p}^2}{2m_e} - \frac{e^2}{r} - \frac{Gm_eM_p}{r} + \frac{GM_p}{2m_er}\boldsymbol{p}^2. \tag{D.52}$$

Now, we make the classical limit of the Hamiltonian and obtain a new potential for the Newton equation with an additional gravitational potential

$$V(r) = -\frac{e^2}{r} - \frac{Gm_eM_p}{r} + \frac{1}{2m_ec^2}\left(\frac{Gm_eM_p}{r}\right)^2. \tag{D.53}$$

If the new potential is applied to the motion of the planets, then this additional gravitational potential turns out to be responsible for the description of the observed period shifts of the earth revolution around the sun. This is discussed in terms of leap second correction in chapter 4.

Appendix E

Wave Propagations in medium and vacuum

The classical wave such as sound can propagate through medium. However, it cannot propagate in vacuum as is well known. This is, of course, clear since the classical wave is the chain of the oscillations of the medium due to the pressure on the density.

On the other hand, quantum wave including photon can propagate in vacuum since it is a particle. Here, we clarify the difference in propagations between the classical wave and quantum wave. The most important point is that the classical wave should be always written in terms of real functions while photon or quantum wave should be described by the complex wave function of the shape e^{ikx} since it should be an eigenstate of the momentum.

E.1 What is wave?

The sound can propagate through medium such as air or water. The wave can be described in terms of the amplitude ϕ in one dimension

$$\phi(x,t) = A_0 \sin(\omega t - kx) \tag{E.1}$$

where ω and k denote the frequency and wave number, respectively. The dispersion relation of this wave can be written as

$$\omega = vk. \tag{E.2}$$

Here, it is important to note that the amplitude is written as the real function, in contrast to the free wave function of electron in quantum mechanics. In fact, the free wave of electron can be described in one dimension as

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - kx)} \tag{E.3}$$

which is a complex function. The electron can propagate by itself and there is no medium necessary for the electron motion.

What is the difference between the real wave amplitude and the complex wave function? Here, we clarify this point in a simple way though this does not contain any new physics.

E.1.1 A real wave function: Classical wave

If the amplitude is real such as (E.1), then it can only propagate in medium. This can be clearly seen since the energy of the wave can be transported in terms of the density oscillation which is a real as the physical quantity. In addition, the amplitude becomes zero at some point, and this is only possible when it corresponds to the oscillation of the medium. This means that the wave function of (E.1) has nothing to do with the probability of wave object. Instead, if it is the oscillation of the medium, then it is easy to understand why one finds the point where the amplitude vanishes to zero. The real amplitude is called a classical wave since it is indeed seen in the world of the classical physics.

E.1.2 A complex wave function: Quantum wave

On the other hand, the free wave function of electron is a complex function, and there is no point where it can vanish to zero. Since this is just the wave function of electron, its probability of finding the wave is always a constant $\frac{1}{V}$ at any space point of volume V.

E.2 Classical wave

The sound propagates in the air, and its propagation should be transported in terms of density wave. The amplitude of this wave can be written in terms of the real function as given in eq.(E.1). This is quite reasonable since the density wave should be described by the real physical quantity. Instead, this requires the existence of the medium (air), and the wave can propagate as long as the air exists. Here, we first write the basic wave equation in one dimension

$$\frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} \tag{E.4}$$

which is similar to the wave equation in quantum mechanics, though it is a real differential equation. Here, v denotes the speed of wave.

E.2.1 Classical waves carry their energy?

In this case, a question may arise as to what is a physical quantity which is carried by the classical wave like sound. It seems natural that the wave carries its energy (or wave length). In fact, the transportation of the energy should be carried out by the compression of the density and successive oscillations of the medium. Therefore this is called compression wave.

E.2.2 Longitudinal and transverse waves

Here, we discuss the terminology of the longitudinal and transverse waves, even though one should not stress its physics too much since there is no special physical meaning.

• **Longitudinal wave :**
The sound propagates as the compressional wave, and the oscillations should be always in the direction of the wave motion. In this case, it is called longitudinal wave. This wave can be easily understood since one can make a picture of the density wave.

• **Transverse wave :**
On the other hand, if the motion of the oscillations is in the perpendicular to the direction of the wave motion, then it is called transverse wave. The tidal wave may be the transverse wave, but its description may not be very simple since the density change may not directly be related to the wave itself.

E.3 Quantum wave

Photon and quantum wave are quite different from the classical wave, and the quantum wave is a particle motion itself. No medium oscillation is involved. For example, a free electron moves with the velocity v in vacuum, and this motion is also called "wave". The reason why we call it wave is due to the fact that the equation of motion that describes electrons looks similar to the classical wave equation of motion. Further, the solution of the wave equation can be described as e^{ikx}, and thus it is the same as the wave behavior in terms of mathematics. But the physical meaning is completely different from the classical wave, and quantum wave is just the particle motion which behaves as the probabilistic motion.

E.3.1 Quantum wave (electron motion)

The wave function of a free electron in one dimension can be described as

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \tag{E.5}$$

which is a solution of the Schrödinger equation of a free electron,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\boldsymbol{\nabla}^2 \psi \tag{E.6}$$

where $k = \sqrt{2m\omega}$, and V denotes the corresponding volume. Since the Schrödinger equation is quite similar to the wave equation in a classical sense, one calls the solution of the Schrödinger equation as a wave. However, the physics of the quantum wave should be understood in terms of the quantum mechanics, and the relation to the classical wave should not be stressed. That is, the quantum wave is completely different from the classical wave, and one should treat the quantum wave as it is. In addition, the behavior and physics of the classical wave are very complicated and it is clear that we do not fully understand the behavior of the classical wave since it involves many body problems in physics.

E.3.2 Photon

The electromagnetic wave is called photon which behaves like a particle and also like a wave. This photon can propagate in vacuum and thus it should be considered to be a particle. Photon can be described by the vector potential $\boldsymbol{A}$.

• **$\boldsymbol{A}$ is real ! :**
However, this $\boldsymbol{A}$ is obviously a real function, and therefore, it cannot propagate like a particle. This can be easily seen since the free Hamiltonian of photon commutes with the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$, and therefore it can be a simultaneous eigenstate of the Hamiltonian. Thus, the $\boldsymbol{A}$ should be an eigenstate of the momentum operator since the free state must be an eigenstate of momentum. However, any real function cannot be an eigenstate of the momentum operator, and thus the vector field in its present shape cannot describe the free particle state.

• **Free solution of vector field :**
What should we do? The only way of solving this puzzle is to quantize a photon field. First, the solution of $\boldsymbol{A}$ can be written as

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k},\lambda} \frac{1}{\sqrt{2\omega_k V}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \left(c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda} e^{ikx} \right) \tag{E.7}$$

with $kx \equiv \omega_k t - \boldsymbol{k}\cdot\boldsymbol{r}$. Here, $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ denotes the polarization vector which will be discussed later more in detail. As one sees, the vector field is indeed a real function.

• **Quantization of vector field :**
Now we impose the following quantization conditions on $c^\dagger_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}$

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'}, \tag{E.8}$$

$$[c_{\boldsymbol{k},\lambda},\ c_{\boldsymbol{k}',\lambda'}] = 0, \qquad [c^\dagger_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = 0. \tag{E.9}$$

In this case, $c^\dagger_{\boldsymbol{k},\lambda},\ c_{\boldsymbol{k},\lambda}$ become operators. Therefore, one should now consider the Fock space on which they can operate. This can be defined as

$$c_{\boldsymbol{k},\lambda}|0\rangle = 0 \tag{E.10}$$

$$c^\dagger_{\boldsymbol{k},\lambda}|0\rangle = |\boldsymbol{k},\lambda\rangle \tag{E.11}$$

where $|0\rangle$ denotes the vacuum state of the photon field. Therefore, if one operates the vector field on the vacuum state, then one obtains

$$\langle \boldsymbol{k},\lambda|\boldsymbol{A}(x)|0\rangle = \frac{1}{\sqrt{2\omega_k V}}\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}e^{-ikx}. \tag{E.12}$$

As one sees, this new state is indeed the eigenstate of the momentum operator and should correspond to the observables. Therefore, photon can be described only after the vector field is quantized. Thus, photon is a particle whose dispersion relation becomes

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|. \tag{E.13}$$

E.4 Polarization vector of photon

Until recently, there is a serious misunderstanding for the polarization vector $\epsilon^\mu_{\boldsymbol{k},\lambda}$. This is related to the fact that the equation of motion for the polarization vector is not solved, and thus there is one condition missing in the determination of the polarization vector.

E.4.1 Equation of motion for polarization vector

Now the equation of motion for $A^\mu = (A^0, \boldsymbol{A})$ without any source terms can be written from the Lagrange equation as

$$\partial_\mu F^{\mu\nu} = 0 \tag{E.14}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This can be rewritten as

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = 0. \tag{E.15}$$

Now, the shape of the solution of this equation can be given as

$$A^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \epsilon^\mu_{\boldsymbol{k},\lambda} \left[c_{\boldsymbol{k},\lambda} e^{-ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{ikx} \right] \tag{E.16}$$

and thus we insert it into eq.(E.15) and obtain

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{E.17}$$

Now the condition that there should exist non-zero solution of $\epsilon^\mu_{\boldsymbol{k},\lambda}$ is obviously that the determinant of the matrix in the above equation should vanish to zero, namely

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{E.18}$$

This leads to $k^2 = 0$, which means $k_0 \equiv \omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. This is indeed a proper dispersion relation for photon.

E.4.2 Condition from equation of motion

Now we insert the condition of $k^2 = 0$ into eq.(E.17), and obtain

$$k_\mu \epsilon^\mu = 0 \tag{E.19}$$

which is a new constraint equation obtained from the basic equation of motion. Therefore, this condition (we call it "Lorentz condition") is most fundamental. It should be noted that the Lorentz gauge fixing is just the same as eq.(E.19). This means that the Lorentz gauge fixing is improper and forbidden for the case of no source term. In this sense, the best gauge fixing should be the Coulomb gauge fixing

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0 \tag{E.20}$$

from which one finds $\epsilon_0 = 0$, and this is indeed consistent with experiment.

• Number of freedom of polarization vector :
Now we can understand the number of degree of freedom of the polarization vector. The Lorentz condition $k_\mu \epsilon^\mu = 0$ should give one constraint on the polarization vector, and the Coulomb gauge fixing $\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0$ gives another constraint. Therefore, the polarization vector has only two degrees of freedom, which is indeed an experimental fact.

• State vector of photon :
The state vector of photon is already discussed. But here we should rewrite it again. This is written as

$$\langle \boldsymbol{k}, \lambda | \boldsymbol{A}(x) | 0 \rangle = \frac{\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}}{\sqrt{2\omega_k V}} e^{-ikx}. \tag{E.21}$$

In this case, the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ has two components, and satisfies the following conditions

$$\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda'} = \delta_{\lambda,\lambda'}, \qquad \boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0. \tag{E.22}$$

E.4.3 Photon is a transverse wave?

People often use the terminology of transverse photon. Is it a correct expression? By now, one can understand that the quantum wave is a particle motion, and thus it has nothing to do with the oscillation of the medium. Therefore, it is meaningless to claim that photon is a transverse wave. The reason of this terminology may well come from the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ which is orthogonal to the direction of photon momentum. However, as one can see, the polarization vector is an intrinsic property of photon, and it does not depend on space coordinates.

• **No rest frame of photon ! :**
In addition, there is no rest frame of photon, and therefore, one cannot discuss its intrinsic property unless one fixes the frame. Even if one says that the polarization vector is orthogonal to the direction of the photon momentum, one has to be careful in which frame one discusses this property.

In this respect, it should be difficult to claim that photon behaves like a transverse wave. Therefore, one sees that photon should be described as a massless particle which has two degrees of freedom with the behavior of a boson. There is no correspondence between classical waves and photon, and even more, there is no necessity of making analogy of photon with the classical waves.

E.5 Poynting vector and radiation

We have clarified that the propagation of the real function requires some medium which can make oscillations. Here, we discuss the Poynting vector how it appears in physics, and show that it cannot propagate in vacuum at all. Also, we present a brief description of the basic radiation mechanism how photon can be emitted.

E.5.1 Field energy and radiation of photon

Before discussing the propagation of the Poynting vector, we should first discuss the mechanism of the radiation of photon in terms of classical electrodynamics. The interaction Hamiltonian can be written as

$$H_I = -\int \boldsymbol{j} \cdot \boldsymbol{A} \, d^3r \tag{E.23}$$

which should be a starting point of all the discussions. Now, we make a time derivative of the interaction Hamiltonian and obtain

$$W \equiv \frac{dH_I}{dt} = -\int \left[\frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} + \boldsymbol{j} \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right] d^3r. \tag{E.24}$$

Since we can safely set $A^0 = 0$ in this treatment, we find

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}. \tag{E.25}$$

Therefore, we can rewrite eq.(E.24) as

$$W = \int \boldsymbol{j} \cdot \boldsymbol{E} \, d^3r - \int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} \, d^3r. \tag{E.26}$$

Defining the first term of eq.(E.24) as W_E, we can rewrite W_E as

$$W_E \equiv \int \boldsymbol{j} \cdot \boldsymbol{E} \, d^3r = -\frac{d}{dt} \left[\int \left(\frac{1}{2\mu_0} |\boldsymbol{B}|^2 + \frac{\varepsilon_0}{2} |\boldsymbol{E}|^2 \right) d^3r \right] - \int \boldsymbol{\nabla} \cdot \boldsymbol{S} \, d^3r \tag{E.27}$$

which is just the energy of electromagnetic fields.

E.5.2 Poynting vector

Here, the last term of eq.(E.27) is Poynting vector $\boldsymbol{S}$ as defined by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} \tag{E.28}$$

which is connected to the energy flow of the electromagnetic field. This Poynting vector is a conserved quantity, and thus it has nothing to do with the electromagnetic wave. In addition, it is a real quantity, and thus there is no way that it can propagate in vacuum. In addition, the Poynting vector cannot be a target of the field quantization, and thus it always remains classical since it is written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. However, there is still some misunderstanding in some of the textbooks on Electromagnetism, and therefore, one should be careful for the treatment of the Poynting vector.

• **Exercise problem:**
Here, we present a simple exercise problem of circuit with condenser with C (disk radius of a and distance of d) and resistance with R. The electric potential difference V is set on the circuit. In this case, the equation for the circuit can be written as

$$V = R\frac{dQ}{dt} + \frac{Q}{C}.$$

This can be easily solved with the initial condition of $Q = 0$ at $t = 0$, and the solution becomes

$$Q = CV\left(1 - e^{-\frac{t}{RC}}\right).$$

Therefore, the electric current J becomes

$$J = \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}.$$

In this case, we find the electric field $\boldsymbol{E}$ and the displacement current $\boldsymbol{j}_d$

$$\boldsymbol{E} = \frac{Q}{\pi a^2}\boldsymbol{e}_z = \frac{VC}{\varepsilon_0 \pi a^2}\left(1 - e^{-\frac{t}{RC}}\right)\boldsymbol{e}_z \tag{E.29}$$

$$\boldsymbol{j}_d = \frac{\partial \boldsymbol{E}}{\partial t} = \frac{V}{R\pi a^2}e^{-\frac{t}{RC}}\boldsymbol{e}_z. \tag{E.30}$$

Thus, the magnetic field $\boldsymbol{B}$ becomes

$$\boldsymbol{B} = \frac{i_d\, r}{2}\boldsymbol{e}_\theta = \frac{r}{2\pi a^2 R}\ e^{-\frac{t}{RC}}\ \boldsymbol{e}_\theta$$

where $\int_C \boldsymbol{B}\cdot d\boldsymbol{r} = \mu_0 i_d \pi r^2$ is used. Therefore, the Poynting vector at the surface (with $r = a$) of the cylindrical space of the disk condenser becomes

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{B} = -\frac{V^2}{2\pi a R d}\ e^{-\frac{t}{RC}}\left(1 - e^{-\frac{t}{RC}}\right)\boldsymbol{e}_r.$$

It should be noted that the energy in the Poynting vector is always flowing into the cylindrical space. Therefore, the electric field energy is now accumlated in the cylindrical space. There is, of course, no electromagnetic wave radiation, and in fact, the Poynting vector is the flow of field energy, and has nothing to do with the electromagnetic wave.

E.5.3 Emission of photon

The emission of photon should come from the second term of eq.(E.26) which can be defined as W_R and thus

$$W_R = -\int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} \, d^3r. \tag{E.31}$$

In this case, we can calculate the $\frac{\partial \boldsymbol{j}}{\partial t}$ term by employing the Zeeman effect Hamiltonian with a uniform magnetic field of $\boldsymbol{B}_0$

$$H_Z = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0. \tag{E.32}$$

The relevant Schrödinger equation for electron with its mass m_e becomes

$$i\frac{\partial \psi}{\partial t} = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0 \, \psi. \tag{E.33}$$

Therefore, we find

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e}{m_e}\left[\frac{\partial \psi^\dagger}{\partial t}\hat{\boldsymbol{p}}\psi + \psi^\dagger \hat{\boldsymbol{p}}\frac{\partial \psi}{\partial t}\right] = -\frac{e^2}{2m_e^2}\boldsymbol{\nabla} B_0(\boldsymbol{r}). \tag{E.34}$$

In order to obtain the photon emission, one should quantize the field $\boldsymbol{A}$ in eq.(E.31).

• Field quantization :

The field quantization in electromagnetic interactions can be done only for the vector potential $\boldsymbol{A}$. The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are classical quantities which are defined before the field quantization.

Appendix F

Lorentz Conditions

Here, we clarify that the Lorentz condition of $k_\mu \epsilon^\mu = 0$ should be obtained from the equation of motion, and therefore it is more fundamental than the requirement of the gauge fixing condition in QED. For the massive vector bosons, the Lorentz condition plays a fundamental role for determining the polarization sum of the vector boson.

F.1 Gauge Field of Photon

We write the Lagrangian density for the free gauge field as

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{F.1}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In this case, the equation of motion becomes

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \tag{F.2}$$

Since the free photon field should have the following solution

$$A^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{\epsilon^\mu(k,\lambda)}{\sqrt{2V\omega_{\boldsymbol{k}}}} \left[c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda} e^{ikx} \right] \tag{F.3}$$

we can insert this solution into eq.(F.2) and obtain the following equation for $\epsilon^\mu(k,\lambda)$

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{F.4}$$

This equation can be written in terms of the matrix equation for the polarization vector ϵ^μ

$$\sum_{\nu=0}^{3} \{ k^2 g^{\mu\nu} - k^\mu k^\nu \} \epsilon_\nu = 0 \tag{F.5}$$

where we write the summation explicitly. In order that the ϵ^μ should have a non-zero solution, the determinant of the matrix should vanish to zero

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{F.6}$$

Now it is easy to prove that $k^2 = 0$ is the only physical solution of eq.(F.6) since we find

$$\det\{-k^\mu k^\nu\} = 0. \tag{F.7}$$

Therefore, putting the solution of $k^2 = 0$ into eq.(F.5), we obtain

$$k_\mu \epsilon^\mu = 0 \tag{F.8}$$

which becomes the constraint equation for the polarization vector. Here, we should note that this process of determining the condition on the wave function of ϵ^μ is just the same as solving the free Dirac equation. Obviously this is the most important process of determining the wave functions in quantum mechanics, and surprisingly, this has been missing in the treatment of determining not only the massive vector boson propagator but also the photon propagator as well.

This constraint equation of eq.(F.8) is obtained from the equation of motion, even though it is just the same equation as Lorentz gauge fixing condition. As can be seen by now, the gauge fixing condition is still left for use. In fact, if we take the Coulomb gauge fixing of $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, then we find $\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0$ which leads to the condition of $\epsilon_0 = 0$. Therefore, we now see that the photon field has only two degrees of freedom which can be naturally obtained from the equation of motion and the gauge fixing condition.

In addition, we realize that the Lorentz gauge fixing is not allowed in the free field gauge theory since the same equation of the Lorentz gauge fixing is already obtained from the equation of motion. Namely, it cannot give a further constraint on the polarization vector. In this respect, we see that the Coulomb gauge fixing gives a proper condition on the polarization vector.

F.2 Massive Vector Fields

The massive vector field can be treated just in the same manner as above. We first write the free Lagrangian density for the vector boson field Z^μ with its mass M

$$\mathcal{L}_W = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 Z_\mu Z^\mu \tag{F.9}$$

with $G^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$. In this case, the equation of motion becomes

$$\partial_\mu(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + M^2 Z^\nu = 0. \tag{F.10}$$

Since the free massive boson field should have the following shape of the solution

$$Z^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{3} \frac{\epsilon^\mu(k,\lambda)}{\sqrt{2V\omega_{\boldsymbol{k}}}} \left[c_{\boldsymbol{k},\lambda} e^{ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} \right] \tag{F.11}$$

we can insert this solution into eq.(F.10) and obtain the following equation for the polarization vector ϵ^μ

$$(k^2 - M^2)\epsilon^\mu - (k_\nu \epsilon^\nu)k^\mu = 0. \tag{F.12}$$

In the same way as above, we can prove that

$$k^2 - M^2 = 0 \tag{F.13}$$

should hold, and this is the only physical solution of eq.(F.12). Therefore we obtain the following equation for the polarization vector ϵ^μ

$$k_\mu \epsilon^\mu = 0 \tag{F.14}$$

which should always hold. This is just the same equation as Lorentz gauge fixing condition in QED. However, there is no gauge freedom for the massive vector boson, and therefore the degrees of freedom of the polarization vector ϵ^μ for the massive vector boson is three, in contrast to the gauge field. Now, we understand that the massive vector field should have a spin of $s = 1$ which has indeed three components as we saw above. In this sense, the photon field is special in that it has a spin of $s = 1$ with only two degrees of freedom. This should be directly related to the massless nature of photon which is required from the gauge invariance of the Lagrangian density of the vector field.

Appendix G

New Derivation of Dirac Equation

Here, we should present a novel method to derive the Dirac equation without making use of the first quantization. It is shown that, from the local gauge invariance and the Maxwell equation, we can derive the Lagrangian density of the Dirac field without involving the first quantization.

G.1 Derivation of Lagrangian Density of Dirac Field

Dirac derived the Dirac equation by factorizing the dispersion relation of energy and momentum such that the field equation becomes the first order in time derivative. Now, we can derive the Lagrangian density of the Dirac field in an alternative way by making use of the local gauge invariance and the Maxwell equation as the most fundamental principle.

G.1.1 Lagrangian Density for Maxwell Equation

We start from the Lagrangian density of the Maxwell equation

$$\mathcal{L} = -gj_\mu A^\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{G.1}$$

where A^μ is the gauge field, and $F_{\mu\nu}$ is the field strength and is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{G.2}$$

Here j_μ denotes the current density of matter field which couples to the electromagnetic field. From the Lagrange equation, we obtain

$$\partial_\mu F^{\mu\nu} = gj^\nu \tag{G.3}$$

which is just the Maxwell equation.

G.1.2 Four Component Spinor

Now, we can derive the kinetic energy term of the fermion Lagrangian density. First, we assume that the Dirac fermion should have four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{G.4}$$

This is based on the observation that electron has spin degree of freedom which is two. In addition, there must be positive and negative energy states since it is a relativistic field, and therefore the fermion field should have 4 components.

• 16 Independent Components

Now, the matrix elements

$$\psi^\dagger \hat{O} \psi \tag{G.5}$$

can be classified into 16 independent Lorentz invariant components as

$$\bullet \begin{cases} \bar{\psi}\psi : & \text{scalar,} \\ \bar{\psi}\gamma_5\psi : & \text{pseudo} - \text{scalar,} \\ \bar{\psi}\gamma_\mu\psi : & \text{4 component vector,} \\ \bar{\psi}\gamma_\mu\gamma_5\psi : & \text{4 component axial} - \text{vector,} \\ \bar{\psi}\sigma_{\mu\nu}\psi : & \text{6 component tensor,} \end{cases} \tag{G.6}$$

where $\bar{\psi}$ is defined for convenience as

$$\bar{\psi} = \psi^\dagger \gamma_0. \tag{G.7}$$

These properties are determined by mathematics.

• Shape of Vector Current

From the invariance consideration, the vector current j_μ must be written as

$$j_\mu = C_0 \bar{\psi}\gamma_\mu\psi \tag{G.8}$$

where C_0 is a constant. Since we can renormalize the constant C_0 into the coupling constant g, we can set without loss of generality

$$C_0 = 1. \tag{G.9}$$

G.2 Shape of Lagrangian Density

By making use of the local gauge invariance of the Lagrangian density, we see that the following shape of the Lagrangian density can keep the local gauge invariance

$$\mathcal{L} = C_1 \bar{\psi} \partial_\mu \gamma^\mu \psi - g\bar{\psi}\gamma_\mu \psi A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{G.10}$$

where C_1 is a constant. At this point, we require that the Lagrangian density should be invariant under the local gauge transformation

$$\bullet \begin{cases} A_\mu \longrightarrow A_\mu + \partial_\mu \chi, \\ \psi \longrightarrow e^{-ig\chi}\psi \end{cases} \tag{G.11}$$

where χ should be an arbitrary function of space and time. In this case, it is easy to find that the constant C_1 must be

$$C_1 = i. \tag{G.12}$$

Here, the constant $\hbar$ should be included implicitly into the constant C_1. The determination of $\hbar$ can be done only when we compare calculated results with experiment such as the spectrum of hydrogen atom.

G.2.1 Mass Term

The Lagrangian density of eq.(G.10) still lacks the mass term. Since the mass term must be a Lorentz scalar, it should be described as

$$C_2 \bar{\psi}\psi \tag{G.13}$$

which is, of course, gauge invariant as well. This constant C_2 should be determined again by comparing the calculated results of hydrogen atom, for example, with experiment. By denoting C_2 as $(-m)$, we arrive at the Lagrangian density of a relativistic fermion which couples with the electromagnetic fields A^μ

$$\mathcal{L} = i\bar{\psi} \partial_\mu \gamma^\mu \psi - g\bar{\psi}\gamma_\mu \psi A^\mu - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{G.14}$$

which is just the Lagrangian density for the Dirac field interacting with electromagnetic fields.

G.2.2 First Quantization

It is important to note that, in the procedure of deriving the Lagrangian density of eq.(G.14), we have not made use of the quantization condition of

$$E \to i\frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i\boldsymbol{\nabla}. \tag{G.15}$$

Instead, the first quantization is automatically done by the gauge condition since the Maxwell equation knows the first quantization in advance. This indicates that there may be some chance to understand the first quantization procedure in depth since this method gives an alternative way of the quantization condition of the energy and momentum.

G.3 Two Component Spinor

The derivation of the Dirac equation in terms of the local gauge invariance shows that the current density that can couple to the gauge field A^μ must be rather limited. Here, we discuss a possibility of finding field equation for the two component spinor. When the field has only two components,

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{G.16}$$

then we can prove that we cannot make the current j_μ that couples with the gauge field A_μ. This can be easily seen since the matrix elements

$$\phi^\dagger \hat{O} \phi \tag{G.17}$$

can be classified into 4 independent variables as

$$\phi^\dagger \phi : \quad \text{scalar}, \quad \phi^\dagger \sigma_k \phi : \quad 3 \text{component} \text{vector}. \tag{G.18}$$

Therefore, there is no chance to make four vector currents which may couple to the gauge field A_μ. This way of making the Lagrangian density indicates that it should be difficult to find a Lagrangian density of relativistic bosons.

Appendix H

Planet Effects on Mercury Perihelion

In this Appendix, we discuss the Mercury perihelion shifts which should come from the gravitational interactions between Mercury and other planets such as Jupiter or Saturn. This calculation can be carried out in the perturbation theory of the Newton dynamics, which is rather new to the classical mechanics. Here, we should comare the numerical results with those calculated by Newcomb in 1898.

H.1 Planet Effects on Mercury Perihelion

The motion of the other planets should affect on the Mercury orbits. However, this is the three body problems, and thus it is not easy to solve the equation of motion in an exact fashion. Here, we develop the perturbative treatment of the other planet motions. Suppose the Mercury and the other planet are orbiting around the sun and in this case, the Lagrangian can be written as

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{GmM}{r} + \frac{1}{2}m_w\dot{\boldsymbol{r}_w}^2 + \frac{Gm_wM}{r_w} + \frac{Gmm_w}{|\boldsymbol{r}-\boldsymbol{r}_w|} \tag{H.1}$$

where $(m,\ \boldsymbol{r})$ and $(m_w,\ \boldsymbol{r}_w)$ denote the mass and coordinate of the Mercury and planet, respectively. The last term in the right side of eq.(H.1) is the gravitational potential between the Mercury and the planet, and therefore it should be much smaller than the gravitational force from the sun.

H.1.1 The Same Plane of Planet Motions

Here, we assume that the motion of the Mercury and the planet must be in the same plane and therefore we rewrite the Lagrangian in terms of polar coordinates

$$\begin{aligned} L &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GmM}{r} + \frac{1}{2}m_w(\dot{r_w}^2 + r_w^2\dot{\varphi_w}^2) + \frac{Gm_wM}{r_w} \\ &+ \frac{Gmm_w}{\sqrt{r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w)}}. \end{aligned} \tag{H.2}$$

Therefore, the Lagrange equation for the Mercury can be written as

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2} - \frac{Gmm_w(r - r_w\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{H.3}$$

$$\frac{d}{dt}(mr^2\dot{\varphi}) = -\frac{GmMrr_w\sin(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{H.4}$$

$$m_w\ddot{r_w} = m_wr_w\dot{\varphi}^2 - \frac{Gm_wM}{r_w^2} - \frac{Gmm_w(r_w - r\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}} \tag{H.5}$$

$$\frac{d}{dt}(m_wr_w^2\dot{\varphi}) = -\frac{Gm_wMrr_w\sin(\varphi_w - \varphi))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{H.6}$$

H.1.2 The Motion of Mercury

If we ignore the interaction between the Mercury and other planets, then the Mercury orbit is just given as the Kepler problem, and the equations of motion become

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{GmM}{r^2} \tag{H.7}$$

$$\frac{d}{dt}(mr^2\dot{\varphi}) = 0. \tag{H.8}$$

Here, the solution of the orbit trajectory is given as

$$r = \frac{A}{1 + \varepsilon\cos\varphi} \tag{H.9}$$

where A and ε are written as

$$A = \frac{\ell^2}{m\alpha}, \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}} \quad \text{with} \quad \alpha = GMm \tag{H.10}$$

which should be taken as the unperturbed solution of the revolution orbit.

H.2 Approximate Estimation of Planet Effects

Now we should make a perturbative calculation of the many body Kepler problem by assuming that the interaction between the Mercury and other planets is sufficiently small. In this case, we can estimate the effects of other planets on the Mercury orbit. Here we write again the equation of motion for the Mercury including the gravity from the other planets

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(r - r_w\cos(\varphi - \varphi_w))}{(r^2 + r_w^2 - 2rr_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{H.11}$$

Now we replace $r,\ r_w$ by the average orbit radius $R,\ R_w$ in the last term of the right side, and thus, the equation becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} - \frac{Gm_w(R - R_w\cos(\varphi - \varphi_w))}{(R^2 + R_w^2 - 2RR_w\cos(\varphi - \varphi_w))^{\frac{3}{2}}}. \tag{H.12}$$

Below we present some approximate solution of eq.(H.12).

H.2.1 Legendre Expansion

First we define the last term of eq.(H.12) by F as

$$F(x) \equiv -\frac{Gm_w(R - R_w x)}{(R^2 + R_w^2 - 2RR_w x))^{\frac{3}{2}}}, \qquad \text{with} \quad x = \cos(\varphi - \varphi_w) \tag{H.13}$$

and we make the Legendre expansion

$$F(x) = -\frac{Gm_w R}{(R^2 + R_w^2)^{\frac{3}{2}}} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}}\, x + \cdots\ . \tag{H.14}$$

Therefore we obtain the equation of motion

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{Gm_w R_w(R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}}\cos(\varphi - \varphi_w) \tag{H.15}$$

where the constant term is irrelevant and thus we do not write it above.

H.2.2 Iteration Method

Now we employ the iteration method in order to solve eq.(H.15). First we make use of the solution of the Kepler problem

$$\varphi = \varphi^{(0)} + \omega t \tag{H.16}$$

$$\varphi_w = \varphi_w^{(0)} + \omega_w t \tag{H.17}$$

and thus eq.(H.15) becomes

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2} + \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{H.18}$$

where $b,\ \beta$ can be given as

$$b = \varphi^{(0)} - \varphi_w^{(0)}, \quad \beta = \omega - \omega_w \tag{H.19}$$

H.2.3 Particular Solution

In order to solve eq.(H.18), we assume that the last term is sufficiently small and therefore r may be written in the following shape as

$$r = r^{(0)} + K \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}}} \cos(b + \beta t) \tag{H.20}$$

where $r^{(0)}$ denotes the Kepler solution of $r^{(0)} = \frac{A}{1+\varepsilon \cos\varphi}$. Now we insert the solution of eq.(H.20) into eq.(H.18) and we find the solution of K as

$$K = -\frac{1}{\beta^2} \tag{H.21}$$

Therefore, we obtain the approximate solution as

$$r = r^{(0)} - \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}} \beta^2} \cos(b + \beta t) \tag{H.22}$$

H.3 Effects of Other Planets on Mercury Perihelion

Therefore we should put the Kepler solution for $r^{(0)}$ and thus the Mercury orbit can be written as

$$\begin{aligned} r &= \frac{A}{1 + \varepsilon \cos\varphi} - \frac{G m_w R_w (R_w^2 - 2R^2)}{(R^2 + R_w^2)^{\frac{5}{2}} \beta^2} \cos(b + \beta t) \\ &\simeq \frac{A}{1 + \varepsilon \cos\varphi + \frac{G m_w R_w (R_w^2 - 2R^2)}{R (R^2 + R_w^2)^{\frac{5}{2}} (\omega - \omega_w)^2} \cos(b + \beta t)} \end{aligned} \tag{H.23}$$

where we take $A \simeq R$ and also $\beta = \omega - \omega_w$. Here as for ε_w, we take

$$\varepsilon_w \equiv \frac{G m_w}{R R_w^2 (\omega - \omega_w)^2} \frac{\left(1 - \frac{2R^2}{R_w^2}\right)}{\left(1 + \frac{R^2}{R_w^2}\right)^{\frac{5}{2}}} \tag{H.24}$$

and using $b + \beta t = \varphi - \varphi_w$, we obtain

$$r \simeq \frac{A}{1 + \varepsilon \cos\varphi + \varepsilon_w \cos(\varphi - \varphi_w)}. \tag{H.25}$$

This equation suggests that the Mercury perihelion may well be affected by the planet motions.

H.3.1 Numerical Evaluations

Now we calculate the Mercury Perihelion shifts due to the planet motions such as Jupiter or Venus. In order to do so, we first rewrite $\varepsilon \cos\varphi + \varepsilon_w \cos(\varphi - \varphi_w)$ terms as

$$\varepsilon \cos\varphi + \varepsilon_w \cos(\varphi - \varphi_w) = c_1 \cos\varphi + c_2 \sin\varphi = \sqrt{c_1^2 + c_2^2}\cos(\varphi + \delta) \tag{H.26}$$

where c_1 and c_2 are defined as

$$c_1 = \varepsilon + \varepsilon_w \cos\varphi_w \tag{H.27}$$

$$c_2 = \varepsilon_w \sin\varphi_w \tag{H.28}$$

where

$$\cos\delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}. \tag{H.29}$$

Here ε_w is much smaller than ε and thus eq.(H.29) becomes

$$\cos\delta = \frac{\varepsilon + \varepsilon_w \cos\varphi_w}{\sqrt{(\varepsilon + \varepsilon_w \cos\varphi_w)^2 + (\varepsilon_w \sin\varphi_w)^2}} \simeq 1 - \frac{1}{2}\left(\frac{\varepsilon_w}{\varepsilon}\right)^2 \sin^2\varphi_w \tag{H.30}$$

H.3.2 Average over One Period of Planet Motion

Now we should make the average over one period of planet motion and therefore we find

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\varphi_w \, d\varphi_w = \frac{1}{2} \tag{H.31}$$

Thus, δ becomes

$$\begin{aligned}
\delta &\simeq \frac{\varepsilon_w}{\sqrt{2}\,\varepsilon} \simeq \frac{1}{\sqrt{2}\,\varepsilon}\frac{GM}{R_w^2}\frac{1}{R(\omega-\omega_w)^2}\left(\frac{m_w}{M}\right)\frac{\left(1-\frac{2R^2}{R_w^2}\right)}{\left(1+\frac{R^2}{R_w^2}\right)^{\frac{5}{2}}} \\
&\simeq \frac{R_w\,\omega_w^2}{\sqrt{2}\,\varepsilon R\,(\omega-\omega_w)^2}\left(\frac{m_w}{M}\right)\frac{\left(1-\frac{2R^2}{R_w^2}\right)}{\left(1+\frac{R^2}{R_w^2}\right)^{\frac{5}{2}}}
\end{aligned} \tag{H.32}$$

where the planet orbits are taken to be just the circle, for simplicity.

H.3.3 Numerical Results

In order to calculate the effects of the planet motions on the δ, we first write the properties of planets in Table 1. Here numbers are shown in units of the earth.

Table 1

	Mercury	Venus	Mars	Jupiter	Saturn	Earth	Sun
Orbit Radius	0.387	0.723	1.524	5.203	9.55	1.0	
Mass	0.055	0.815	0.107	317.8	95.2	1.0	332946.0
Period	0.241	0.615	1.881	11.86	29.5	1.0	
ω	4.15	1.626	0.532	0.0843	0.0339	1.0	

In Table 2, we present the calculations of the values δ for one hundred years of averaging and the calculations are compared with the calculated results by Newcomb.

Table 2 The values of δ for one hundred years

Planets	Venus	Earth	Mars	Jupiter	Saturn	Sum of Planets
δ by eq.(H.32)	49.7	27.4	0.77	32.1	1.14	111.1
δ by Newcomb	56.8	18.8	0.51	31.7	1.5	109.3

As one sees, the agreement between the present calculation and Newcomb results is surprisingly good [9]. Here we do not review the calculation of Newcomb for the other planet effects on the Mercury perihelion shifts, and instead we simply employ his calculated results.

H.3.4 Comparison with Experiments

The observed values of the Mercury perihelion shifts are often quoted in some of the old textbooks. However, it should be very difficult to find some reliable numbers of the Mercury perihelion shifts since these values are determined for 100 years of observation period in 19 century. In this respect, the comparison between the calculation and observation should be a homework problem for readers.

Appendix I

Path Integral and Sommerfeld Quantization

In this appendix, we should explain the basic problem in the path integral formulation. To do so, we simply copy the recent paper with its title

"Path integral and Sommerfeld quantization".

arXiv:1809.04416, Path integral and Sommerfeld quantization
Mikoto Matsuda, Takehisa Fujita
Comments: 7 pages
Subjects: General Physics (physics.gen-ph)

arXiv:1809.04416: " Path integral and Sommerfeld quantization"

Path integral and Sommerfeld quantization

Mikoto Matsuda and Takehisa Fujita

The path integral formulation can reproduce the right energy spectrum of the harmonic oscillator potential, but it cannot resolve the Coulomb potential problem. This is because the path integral cannot properly take into account the boundary condition, which is due to the presence of the scattering states in the Coulomb potential system. On the other hand, the Sommerfeld quantization can reproduce the right energy spectrum of both harmonic oscillator and Coulomb potential cases since the boundary condition is effectively taken into account in this semiclassical treatment. The basic difference between the two schemes should be that no constraint is imposed on the wave function in the path integral while the Sommerfeld quantization rule is derived by requiring that the state vector should be a single-valued function. The limitation of the semiclassical method is also clarified in terms of the square well and $\delta(x)$ function potential models.

I.1 Introduction

Quantum field theory is the basis of modern theoretical physics and it is well established by now [1-4]. If the kinematics is non-relativistic, then one obtains the equation of quantum mechanics which is the Schrödinger equation. In this respect, if one solves the Schrödinger equation, then one can properly obtain the energy eigenvalue of the corresponding potential model .

Historically, however, the energy eigenvalue is obtained without solving the Schrödinger equation, and the most interesting method is known as the Sommerfeld quantization rule which is the semiclassical method [5-7]. In this case, the Sommerfeld quantization rule is to first assume

$$\oint p dr = nh \tag{I.1}$$

where n denotes an integer and h is the Planck constant. This method can be applied to the Coulomb potential problem of $V(r) = -\frac{Ze^2}{r}$. In fact, it can be solved exactly by the Sommerfeld quantization. Here, the momentum p is related to the energy E as

$$E = \frac{p^2}{2m} - \frac{Ze^2}{r} \tag{I.2}$$

where m denotes the mass of electron which is bound in the Coulomb potential. In this case, eq. (I.1) can be written as

$$\oint pdr = 2\sqrt{2m|E|}\int_0^{r_0}\sqrt{\frac{r_0 - r}{r}}dr = nh \tag{I.3}$$

where $r_0 = \frac{Ze^2}{|E|}$ is introduced. After the integration of eq.(I.3), we obtain the energy eigenvalue E as

$$E = -\frac{m(Ze^2)^2}{2\hbar^2 n^2} \tag{I.4}$$

which is a correct expression. The advantage of this method is clear in that it does not require the solution of the differential equation. Instead, one should make an integration of the space coordinates over the limited range of space.

The similar type of advantage is also known in the path integral formulation, and indeed the path integral method is formulated so that the spectrum can be obtained by the infinitely many dimensional integration over the discretized space coordinates. However, the path integral method can be successfully applied only to the harmonic oscillator potential case. The reason of the successful description of the harmonic oscillator potential is clear because there is no scattering state in the harmonic oscillator potential system. On the other hand, the path integral cannot reproduce the energy spectrum of the Coulomb potential case. This is related to the fact that the Coulomb case contains scattering states, and thus the value of $\int_{-\infty}^{\infty}|\psi(x)|^2dx$ is not necessarily unity. Therefore, the boundary condition cannot be properly taken into account in the path integral method, which should be an intrinsic defect arising from the formulation itself.

In addition, we discuss the Sommerfeld quantization rule [5-7] and clarify why it is successful for obtaining the energy spectrum of both the Coulomb and the harmonic oscillator potential systems. As we see below, the integration of the Sommerfeld quantization rule over coordinate space can be limited to the finite range of space. Therefore, the boundary condition is effectively taken into account in this treatment. However, the scattering problem cannot be treated in this semiclassical method, and the scattering states can be obtained by the WKB method which is the semiclassical approximation for the Schrödinger wave function. This WKB method is, however, not exact and only valid for the limited range of applications. The exact calculation of the scattering problem should be possible only when one solves the Schrödinger equation.

I.2 Boundary condition in path integral formulation

Here, we discuss the problem of the boundary condition how we can consider the physics of the boundary condition into the path integral expression. In fact, there is no way to include the boundary condition in the path integral formulation since the path integral does not make use of any information on the wave function. This is the basic reason why one cannot solve the Coulomb potential problem in the path integral calculation since the Coulomb case contains the scattering states in which the boundary condition in the scattering state is completely different from the bound state problem. In this section, we take the representation of $\hbar = 1$.

I.2.1 Path integral in quantum mechanics

Here, we briefly describe the path integral formulation in quantum mechanics [8-10]. First, we define the transition amplitude $K(x', x : t)$ as

$$K(x', x : t) = \langle x' | e^{-iHt} | x \rangle = \sum_n \psi_n^*(x')\, \psi_n(x)\, e^{-iE_n t} \tag{I.5}$$

where Hamiltonian H is defined as

$$H = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{I.6}$$

and $\psi_n(x)$ is an eigenstate of the Hamiltonian H

$$H\psi_n(x) = E_n \psi_n(x). \tag{I.7}$$

In this case, the path integral formulation is written as

$$K(x', x : t) = \int \mathcal{D}x \ \exp\left\{ i\Delta t \sum_{i=1}^{n} \left(\frac{m(x_i - x_{i-1})^2}{2(\Delta t)^2} - V(x_i) \right) \right\} \tag{I.8}$$

where many dimensional integration $\int \mathcal{D}x$ is introduced as

$$\int \mathcal{D}x \equiv \lim_{n\to\infty} \left(\frac{m}{2i\pi\Delta t} \right)^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \tag{I.9}$$

which is a symbolic notation in the path integral formulation. Therefore, the transition amplitude can be written in terms of the path integral formulation as

$$K(x', x : t) = \langle x' | e^{-iHt} | x \rangle = \int \mathcal{D}x \ \exp\left(i \int_0^t L(x, \dot{x}) dt \right) \tag{I.10}$$

where $L(x, \dot{x})$ is defined as

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - V(x) \tag{I.11}$$

which is the Lagrangian of the corresponding classical system.

I.2.2 Boundary condition in path integral formulation

Now a question may arise as to whether we may take into account the boundary condition for the wave function $\psi(x)$ in order to solve a bound state problem. That is, we must take into account the condition

$$\psi(\pm\infty) = 0 \tag{I.12}$$

in the path integral expression of eq.(I.5) and eq.(I.10) in some way or other. However, one sees that the boundary condition cannot be taken into account in the path integral formulation in which any condition on the wave function never appears in this formulation. This will be shown below more in detail.

I.2.3 Boundary condition in the harmonic oscillator potential system

There is one example which is solved exactly in terms of the path integral formulation, that is, the harmonic oscillator potential system. Why is it that the harmonic oscillator potential system can be successfully solved? The reason is, of course, simple in that the harmonic oscillator potential system does not contain any scattering states, and in fact, all the states are bound. Therefore, it is clear that

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1 \tag{I.13}$$

is always guaranteed. Thus, we obtain from eq.(I.5)

$$\int K(x, x : t)dx = \int \langle x|e^{-iHt}|x\rangle dx = \int dx \sum_n |\psi_n(x)|^2 e^{-iE_n t} = \sum_n e^{-iE_n t}. \tag{I.14}$$

In fact, we obtain the $K(x', x : t)$ for the harmonic oscillator potential system as

$$K(x', x : t) = \sqrt{\frac{m\omega}{2i\pi \sin\omega t}} \exp\left\{ i\frac{m\omega}{2}\left[(x'^2 + x^2)\cot\omega t - \frac{2x'x}{\sin\omega t} \right] \right\} \tag{I.15}$$

and therefore

$$\begin{aligned} \int K(x, x : t)dx &= \int_{-\infty}^{\infty} dx \sqrt{\frac{m\omega}{2i\pi\sin\omega t}} e^{-im\omega x^2 \tan\frac{\omega t}{2}} \\ &= \frac{1}{2i\sin\frac{\omega t}{2}} = \sum_{n=0}^{\infty} e^{-iE_n t}. \end{aligned} \tag{I.16}$$

Here, we make use of the following equation

$$\frac{1}{2i\sin\frac{\omega t}{2}} = \frac{e^{-\frac{i}{2}\omega t}}{1 - e^{-i\omega t}} = \sum_{n=0}^{\infty} e^{-i\omega t(n+\frac{1}{2})} \tag{I.17}$$

and thus find

$$E_n = \omega\left(n + \frac{1}{2}\right) \tag{I.18}$$

which is a correct energy for the harmonic oscillator potential system.

I.2.4 Boundary condition with scattering states

The harmonic oscillator potential is a very special and exceptional case in quantum mechanics, and it is, of course, unrealistic and cannot be applied to the description of nature. The reason why we often find the harmonic oscillator potential is simple. If we treat many body systems and make some approximations to obtain an effective one body potential, then we always obtain the harmonic oscillator potential near the minimum point. Namely, if we treat it in terms of the small vibration around the minimum point $x = x_0$ of the complicated potential, then we find

$$V(x) = V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots. \tag{I.19}$$

and here the first term is a constant and the second term is just the harmonic oscillator potential. However, it is clear that this is only valid for the small vibrations.

The complicated realistic potential should contain the scattering states since the potential $V(x)$ should vanish at the large value of x. For any systems with scattering states $\psi_s(x)$, the condition

$$\int |\psi_s(x)|^2 dx < \infty \tag{I.20}$$

is not necessarily satisfied. Therefore, eq.(I.14) cannot be used, and thus there is no way to obtain the energy eigenvalues of the corresponding system from the path integral method. This indicates that eq.(I.5) should normally contain some infinity in the integrations of the left side.

I.3 Sommerfeld quantization and boundary condition

As we see in the Introduction, the Sommerfeld quantization scheme is quite successful for obtaining the energy eigenvalue in some of the potential problems. Here, we briefly review the derivation of the Sommerfeld quantization rule from the Schrödinger equation in terms of the semiclassical approximation.

I.3.1 Sommerfeld quantization rule

We start from the Schrödinger equation in one dimension

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x) = E\psi(x). \tag{I.21}$$

Now we assume the wave function in the following shape

$$\psi(x) = Ae^{i\frac{S}{\hbar}} \tag{I.22}$$

and expand the S in terms of $\hbar$ as $S = S_0 + \hbar S_1 + \cdots$. By inserting the S into the Schrödinger equation, we obtain the S_0 to the lowest order of $\hbar$

$$\frac{dS_0(x)}{dx} = \pm\sqrt{2m(E - V(x))}. \tag{I.23}$$

Therefore, the wave function $\psi(x)$ becomes

$$\psi(x) = Ae^{\pm\frac{i}{\hbar}\int\sqrt{2m(E-V(x))}dx}. \tag{I.24}$$

At this point, we require that the wave function should be a single-valued function, and therefore obtain the following constraint equation

$$\oint pdx \equiv \oint \sqrt{2m(E - V(x))}dx = 2\pi\hbar n, \qquad (n: \text{ integer or half integer}) \tag{I.25}$$

which is just the Sommerfeld quantization rule.

I.3.2 Boundary condition in the Sommerfeld quantization

The constraint equation of eq.(I.25) is obtained from the requirement that the wave function should be a single-valued function. This should not necessarily correspond to a boundary condition at infinity, but should be an important physical imposition on the state vector, and therefore we can obtain the energy spectrum from this condition. However, it should be noted that the Sommerfeld quantization scheme can reproduce the right description of the energy eigenvalue only if the system has some semiclassical nature like the Coulomb potential system.

The boundary condition at large x should reflect the quantum effect in the wave function of the corresponding particle. This can be viewed from the uncertainty law that the large x corresponds normally to the small momentum regions in which the quantum effect becomes most important.

• **Coulomb potential ($V(r) = -\frac{Ze^2}{r}$) :**
In the Coulomb potential, the strong part should be found in the small r region, and therefore the large momentum region is most important, and thus it can be considered to be almost semiclassical. Indeed, the Sommerfeld quantization scheme can give the right answer for the energy spectrum of the Coulomb potential system. We show the calculated result which is given in Introduction

$$\oint pdr = 2\sqrt{2m|E|}\int_0^{\frac{Ze^2}{|E|}}\sqrt{\frac{\frac{Ze^2}{|E|} - r}{r}}dr = nh \qquad \Longrightarrow \qquad E = -\frac{m(Ze^2)^2}{2\hbar^2 n^2}. \tag{I.26}$$

The boundary condition of eq.(I.12) is effectively included in the above equation. This can be seen since the integration is limited to the finite region of space coordinate which is physically acceptable area of space.

• **Harmonic oscillator potential** $(V(x) = \frac{1}{2}m\omega^2 x^2)$ **:**

$$\oint pdx \equiv \oint \sqrt{2m(E - V(x))}dx = 2m\omega \oint_{-\sqrt{\frac{2E}{m\omega^2}}}^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{\frac{2E}{m\omega^2} - x^2}dx \tag{I.27}$$

$$= 2m\omega \frac{2E}{m\omega^2}\frac{n}{2} = 2\pi\hbar n. \tag{I.28}$$

Thus, by considering the fact that the value of n in eq.(I.25) can be an integer as well as a half integer, we obtain the energy E

$$E = \hbar\omega\left(n + \frac{1}{2}\right), \qquad (n = 0, 1, 2, \cdots). \tag{I.29}$$

• **Square well potential** $(V(x) = -V_0$ **for** $|x| < a$, **otherwise** $V(x) = 0)$ **:**

$$\oint pdx \equiv \oint \sqrt{2m(E - V(x))}dx = 2\sqrt{2m(E + V_0)}2a = 2\pi\hbar n. \tag{I.30}$$

Thus, we obtain

$$E = -V_0 + \frac{\hbar^2 \left(\frac{\pi n}{2}\right)^2}{2ma^2}. \tag{I.31}$$

This can be compared with the result obtained from the quantum mechanics calculation for the square well potential model. This energy cannot be given in terms of the analytical expression. Instead, one should solve the following equation for E

$$\alpha = \sqrt{\frac{2m\,(E + V_0)\,a^2}{\hbar^2}}, \quad \beta = \sqrt{-\frac{2mEa^2}{\hbar^2}} \tag{I.32}$$

$$\beta = \alpha \tan\alpha. \tag{I.33}$$

Approximate solutions can be found when $\alpha \simeq \frac{\pi}{2}n$ which should be expected from $\tan\alpha \simeq 0$. In this case, we can immediately obtain the energy E as

$$E \simeq -V_0 + \frac{\hbar^2 \left(\frac{\pi n}{2}\right)^2}{2ma^2} \tag{I.34}$$

which is just the same as the semiclassical result of eq.(I.31). In this case, however, eq.(I.33) is not necessarily satisfied, and thus the result of the bound state energy should be far from reliable.

• δ **function potential** $(V(x) = -V_\delta\delta(x))$ **:**
The limitation of the Sommerfeld quantization rule can be well exhibited if one solves the energy spectrum of the δ function potential with the semiclassical method. The answer is that there is no way to express the energy E in terms of V_δ. This is partly

because of the mathematical difficulty due to the generalized function of $\delta(x)$ in the square root, but mainly because of the limitation of the Sommerfeld quantization rule itself. This difficulty is better understood if we calculate the energy spectrum of the δ function potential from the square well potential result. In this case, we start from eq.(I.33)

$$\sqrt{-\frac{2mEa^2}{\hbar^2}} = \sqrt{\frac{2m(E+V_0)a^2}{\hbar^2}} \tan\sqrt{\frac{2m(E+V_0)a^2}{\hbar^2}}. \tag{I.35}$$

Here, we make $a \to 0$, $V_0 \to \infty$, but keep $V_\delta = 2aV_0$ finite. Therefore, eq.(I.35) can be written to a good approximation as

$$\sqrt{\frac{-2mE}{\hbar^2}} \simeq \sqrt{\frac{mV_\delta}{\hbar^2}} \sqrt{\frac{mV_\delta}{\hbar^2}} \tag{I.36}$$

which leads to the energy E

$$E = -\frac{mV_\delta^2}{2\hbar^2} \tag{I.37}$$

and this is indeed a correct energy eigenvalue of the δ function potential in quantum mechanics problem.

As we note above, the energy of the square well potential from the Sommerfeld quantization rule is obtained without making use of eq.(I.33) while the energy of the δ function potential is obtained only if we make use of eq.(I.33). From this discussion, we see that the Sommerfeld quantization rule cannot be applied to the δ function potential.

I.4 Summary

We have clarified the similarity and difference between the path integral formulation and the Sommerfeld quantization rule. The similarity of the two methods is concerned with the classical mechanics since both approaches start from the equations of classical mechanics.

On the other hand, there is a significant difference between them. The Sommerfeld quantization rule can properly take into account the boundary condition which is crucial for solving the bound state problem. On the other hand, the path integral method cannot include any of the wave function information, and therefore it cannot consider the boundary condition at infinity. Therefore, the path integral method can only solve the harmonic oscillator potential problem in which there is no need of the boundary condition.

The investigation of the present paper may not contain any new physics except that the limitation of the path integral method is made clear for the first time. In this respect, it should present an important step for readers to realize that any numerical calculations based on the path integral formulation cannot give the right energy spectrum of physically interesting potential models.

In addition, it is shown that the Sommerfeld quantization rule can be quite useful for calculating the spectrum of some quantum mechanics models, but at the same time, we explicitly present the limitation of the semiclassical picture in terms of the square well potential calculation.

References

[1] J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics", (McGraw-Hill Book Company,1964)

[2] J.J. Sakurai, "Advanced Quantum Mechanics", (addison-Wesley,1967)

[3] K. Nishijima, "Fields and Particles", (W.A. Benjamin, INC, 1969)

[4] T. Fujita and N. Kanda, "Fundamental Problems in Quantum Field Theory" (Bentham Publishers, 2013)

[5] A. Sommerfeld, Ann. Physik **28**, 665 (1909)

[6] A. Sommerfeld, Ann. Physik **62**, 95 (1920)

[7] A. Sommerfeld, Ann. Physik **81**, 1135 (1926)

[8] R.P. Feynman, Phys. Rev. **76**, 749 (1949)

[9] R.P. Feynman, Phys. Rev. **76**, 769 (1949)

[10] R.P. Feynman, Phys. Rev. **80**, 440 (1950)

Appendix J

Non-integrable Potential

When the non-integrable potential appears as the small perturbation on the Newton equation, what should be the best way to take into account this small potential effect?

J.1 Non-integrable Potential

Here we discuss the physical effects of the non-integrable potential. The additional potential from the new gravity model has the shape of $\frac{B_0}{r^2}$, and, therefore, we can write the non-integrable potentials into the simple shape in the following way

$$V_a(r) = \frac{q}{2mc^2}\left(\frac{GmM}{r}\right)^2 \tag{J.1}$$

where

$$q = \begin{cases} -6 & \text{for General Relativity} \\ 1 & \text{for New Gravity} \end{cases} . \tag{J.2}$$

In this case, the differential equation for the orbit with the additional potential becomes

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2 r} - \frac{1}{r^2} - \frac{q}{\ell^2 c^2}\left(\frac{GmM}{r}\right)^2}. \tag{J.3}$$

This equation can be solved exactly and the effect due to the correction appears in $\cos\varphi$ term and is written as

$$r = \frac{A_g}{1 + \varepsilon\cos\left(\frac{L_g}{\ell}\varphi\right)} \tag{J.4}$$

where A_g and L_g are given as

$$A_g = \frac{L_g^2}{GMm^2}, \quad L_g \equiv \sqrt{\ell^2 + \frac{qG^2M^2m^2}{c^2}} \equiv \ell\sqrt{1+\eta} \simeq \ell\left(1 + \frac{1}{2}\eta\right). \tag{J.5}$$

Here, the η is defined as

$$\eta \equiv \frac{qG^2M^2}{c^2R^4\omega^2} \tag{J.6}$$

which is a very small number. It is around 10^{-8} for the planet motion such as the earth or Mercury.

J.1.1 Effects of Non-integrable Potential on Solution

The solution of eq.(J.4) has a serious problem in that the orbit is not closed. This is quite well known that the potential with the non-integrable shape such as $V_c(r) = \frac{C}{r^2}$ gives rise to the orbit which is not closed. It is, of course, clear that this type of orbits should not happen in nature.

The abnormal behavior of the solution eq.(J.4) can also be seen from the following term

$$\cos\left(\frac{L_g}{\ell}\varphi\right) \simeq \cos(\varphi + \frac{1}{2}\eta\varphi). \tag{J.7}$$

It should be interesting to see that this term cannot be described in terms of the cartesian coordinates of $x = r\cos\varphi,\ y = r\sin\varphi$. In fact, $\cos(\varphi+\frac{1}{2}\eta\varphi)$ term becomes

$$\cos(\varphi + \frac{1}{2}\eta\varphi) = \frac{x}{r}\cos\frac{1}{2}\eta\varphi - \frac{y}{r}\sin\frac{1}{2}\eta\varphi \tag{J.8}$$

and there is no way to transform the $\cos\frac{1}{2}\eta\varphi$ term into $x,\ y$ coordinates even though we started from this cartesian coordinate. This is very serious since the solution expressed by polar coordinates cannot be written any more in the cartesian coordinates. This is related to the fact that the orbit is not closed due to the non-integrable potential effects.

J.1.2 Discontinuity of Orbit

The effect of the non-integral potential can be further seen as the discontinuity of the orbit trajectory since the orbit is not closed. In order to see this discontinuity of the orbit, we first start from the orbit solution with the non-integral potential, which is eq.(J.4)

$$r = \frac{A_g}{1+\varepsilon\cos\left(1+\frac{1}{2}\eta\right)\varphi}.$$

In this case, we find the radius r at $\varphi = 0$ and $\varphi = 2\pi$ as

$$r = \frac{A_g}{1+\varepsilon}, \qquad \varphi = 0 \tag{J.9}$$

$$r = \frac{A_g}{1+\varepsilon\cos\pi\eta}, \qquad \varphi = 2\pi. \tag{J.10}$$

Therefore the difference Δr becomes

$$\Delta r \equiv r_{(\varphi=2\pi)} - r_{(\varphi=0)} \simeq \frac{1}{2} A_g \pi^2 \eta^2 \varepsilon \simeq 0.15 \text{ cm} \tag{J.11}$$

for the Mercury orbit case of the general relativity as an example. This means that the orbit is discontinuous when φ becomes 2π. This is not acceptable for the classical mechanics, and indeed it disagrees with the observation. In addition, eq.(J.4) cannot generate the perihelion shift, and this can be easily seen from the orbit trajectory of eq.(J.4).

J.2 Perturbative Treatment of Non-integrable Potential

Here we should present a perturbative treatment of the non-integrable potential. This must be the only way to reliably treat the non-integrability in classical mechanics.

J.2.1 Integrable Expression

The equation for the orbit determination becomes

$$\begin{aligned}\frac{dr}{d\varphi} &= \frac{\dot{r}}{\dot{\varphi}} = r^2\sqrt{\frac{2mE}{\ell^2}+\frac{2m\alpha}{\ell^2 r}-\frac{1}{r^2}-\frac{q}{\ell^2c^2}\left(\frac{GmM}{r}\right)^2} \\ &= r^2\sqrt{1+\eta}\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}}. \end{aligned} \tag{J.12}$$

Therefore, we can rewrite the above equation as

$$\sqrt{1+\eta}\,d\varphi = \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}}}. \tag{J.13}$$

Here we note that $\eta = \frac{q}{\ell^2c^2}\left(GmM\right)^2$ is a very small number which is of the order $\eta \sim 10^{-8}$. Now in order to keep the effect of the non-integrable potential in terms of integrable expression, we should make an approximation as

$$\sqrt{1+\eta}\,d\varphi \simeq d\varphi. \tag{J.14}$$

The reason why we should make this approximation is because we should consider the dynamical effect as the perturbation while the η in the right hand side of eq.(J.13) should only change the value of constants such as E or α in the differential equation. In this way, the equation to determine the orbit becomes

$$\frac{dr}{d\varphi} = r^2\sqrt{\frac{2mE}{\ell^2(1+\eta)}+\frac{2m\alpha}{\ell^2(1+\eta)r}-\frac{1}{r^2}} \tag{J.15}$$

which gives the right orbit solution. Now the orbit is closed, and the solution can be written as

$$r = \frac{A_g}{1+\varepsilon\cos\varphi} \tag{J.16}$$

where A_g is given as

$$A_g = \frac{\ell^2}{GMm^2}(1+\eta). \tag{J.17}$$

Note that the ε is also changed due to the η term, but here we can safely neglect this effect since it does not play any role for physical observables. Therefore, the effect of the additional potential is to change the radius A_g of the orbit even though this change is very small indeed. Now eq.(J.16) clearly shows that there is no perihelion shift, and this is very reasonable since the additional potential cannot shift the main axis of the orbit.

J.2.2 Higher Order Effect of Perturbation

Here we should estimate the higher order effect of the perturbation in eq.(J.13). Denoting the solution of eq.(J.16) by $r^{(0)}$

$$r^{(0)} = \frac{A_g}{1 + \varepsilon \cos\varphi}$$

and the perturbative part of the radius by r' ($r = r^{(0)} + r'$), we can write the equation for r' as

$$\frac{dr'}{d\varphi} = \frac{1}{2}\eta(r^{(0)})^2 \sqrt{\frac{2mE}{\ell^2(1+\eta)} + \frac{2m\alpha}{\ell^2(1+\eta)r^{(0)}} - \frac{1}{(r^{(0)})^2}} \tag{J.18}$$

where the right side depends only on φ. Here, we should make a rough estimation and only consider the case in which the eccentricity ε is zero. In this case, the right side does not depend on the variable ε, and thus we can prove that the right side is zero. Therefore, the higher order correction of r' should be proportional to the eccentricity ε and can be written as

$$r' \simeq C_0 \eta \varepsilon A_g \tag{J.19}$$

where C_0 should be some numerical constant. For the earth revolution, the value of ε is very small ($\varepsilon \simeq 0.0167$) and thus we can safely ignore this higher order perturbative effect.

J.3 Period Corrections from General Relativity

Here we discuss briefly the period corrections generated by the additional potential of the general relativity. The gravitational potential together with the additional potential from the general relativity is given as

$$V(r) = -\frac{GMm}{r} - \frac{3}{mc^2}\left(\frac{GmM}{r}\right)^2. \tag{J.20}$$

Therefore the Newton equation becomes

$$m\ddot{r} = -\frac{GmM}{r^2} + \frac{L_g^2}{mr^3} \tag{J.21}$$

where L_g^2 is defined as

$$L_g^2 \equiv \ell^2 - \frac{6G^2M^2m^2}{c^2}. \tag{J.22}$$

The solution of the differential equation is given by taking into account the perturbative treatment of the non-integrable potential

$$r = \frac{A_g}{1+\varepsilon\cos\varphi} \tag{J.23}$$

where A_g is given as

$$A_g = \frac{L_g^2}{GMm^2}. \tag{J.24}$$

Therefore, the period T can be determined when we integrate $\dot{\varphi} = \frac{\ell}{mr^2}$ over the orbit period as

$$\frac{\ell}{m}\int_0^T dt = \int_0^{2\pi} r^2 d\varphi = A_g^2 \int_0^{2\pi} \frac{1}{(1+\varepsilon\cos\varphi)^2} d\varphi \tag{J.25}$$

which can be calculated to be

$$\omega T = 2\pi(1-2\gamma). \tag{J.26}$$

In this case, the correction ΔT to the period can be written as

$$\left(\frac{\Delta T}{T}\right)_{GR} \simeq -2\gamma. \tag{J.27}$$

J.3.1 Earth Revolution Period

For the earth revolution around the sun, the correction to the period T due to the general relativity becomes

$$\Delta T_{GR} = -3.8 \quad \text{[s/year]} \tag{J.28}$$

which is in the wrong direction as compared to the observation in terms of the leap second delay. In addition, this value is, by far, too large compared to the leap second, and in fact, the observed value of the leap second is around 0.62 [s/year]. Therefore, the correction to the earth period from the general relativity should be completely ruled out from the observation.

J.4 Gravitational Wave

It is really a shame as a theoretical physicist that we have to make a brief explanation about the gravitational wave. It is beyond imagination that some group of people insisted that they observed a signal of the gravitational wave. Those people who claimed a "discovery" of the gravitational wave should be far from physicists, and their standard of understanding physics must be lower than the fourth grade student of university. The physical observation can be done only if the object should have any interactions with matters whatever it can be. However, the gravitational wave which is a classical wave has no interaction with any physical objects. This means that its observation of their claim does not make sense.

When a physical object can propagate in vacuum, then it must be a particle like photon whatever it may be, even though massless. This is confirmed from the vast amount of experiments, and by now, "the ether hypothesis" is completely excluded. In fact, all modern physics is based on the relativity principle, and there is no experiment which contradicts the relativity.

Appendix K

Why Is General Relativity Meaningless?

The Einstein equation is a differential equation for the metric tensor of $g^{\mu\nu}$. This metric tensor is defined when the Lorentz invariant quantity $(ds)^2$ is expressed in terms of generalized formula as $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$. However, there is no special physical meaning in this generalization, and thus we cannot find any physics related to the metric tensor of $g^{\mu\nu}$. This problem of the general relativity has nothing to do with physics, but it is important in the science history. Therefore, we should explain why the general relativity was accepted to physicists for such a long time, even though it is a meaningless theory in physics.

K.1 Relativity Principle

Relativity principle should require that equations of motion in any inertial system should have the same form of differential equations, and, thus, all of the physical observables must be the same in every inertial system. This is the essence of the relativity, and nature can be understood in terms of four basic Lagrangian densities of electromagnetic, weak, strong eand gravitational interactions. Indeed, all the field theory models satisfy the relativistic invariance of Lorentz transformation.

K.1.1 Lorentz Transformation

Let us consider the moving frame $S(t', x', y', z')$ which is moving with linear motion of constant velocity v along $x-$axis with respect to the rest frame $R(t, x, y, z)$. In this case, the requirement that the equation of motion must be equivalent to each other in both systems can be written in terms of Lorentz transformation

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \quad y = y', \quad z = z'. \tag{K.1}$$

K.1.2 Lorentz Invariance

This Lorentz transformation is the necessary and sufficient condition for relativity principle. However, if we consider only the invariance of Lorentz transformation, then there should be many other physical quantities. Here, we should discuss the small distance square of $(ds)^2$ in four dimensions, which is defined as

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2.$$

K.1.3 Minkowski Space

This $(ds)^2$ is introduced by Minkowski as a Lorentz invariant quantity

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{K.2}$$

which is indeed invariant under the Lorentz transformation of

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right), \quad y = y', \quad z = z'. \tag{K.3}$$

Minkowski extended mathematically $(ds)^2$ to

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \equiv g^{\mu\nu} dx_\mu dx_\nu \tag{K.4}$$

even though there is no physical reason for this generalization. In this case, dx^μ and dx_μ are introduced as

$$dx^\mu = (cdt, dx, dy, dz), \quad dx_\mu = (cdt, -dx, -dy, -dz). \tag{K.5}$$

Further, the metric tensor $g^{\mu\nu}$ is defined as

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This extension of $(ds)^2$ is not incorrect. However, the naming of $g^{\mu\nu}$ as metric tensor is wrong since it is a dimensionless quantity and, therefore, it cannot be taken as any measure of space and time.

K.2 Risk of Generalization

It indeed makes sense that $(ds)^2$ can be taken as a test of Lorentz invariance, and it is also understandable that $(ds)^2$ is expressed in terms of eq.(K.4). However, it should be important to realize that this generalization is physically meaningless since $(ds)^2$ itself is far from any essential quantity in physics.

K.2.1 Invariance of $(ds)^2$

Here, we should explain some important point of $(ds)^2$. This $(ds)^2$ is certainly Lorentz invariant, but it is the result of the Lorentz transformation, and not the condition. In fact, there should be many other transformations that can make $(ds)^2$ invariant. This point is quite important since it is related to the essence of relativity. The theory of relativity is a theoretical frame work in which any equation of motion must be the same in any inertial system. The Lorentz transformation satisfies this necessary and sufficient conditions. On the other hand, $(ds)^2$ can serve as a sufficient condition of the relativity requirement, but it is not necessary.

K.2.2 Generalized Expression of $(ds)^2$

For a long time, people believed that the generalized expression of $(ds)^2$

$$(ds)^2 = g^{\mu\nu} dx_\mu dx_\nu \tag{K.6}$$

must be basic and essential for $(ds)^2$. This is, of course, an illusion. However, most of physicists may well have been trapped for a long time in a blind state, and this is quite unfortunate.

K.2.3 Physical Meaning of $g^{\mu\nu}$

In physics, the expression of (K.2) is essential, and it is impossible to find any physical meaning for the metric tensor of $g^{\mu\nu}$. Indeed, $g^{\mu\nu}$ must be mathematically all right, but it has no physical meaning, and it is just useless.

K.3 General Relativity

The Einstein equation is the differential equation for this useless metric tensor $g^{\mu\nu}$ [17], and therefore, we cannot find any physical meaning in this equation.

In fact, even if the metric tensor $g^{\mu\nu}$ becomes some function of space and time, there is no effect on the relativity. In case the $(ds)^2$ which is expressed by $g^{\mu\nu}$ in eq.(K.6) has lost the Lorentz invariance, we should make use of $(ds)^2$ as expressed in eq.(K.2). Therefore, there is no physical effect of $g^{\mu\nu}$ in nature at all.

This clearly shows that the Einstein equation has nothing to do with physics, and it is simply a mathematical equation which may help young people learn geometrical differential equation as an exercise problem.

K.4 Negative Legacy

It is a shame that we could not clarify 30 years ago, for example, that the Einstein equation has nothing to do with physics. Many young people wasted their time by learning this general relativity which is completely meaningless in physics. This is quite unfortunate and serious.

Incidentally, there was a claim at one point that the Mercury perihelion shifts could be described by the metric tensor which is, by hand, connected to gravity. However, this shift is identified by the discontinuity of Mercury orbit, and, therefore, this prediction is both physically and mathematically meaningless. In this sense, this claim may well be one of the worst theoretical predictions in physics.

Bibliography

[1] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Mechanics”, (McGraw-Hill Book Company,1964)

[2] J.J. Sakurai, “Advanced Quantum Mechanics”, (addison-Wesley,1967)

[3] K. Nishijima, “Fields and Particles”, W.A. Benjamin, INC, 1969

[4] T. Fujita, “Symmetry and Its Breaking in Quantum Field Theory” Nova Science Publishers, 2011 (2nd edition)

[5] T. Fujita and N. Kanda, “Fundamental Problems in Quantum Field Theory” Bentham Publishers, 2013

[6] L. D. Landau, Dokl. Akad. Nawk., USSR vol. 60, pp. 207, 1948.

[7] C. N. Yang, “Selection Rules for the Dematerialization of a Particle into Two Photons,” Phys. Rev. vol. 77, pp. 242−245, Jan. 1950.

[8] Dirac, AIP Conference Proceedings **74**, 129 (1981)

[9] Simon Newcomb, ”Tables of the Four Inner Planets”, 2nd ed. (Washington: Bureau of Equipment, Navy Dept., 1898).

[10] B.G. Bills and R.D. Ray. (1999), “ Lunar Orbital Evolution: A Synthesis of Recent ResultGeophysical Research Letters 26 (19): 3045-3048

[11] A. Sommerfeld, Ann. Physik **28**, 665 (1909)

[12] A. Sommerfeld, Ann. Physik **62**, 95 (1920)

[13] A. Sommerfeld, Ann. Physik **81**, 1135 (1926)

[14] R.P. Feynman, Phys. Rev. **76**, 749 (1949)

[15] R.P. Feynman, Phys. Rev. **76**, 769 (1949)

[16] R.P. Feynman, Phys. Rev. **80**, 440 (1950)

[17] A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie,” Annalen der Physik vol. 49, pp. 769−822, März. 1916.

Index