# 第5章　Weinberg-Salam の標準模型

弱い相互作用に関する実験結果は基本的に CVC 理論によって非常にうまく記述されていた [11]．しかしながらこの理論には重大な理論的な欠陥があることが知られていた．それは相互作用 Hamiltonian がカレント・カレント形式であるため，これは４点相互作用となっていると言う点である．そしてこれだと２次の摂動論で計算すると２次発散が出てしまうと言う問題があり，この点を克服することは理論上の重要問題であった．弱い相互作用の結合定数は非常に小さいので，ほとんどの実験結果は１次の摂動論によって記述されていた．しかしながら，理論スキームとしては欠陥があることは明らかであった．このため，何らかの修正が必要であったが，１９６０年代後半に Weinberg-Salam が非可換ゲージ理論による弱い相互作用の模型を提案したのである [12, 13]．ところが非可換ゲージ理論では構成粒子が観測量ではなく，さらにゲージ粒子は質量ゼロであるため，出発点が間違っていたのは明らかであった．その上，Higgs 機構と言うさらに意味不明の模型を採用したため，理論体系としてはあまりにも稚拙な間違いだらけの理論模型となっていたのである．しかしながらこの模型は最終的には CVC 理論を再現するように手直ししているため，パラメータをうまく選べば実験を再現できる理論模型となっている．

## 5.1　非可換ゲージ理論

量子電磁力学において，繰りこみ理論がうまく機能したと人々は考えたため，これは QED がゲージ理論であることに依っていると言うほとんど根拠のない話が一般的に浸透してしまった．このため弱い相互作用もゲージ理論で構築しようと言う事が１９６０年代には主流になっていた．この場合，弱い相互作用では SU(2) を考える必要があったため，非可換ゲージ理論が採用されることになったのである．

当時，非可換ゲージ理論はこれまでの U(1) ゲージ理論と大きな違いはないと人々は考えたものと思われる．しかしながら実際には，非可換ゲージ理論における構成粒子の電荷がゲージに依ってしまうため，そのままではこれらの粒子が観測量にはならないことが証明されている．

### 5.1.1 ゲージ粒子の質量

ゲージ不変性がある理論体系の場合，そのゲージ粒子の質量はゼロである．一方，弱い相互作用で必要とされていた弱ベクトルボソンの質量は当時から核子の質量よりはるかに重いものであると言うことは実験的にも知られていた事実である．実際に弱ベクトルボソンが発見され，その質量が実験的に決められたのは１９８０年代に入ってからではある．しかしながら弱い相互作用の理論体系を作る際，ゲージ理論から出発することは明らかに無謀な試みであったのである．

## 5.2 Higgs 機構

ゲージボソンに質量を与えると言うほとんど奇術的な手法を用いざるを得なかったのは勿論，出発点が間違えているからである．この奇術が Higgs 機構である．この模型はあまりにも稚拙な理論ではあるが，ここでは簡単に解説しておこう．

### 5.2.1 Higgs ポテンシャル

Higgs 機構の Lagrangian 密度は [14]

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^\dagger (D^\mu \phi) - U(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.1}$$

である．ここで $U(\phi),\ D^\mu,\ F^{\mu\nu}$

$$\begin{aligned} U(\phi) &= -\frac{1}{4} u_0 \left(|\phi|^2 - \lambda^2\right)^2 &(5.2)\\ D^\mu &= \partial^\mu + igA^\mu &(5.3)\\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu. &(5.4)\end{aligned}$$

と定義されている．ここで $u_0,\ \lambda$ は定数．この Lagrangian 密度はゲージ変換

$$\begin{aligned} A^\mu &\to A^\mu + \partial^\mu \chi &(5.5)\\ \phi &\to e^{-ig\chi}\phi &(5.6)\end{aligned}$$

に対して不変となっている．また，場のポテンシャル $U(\phi)$ は Higgs ポテンシャルと呼ばれているが，この出どころは不明であり，基本的な物理量ではない．

## 5.3 保存カレントと非保存カレント

ここでは U(1) の場合のみ考えよう．この場合，スカラー場 $\phi$ に対する方程式は

$$\partial_\mu(\partial^\mu + igA^\mu)\phi = -u_0\phi\left(|\phi|^2 - \lambda^2\right) - igA_\mu(\partial^\mu + igA^\mu)\phi \tag{5.7}$$

となる．一方，ゲージ場 $A_\mu$ に対する方程式は

$$\partial_\mu F^{\mu\nu} = gJ^\nu \tag{5.8}$$

となる．

### 5.3.1 保存カレント

ここで式 (5.8) の右辺の $J^\mu$ は

$$J^\mu = \frac{i}{2}\left\{\phi^\dagger(\partial^\mu + igA^\mu)\phi - \phi(\partial^\mu - igA^\mu)\phi^\dagger\right\}. \tag{5.9}$$

と定義されている．この場合，

$$\partial_\mu J^\mu = 0 \tag{5.10}$$

が成り立っている．従って，$J^\mu$ はこの系全体の保存カレントとなっている．しかしこのカレントにはベクトルポテンシャル $A^\mu$ が含まれている事に注意する必要がある．

### 5.3.2 複素スカラーボソンのカレント

一方，複素スカラーボソンのカレント $J^\mu_{CSB}$ は

$$J^\mu_{CSB} = \frac{i}{2}\left\{\phi^\dagger(\partial^\mu\phi) - \phi(\partial^\mu\phi^\dagger)\right\} \tag{5.11}$$

であり，このカレントはゲージ変換

$$\phi \to e^{-ig\chi}\phi \tag{5.12}$$

に対して不変ではない．すなわち，複素スカラーボソンのカレント $J^\mu_{CSB}$ はゲージ依存となっている [15]．さらに，このカレントは

$$\partial_\mu J^\mu_{CSB} \neq 0 \tag{5.13}$$

であり，これは保存していない．従ってこの複素スカラーボソンは物理的な観測量とはなっていない．

## 5.4　ユニタリーゲージ

一方，Higgs 機構の模型計算において人々は Lagrangian 密度の段階でユニタリーゲージ固定をしている．これは

$$\phi = \phi^\dagger \tag{5.14}$$

とする事に対応している．こうすると最終的な Lagrangian 密度が

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \frac{1}{4}u_0 \left(|\lambda + \eta(x)|^2 - \lambda^2\right)^2 + \frac{1}{2}g^2(\lambda + \eta(x))^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

となる．ここで Higgs 場を

$$\phi = \phi^\dagger = \lambda + \eta(x) \tag{5.15}$$

と仮定している．

### 5.4.1　2 次発散項

Higgs 模型の Lagrangian 密度において，第３項を

$$\mathcal{L}_I = \frac{1}{2}g^2(\lambda + \eta(x))^2 A_\mu A^\mu \tag{5.16}$$

と置こう．これは，ベクトル場 $A_\mu$ を量子化すると，物理的な観測量に対して，２次の摂動計算では２次発散を与える項である．従って，この項が存在する事はこの理論形式が本質的な欠陥を持っていることに対応している．

## 5.5 自発的対称性の破れ

Higgs 模型の基礎になっている模型は自発的対称性の破れの理論模型である．しかしながら，この理論模型はさらに稚拙な模型計算であり，詳細な解説は必要とは言えないものである．それでここでは簡単にその問題点を解説しておこう．この問題に興味がある読者は参考文献 [3, 16] を参照して貰うことにしよう．

### 5.5.1 自発的対称性の破れの模型

南部達は対称性を議論するにあたり，次のような模型の Lagrangian 密度から議論を進めている [17]．それは

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi + \frac{1}{2}G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right] \tag{5.17}$$

である．ここでフェルミオンの質量はゼロとしている．従ってこの式 (5.17) は次のカイラル変換

$$\psi' = e^{i\alpha\gamma_5}\psi \tag{5.18}$$

に対して不変である．従って，この系はカイラル対称性を持っている．しかしながらフェルミオンの質量がゼロの場合、その系を測るものが存在していないため、フェルミオン模型としては物理的な意味はない。

### 5.5.2 Bogoliubov 変換

ここで南部達は Bogoliubov 変換 [18]

$$c_n = e^{\mathcal{A}}a_n e^{-\mathcal{A}} = \cos\theta_n\, a_n - \sin\theta_n\, b_n, \tag{5.19}$$

$$d^\dagger_{-n} = e^{\mathcal{A}}b_n e^{-\mathcal{A}} = \cos\theta_n\, b_n + \sin\theta_n\, a_n \tag{5.20}$$

を使ってフェルミオン演算子 $a_n, b_n$ から新しいフェルミオン演算子 $c_n, d_n$ に変換している．ここで

$$\mathcal{A} = \sum_n \theta_n(a_n^\dagger b_n - b_n^\dagger a_n) \tag{5.21}$$

であり，$\theta_n$ は Bogoliubov 角である．

## 5.6 カイラル対称性の自発的破れ？

この方法により，南部達は Hamiltonian 密度を書き直している．この場合，Hamiltonian 密度の中に見かけ上，質量項に対応する項が現れている．彼らはこの項が現われた事により，カイラル対称性が破れたと誤解してしまったのである．Bogoliubov 変換はユニタリー変換なので，厳密に行えば正しい変換となっている．しかし彼らは高次項を考慮しないで議論を進めてしまったため，対称性が破れたと思い込んでしまったのであろう．系が持っている対称性が自然に破れるとしたら，その原因をきちんと調べる必要があるが，しかし検証された形跡はない．このため彼らはこの状況を『自発的破れ』としてしまったのであろう．物理学においては基本的な対称性が外力なしに破れると言う事はない．その意味において，これは相当，お粗末な計算である事は間違いない．さらに言えば，彼らは自発的対称性の破れに対応して『massless boson』が現れると主張しているが，これは S-行列のポールから『massless boson』が存在するはずであると言う主張であった．ところが，これには理論的な根拠は全くない事がわかっている．

### 5.6.1 カイラル対称性模型の厳密解

２次元の Thirring 模型はフェルミオンの質量がゼロの場合 [19]，Lagrangian 密度は

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{1}{2}\,gj^\mu j_\mu \tag{5.22}$$

と書かれている．ここで $j_\mu$ はフェルミオンカレントである．この Lagrangian 密度はカイラル変換

$$\psi' = e^{i\alpha\gamma_5}\psi \tag{5.23}$$

に対して不変であり，カイラル対称性がある模型となっている．そしてこの Hamiltonian は

$$\hat{H} = \int dx \left\{ -i\left(\psi_a^\dagger \frac{\partial}{\partial x}\,\psi_a - \psi_b^\dagger \frac{\partial}{\partial x}\,\psi_b\right) + 2g\psi_a^\dagger\psi_b^\dagger\psi_b\psi_a \right\} \tag{5.24}$$

と書かれていて，これは Bethe 仮設により厳密に解かれている．

### 5.6.2　Thirring 模型の真空の厳密解

Thirring 模型の厳密解により作られた真空状態のエネルギーが解析的に解かれている．このため，カイラル対称性に関して極めて重要な性質を知ることができている．この仕事は平本誠，本間崇司，高橋秀典３氏との共同研究により，幸運にも真空状態のエネルギーの解析解が見つかったものである [20]．これは自発的対称性の破れに関する反証の論文として決定的な役割を果たしたことは確かである．これらの結果の詳細は参考文献 [16] に解説されているので詳細はこの文献を参照して貰うことにしよう．

### 5.6.3　厳密解による Thirring 模型の真空の性質

ここでは厳密解による Thirring 模型の真空の諸性質について簡単な説明だけをしておこう [16]．

#### ● Thirring 模型の真空のエネルギー

Thirring 模型の真空は勿論，カイラル対称性を破ることはない．この場合，自由場の真空と比べて厳密解の真空はより低いエネルギー状態になっていることが示されている．2 次元模型なので実際の自然界との接点はないが，しかしこの真空が実現されていることは確かである．

#### ● 真空におけるカイラル電荷の固有値

真空状態を記述する固有値にカイラル電荷がある．これは

$$Q_5 = \int j_5^0(x)\, d^3r, \qquad (j_5^\mu = \bar{\psi}\gamma_\mu\gamma_5\psi) \tag{5.25}$$

と定義されている．この場合，自由場の真空は左右の対称性があるため，この真空のカイラル電荷はゼロである．一方，厳密解の真空のカイラル電荷の固有値は $\pm 1$ であることがわかっている．

#### ● 『注意書き』

系のカイラル対称性により，カイラル電荷が保存量となっている．そして，カイラル電荷の固有値がゼロでなく有限値である場合，これは勿論，カイラル対称の破れとは無関係である．昔，ある時期に誤解があったので，コメントしている．

## 5.7　負の遺産

現在まで標準理論として評価されてきた Weinberg-Salam の模型は CVC 理論を再現するように手を加えられていたので，最終的には弱い相互作用の実験を再現できる模型ではあった．その意味では，現代物理に残した負の遺産としてはそれ程，大きいとは言えないかも知れない．しかしながら Higgs 機構にせよ自発的対称性の破れにせよ，理論物理としては極めて稚拙な理論模型であり，これらが理論物理に与えた負の遺産は到底，小さいとは言えないであろう．

そして，さらに現在においてさえもまだ，CERN では Higgs 粒子探索実験を継続している．これは実験物理学に対して甚大な負の遺産を残したことは間違いない事である．この傷跡を回復するためにはかなり長い時間が必要となっているものと考えられるが，どうしたら良いのかこればかりは良くわからない．

# 付 録D Basic Notations in Field Theory

**In field theory, one often employs special notations which are by now commonly used. In this Appendix, we explain some of the notations which are particularly useful in field theory calculations.**

## D.1 Natural Units and Constants

**Here, we employ the natural units because of its simplicity**

$$c = 1, \quad \hbar = 1. \qquad (D.1.1)$$

**If one wishes to get the right dimensions out, one should use**

$$\hbar c = 197.33 \ \text{MeV} \cdot \text{fm}. \qquad (D.1.2)$$

**For example, pion mass is $m_\pi \simeq 140$ MeV/c$^2$. Its Compton wave length is**

$$\frac{1}{m_\pi} = \frac{\hbar c}{m_\pi c^2} = \frac{197 \ \text{MeV} \cdot \ \text{fm}}{140 \ \text{MeV}} \simeq 1.4 \ \text{fm}.$$

**Fine structure constant:** $\alpha = e^2 = \dfrac{e^2}{\hbar c} = \dfrac{e^2}{4\pi} = \dfrac{e^2}{4\pi\hbar c} = \dfrac{1}{137.036}$.

**Some constants:**
$$\begin{cases} \textbf{Electron mass}: & m_e = 0.511 \ \ \text{MeV}/c^2 \\ \textbf{Muon mass}: & m_\mu = 105.66 \ \ \text{MeV}/c^2 \\ \textbf{Proton mass}: & M_p = 938.28 \ \ \text{MeV}/c^2 \\ \textbf{Bohr radius}: & a_0 = \dfrac{1}{m_e e^2} = 0.529 \times \ 10^{-8} \ \ \text{cm} \end{cases}$$

**Gravitational constant:** $G = 5.906 \times 10^{-39} \ \frac{1}{M_p^2}$

**Weak coupling Constant:** $G_F = 1.166 \times 10^{-5} \ \ (\mathrm{GeV})^{-2}$

**Magnetic moments :**
$$\begin{cases} \textbf{Electron}: & \mu_e = 1.00115965219 \ \dfrac{e\hbar}{2m_e c} \\ \textbf{Muon}: & \mu_\mu = 1.001165920 \ \dfrac{e\hbar}{2m_\mu c} \end{cases}$$

**Weak bosons :**
$$\begin{cases} W^\pm - \textbf{boson}: M_W = 80.4 \ \ \mathrm{GeV}/c^2, & \alpha_W \simeq 4.3 \times 10^{-3} \\ Z^0 - \textbf{boson}: \ M_z = 91.2 \ \ \mathrm{GeV}/c^2, & \alpha_Z \simeq 2.73 \times 10^{-3} \end{cases}$$

## D.2 Hermite Conjugate and Complex Conjugate

**For a complex c-number** $A$

$$A = a + bi \ \ (a, b: \ \text{real}). \tag{D.2.1}$$

**Its complex conjugate** $A^*$ **is defined as**

$$A^* = a - bi. \tag{D.2.2}$$

**Matrix** $A$

**If** $A$ **is a matrix, one defines the hermite conjugate** $A^\dagger$

$$(A^\dagger)_{ij} = A^*_{ji}. \tag{D.2.3}$$

**Differential Operator** $\hat{A}$

**If** $\hat{A}$ **is a differential operator, then the hermite conjugate can be defined only when the Hilbert space and its scalar product are defined. For example, suppose** $\hat{A}$ **is written as**

$$\hat{A} = i\frac{\partial}{\partial x}. \tag{D.2.4}$$

**In this case, its hermite conjugate $\hat{A}^\dagger$ becomes**

$$\hat{A}^\dagger = -i\left(\frac{\partial}{\partial x}\right)^T = i\,\frac{\partial}{\partial x} = \hat{A} \qquad (D.2.5)$$

**which means $\hat{A}$ is Hermitian. This can be easily seen in a concrete fashion since**

$$\langle\psi|\hat{A}\psi\rangle = \int_{-\infty}^{\infty} \psi^\dagger(x) i\,\frac{\partial}{\partial x}\,\psi(x)\,dx = -i\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\,\psi^\dagger(x)\right)\psi(x)\,dx = \langle\hat{A}\psi|\psi\rangle, \qquad (D.2.6)$$

**where $\psi(\pm\infty) = 0$ is assumed. The complex conjugate of $\hat{A}$ is simply**

$$\hat{A}^* = -i\,\frac{\partial}{\partial x} \neq \hat{A}. \qquad (D.2.7)$$

**Field $\psi$**

**If the $\psi(x)$ is a c-number field, then the hermite conjugate $\psi^\dagger(x)$ is just the same as the complex conjugate $\psi^*(x)$. However, when the field $\psi(x)$ is quantized, then one should always take the hermite conjugate $\psi^\dagger(x)$. When one takes the complex conjugate of the field as $\psi^*(x)$, one may examine the time reversal invariance.**

## D.3 Scalar and Vector Products (Three Dimensions) :

**Scalar Product**

**For two vectors in three dimensions**

$$\boldsymbol{r} = (x, y, z) \equiv (x_1, x_2, x_3), \quad \boldsymbol{p} = (p_x, p_y, p_z) \equiv (p_1, p_2, p_3) \qquad (D.3.1)$$

**the scalar product is defined**

$$\boldsymbol{r}\cdot\boldsymbol{p} = \sum_{k=1}^{3} x_k p_k \equiv x_k p_k, \qquad (D.3.2)$$

**where, in the last step, we omit the summation notation if the index $k$ is repeated twice.**

**Vector Product**

**The vector product is defined as**

$$\boldsymbol{r} \times \boldsymbol{p} \equiv (x_2p_3 - x_3p_2, x_3p_1 - x_1p_3, x_1p_2 - x_2p_1). \tag{D.3.3}$$

**This can be rewritten in terms of components,**

$$(\boldsymbol{r} \times \boldsymbol{p})_i = \epsilon_{ijk} x_j p_k, \tag{D.3.4}$$

**where** $\epsilon_{ijk}$ **denotes anti-symmetric symbol with**

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \quad \text{otherwise} = 0.$$

## D.4 Scalar Product (Four Dimensions)

**For two vectors in four dimensions,**

$$x^\mu \equiv (t, x, y, z) = (x_0, \boldsymbol{r}), \quad p^\mu \equiv (E, p_x, p_y, p_z) = (p_0, \boldsymbol{p}) \tag{D.4.1}$$

**the scalar product is defined**

$$x \cdot p \equiv Et - \boldsymbol{r} \cdot \boldsymbol{p} = x_0p_0 - x_kp_k. \tag{D.4.2}$$

**This can be also written as**

$$x_\mu p^\mu \equiv x_0p^0 + x_1p^1 + x_2p^2 + x_3p^3 = Et - \boldsymbol{r} \cdot \boldsymbol{p} = x \cdot p, \tag{D.4.3}$$

**where** $x_\mu$ **and** $p_\mu$ **are defined as**

$$x_\mu \equiv (x_0, -\boldsymbol{r}), \quad p_\mu \equiv (p_0, -\boldsymbol{p}). \tag{D.4.4}$$

**Here, the repeated indices of the Greek letters mean the four dimensional summation** $\mu = 0, 1, 2, 3$**. The repeated indices of the roman letters always denote the three dimensional summation throughout the text.**

**Metric Tensor**

It is sometimes convenient to introduce the metric tensor $g^{\mu\nu}$ which has the following properties

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{D.4.5}$$

In this case, the scalar product can be rewritten as

$$x \cdot p = x^\mu p^\nu g_{\mu\nu} = Et - \boldsymbol{r} \cdot \boldsymbol{p}. \tag{D.4.6}$$

## D.5 Four Dimensional Derivatives $\partial_\mu$

The derivative $\partial_\mu$ is introduced for convenience

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \tag{D.5.1}$$

where the lower index has the positive space part. Therefore, the derivative $\partial^\mu$ becomes

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \tag{D.5.2}$$

### D.5.1 $\hat{p}^\mu$ and Differential Operator

Since the operator $\hat{p}^\mu$ becomes a differential operator as

$$\hat{p}^\mu = (\hat{E}, \hat{\boldsymbol{p}}) = \left( i\frac{\partial}{\partial t},\ -i\boldsymbol{\nabla}\right) = i\partial^\mu$$

the negative sign, therefore, appears in the space part. For example, if one defines the current $j^\mu$ in four dimension as

$$j^\mu = (\rho, \boldsymbol{j}),$$

then the current conservation is written as

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \frac{1}{i}\,\hat{p}_\mu j^\mu = 0. \tag{D.5.3}$$

### D.5.2 Laplacian and d'Alembertian Operators

The Laplacian and d'Alembertian operators, $\Delta$ and $\Box$ are defined as

$$\Delta \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$
$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta.$$

## D.6 $\gamma$-Matrix

Here, we present explicit expressions of the $\gamma$-matrices in two and four dimensions. Before presenting the representation of the $\gamma$-matrices, we first give the explicit representation of Pauli matrices.

### D.6.1 Pauli Matrix

Pauli matrices are given as

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (D.6.1)$$

Below we write some properties of the Pauli matrices.

**Hermiticity**

$$\sigma_1^\dagger = \sigma_1, \quad \sigma_2^\dagger = \sigma_2, \quad \sigma_3^\dagger = \sigma_3.$$

**Complex Conjugate**

$$\sigma_1^* = \sigma_1, \quad \sigma_2^* = -\sigma_2, \quad \sigma_3^* = \sigma_3.$$

**Transposed**

$$\sigma_1^T = \sigma_1, \quad \sigma_2^T = -\sigma_2, \quad \sigma_3^T = \sigma_3 \quad (\sigma_k^T = \sigma_k^*).$$

**Useful Relations**

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{D.6.2}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{D.6.3}$$

## D.6.2 Representation of $\gamma$-matrix

**(a) Two dimensional representations of $\gamma$-matrices**

**Dirac :** $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

**Chiral :** $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

**(b) Four dimensional representations of gamma matrices**

**Dirac :** $\gamma^0 = \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$$

**Chiral :** $\gamma^0 = \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix},$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}.$$

**where** $\mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

## D.6.3 Useful Relations of $\gamma$-Matrix

**Here, we summarize some useful relations of the $\gamma$-matrices.**

**Anti-commutation relations**

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^5, \gamma^\nu\} = 0. \qquad (D.6.4)$$

**Hermiticity**

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \ \ (\gamma_0^\dagger = \gamma_0, \ \ \gamma_k^\dagger = -\gamma_k), \qquad \gamma_5^\dagger = \gamma_5. \qquad (D.6.5)$$

**Complex Conjugate**

$$\gamma_0^* = \gamma^0, \quad \gamma_1^* = \gamma_1, \quad \gamma_2^* = -\gamma_2, \quad \gamma_3^* = \gamma_3, \qquad \gamma_5^* = \gamma_5. \qquad (D.6.6)$$

**Transposed**

$$\gamma_\mu^T = \gamma^0\gamma_\mu^\dagger\gamma^0, \quad \gamma_5^T = \gamma_5. \qquad (D.6.7)$$

## D.7 Transformation of State and Operator

**When one transforms a quantum state $|\psi\rangle$ by a unitary transformation $U$ which satisfies**

$$U^\dagger U = 1$$

**one writes the transformed state as**

$$|\psi'\rangle = U|\psi\rangle. \qquad (D.7.1)$$

**The unitarity is important since the norm must be conserved, that is,**

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1.$$

**In this case, an arbitrary operator $\mathcal{O}$ is transformed as**

$$\mathcal{O}' = U\mathcal{O}U^{-1}. \qquad (D.7.2)$$

**This can be obtained since the expectation value of the operator $\mathcal{O}$ must be the same between two systems, that is,**

$$\langle\psi|\mathcal{O}|\psi\rangle = \langle\psi'|\mathcal{O}'|\psi'\rangle. \qquad (D.7.3)$$

**Since**

$$\langle\psi'|\mathcal{O}'|\psi'\rangle = \langle\psi|U^\dagger\mathcal{O}'U|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

**one finds**

$$U^\dagger\mathcal{O}'U = \mathcal{O}$$

**which is just eq.(D.7.2).**

# D.8 Fermion Current

**We summarize the fermion currents and their properties of the Lorentz transformation. We also give their nonrelativistic expressions since the basic behaviors must be kept in the nonrelativistic expressions. Here, the approximate expressions are obtained by making use of the plane wave solutions for the Dirac wave function.**

$$\text{Fermion currents}: \begin{cases} \text{Scalar}: & \bar{\psi}\psi \simeq 1 \\ \text{Pseudoscalar}: & \bar{\psi}\gamma^5\psi \simeq \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} \\ \text{Vector}: & \bar{\psi}\gamma^\mu\psi \simeq \left(1, \dfrac{\boldsymbol{p}}{m}\right) \\ \text{Axialvector}: & \bar{\psi}\gamma^\mu\gamma^5\psi \simeq \left(\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}, \boldsymbol{\sigma}\right) \end{cases} \qquad (D.8.1)$$

**Therefore, under the parity $\hat{P}$ and time reversal $\hat{T}$ transformation, the currents behave**

$$\text{Parity } \hat{P} \quad : \begin{cases} \bar{\psi}'\psi' = \bar{\psi}\hat{P}^{-1}\hat{P}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_5\hat{P}\psi = -\bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\hat{P}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi \end{cases} \qquad (D.8.2)$$

$$\text{Time Reversal } \hat{T}: \begin{cases} \bar{\psi}'\psi' = \bar{\psi}\hat{T}^{-1}\hat{T}\psi = \bar{\psi}\psi \\ \bar{\psi}'\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_5\hat{T}\psi = \bar{\psi}\gamma_5\psi \\ \bar{\psi}'\gamma_k\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\hat{T}\psi = -\bar{\psi}\gamma_k\psi \\ \bar{\psi}'\gamma_k\gamma_5\psi' = \bar{\psi}\hat{T}^{-1}\gamma_k\gamma_5\hat{T}\psi = -\bar{\psi}\gamma_k\gamma_5\psi \end{cases} \tag{D.8.3}$$

## D.9 Trace in Physics

### D.9.1 Definition

**The trace of $N \times N$ matrix $A$ is defined as**

$$\mathrm{Tr}[A] = \sum_{i=1}^{N} A_{ii}. \tag{D.9.1}$$

**It is easy to prove**

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]. \tag{D.9.2}$$

### D.9.2 Trace in Quantum Mechanics

**The trace of the Hamiltonian $H$ becomes**

$$\mathrm{Tr}[H] = \mathrm{Tr}[UHU^{-1}] = \sum_{n=1} E_n, \tag{D.9.3}$$

**where $U$ is a unitary operator, and $E_n$ denotes the energy eigenvalue of the Hamiltonian.**

### D.9.3 Trace in $SU(N)$

**In $SU(N)$, the element $U^a$ can be described in terms of the generator $T^a$**

$$U^a = e^{i\alpha T^a} \tag{D.9.4}$$

**where the generator must be hermitian and traceless since**

$$\det U^a = \exp\left(\mathrm{Tr}\left[\ln U^a\right]\right) = \exp\left(i\alpha\,\mathrm{Tr}\left[T^a\right]\right) = 1 \tag{D.9.5a}$$

$$\mathrm{Tr}\,[T^a] = 0. \tag{$D.9.5b$}$$

**The generators of $SU(N)$ group satisfy the following commutation relations**

$$[T^a, T^b] = iC^{abc}T^c, \tag{$D.9.6$}$$

**where $C^{abc}$ denotes a structure constant. The generators are normalized such that**

$$\mathrm{Tr}\,[T^aT^b] = \frac{1}{2}\,\delta^{ab}. \tag{$D.9.7$}$$

## D.9.4 Trace of $\gamma$-Matrices and $\not p$

**Trace of $\gamma$-matrices :**

$$\mathrm{Tr}\,[1] = 4, \quad \mathrm{Tr}\,[\gamma_\mu] = 0, \quad \mathrm{Tr}\,[\gamma_5] = 0. \tag{$D.9.8$}$$

**Symbol $\not p$ :** $\qquad \not p \equiv p_\mu \gamma^\mu$

**Useful Relations:**

$$\gamma_\mu \not p \gamma^\mu = -2\not p \tag{$D.9.9$}$$

$$\not p \not q = p\cdot q - i\sigma_{\mu\nu}p^\mu q^\nu \tag{$D.9.10$}$$

$$\mathrm{Tr}\,[\not p \not q] = 4p\cdot q \tag{$D.9.11$}$$

$$\mathrm{Tr}\,[\gamma_5 \not p \not q] = 0 \tag{$D.9.12$}$$

$$\mathrm{Tr}\,[\not p_1 \not p_2 \not p_3 \not p_4] = 4\Big\{(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\Big\} \tag{$D.9.13$}$$

$$\mathrm{Tr}\,[\gamma^5 \not p_1 \not p_2 \not p_3 \not p_4] = -4i\varepsilon_{\alpha\beta\gamma\delta}\; p_1^\alpha\; p_2^\beta\; p_3^\gamma\; p_4^\delta \tag{$D.9.14$}$$

$$\begin{aligned}\mathrm{Tr}\,[\gamma^5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}] &= -4i\,[g_{\mu_1\mu_2}\varepsilon_{\mu_3\mu_4\mu_5\mu_6} - g_{\mu_1\mu_3}\varepsilon_{\mu_2\mu_4\mu_5\mu_6} \\ &+ g_{\mu_2\mu_3}\varepsilon_{\mu_1\mu_4\mu_5\mu_6} + g_{\mu_4\mu_5}\varepsilon_{\mu_1\mu_2\mu_3\mu_6} - g_{\mu_4\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_5} + g_{\mu_5\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}]\end{aligned} \tag{$D.9.15$}$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'} = -\begin{vmatrix} \delta^\mu{}_{\mu'} & \delta^\mu{}_{\nu'} & \delta^\mu{}_{\alpha'} & \delta^\mu{}_{\beta'} \\ \delta^\nu{}_{\mu'} & \delta^\nu{}_{\nu'} & \delta^\nu{}_{\alpha'} & \delta^\nu{}_{\beta'} \\ \delta^\alpha{}_{\mu'} & \delta^\alpha{}_{\nu'} & \delta^\alpha{}_{\alpha'} & \delta^\alpha{}_{\beta'} \\ \delta^\beta{}_{\mu'} & \delta^\beta{}_{\nu'} & \delta^\beta{}_{\alpha'} & \delta^\beta{}_{\beta'} \end{vmatrix} \tag{$D.9.16$}$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu'\alpha'\beta'} = -\begin{vmatrix} \delta^{\nu}{}_{\nu'} & \delta^{\nu}{}_{\alpha'} & \delta^{\nu}{}_{\beta'} \\ \delta^{\alpha}{}_{\nu'} & \delta^{\alpha}{}_{\alpha'} & \delta^{\alpha}{}_{\beta'} \\ \delta^{\beta}{}_{\nu'} & \delta^{\beta}{}_{\alpha'} & \delta^{\beta}{}_{\beta'} \end{vmatrix} \qquad (D.9.17)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha'\beta'} = -2\begin{vmatrix} \delta^{\alpha}{}_{\alpha'} & \delta^{\alpha}{}_{\beta'} \\ \delta^{\beta}{}_{\alpha'} & \delta^{\beta}{}_{\beta'} \end{vmatrix} \qquad (D.9.18)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta'} = -6\delta^{\beta}{}_{\beta'} \qquad (D.9.19)$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} = -24 \qquad (D.9.20)$$

# D.10 Lagrange Equation

**In classical field theory, the equation of motion is most important, and it is derived from the Lagrange equation. Therefore, we review briefly how we can obtain the equation of motion from the Lagrangian density.**

## D.10.1 Lagrange Equation in Classical Mechanics

**Before going to the field theory treatment, we first discuss the Lagrange equation (Newton equation) in classical mechanics. In order to obtain the Lagrange equation by the variational principle in classical mechanics, one starts from the action $S$ as defined**

$$S = \int L(q, \dot{q})\, dt, \qquad (D.10.1)$$

**where the Lagrangian $L(q, \dot{q})$ depends on the general coordinate $q$ and its velocity $\dot{q}$. At the time of deriving equation of motion by the variational principle, $q$ and $\dot{q}$ are independent as the function of $t$. This is clear since, in the action $S$, the functional dependence of $q(t)$ is unknown and therefore one cannot make any derivative of $q(t)$ with respect to time $t$. Once the equation of motion is established, then one can obtain $\dot{q}$ by time differentiation of $q(t)$ which is a solution of the equation of motion. The Lagrange equation can be obtained by requiring that the action $S$ should be a minimum with respect to the variation of $q$ and $\dot{q}$.**

$$\delta S = \int \delta L(q, \dot{q})\, dt = \int \left( \frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q} \right) dt$$

$$= \int \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt = 0, \tag{D.10.2}$$

where the surface terms should vanish. Thus one obtains the Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0. \tag{D.10.3}$$

**Hamiltonian in Classical Mechanics**

The Lagrangian must be invariant under the infinitesimal time displacement $\epsilon$ of $q(t)$ as

$$q(t+\epsilon) \to q(t) + \dot{q}\epsilon, \quad \dot{q}(t+\epsilon) \to \dot{q}(t) + \ddot{q}\epsilon + \dot{q} \frac{d\epsilon}{dt}. \tag{D.10.4}$$

Therefore, one finds

$$\delta L(q, \dot{q}) = L(q(t+\epsilon), \dot{q}(t+\epsilon)) - L(q, \dot{q}) = \frac{\partial L}{\partial q} \dot{q}\epsilon + \frac{\partial L}{\partial \dot{q}} \ddot{q}\epsilon + \frac{\partial L}{\partial \dot{q}} \dot{q} \frac{d\epsilon}{dt} = 0. \tag{D.10.5}$$

Since the surface term vanishes, one obtains

$$\delta L(q, \dot{q}) = \left[ \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \dot{q} \right) \right] \epsilon = \left[ \frac{d}{dt} \left( L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right) \right] \epsilon = 0 \tag{D.10.6}$$

where the term in bracket is a conserved quantity, and thus the Hamiltonian $H$ is defined as

$$H \equiv \frac{\partial L}{\partial \dot{q}} \dot{q} - L. \tag{D.10.7}$$

## D.10.2 Lagrange Equation for Fields

The Lagrange equation for fields can be obtained almost in the same way as the particle case. For fields, we should start from the Lagrangian density $\mathcal{L}$ and the action is written as

$$S = \int \mathcal{L} \left( \psi, \dot{\psi}, \frac{\partial \psi}{\partial x_k} \right) d^3 r \, dt, \tag{D.10.8}$$

where $\psi(x)$, $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x_k}$ are independent functional variables. Hereafter, we use the notation of $\dot{\psi}(x) \equiv \frac{\partial \psi}{\partial t}$. The Lagrange equation can be obtained

**by requiring that the action $S$ should be a minimum with respect to the variation of $\psi$, $\dot{\psi}$ and $\frac{\partial\psi}{\partial x_k}$,**

$$\delta S = \int \delta\mathcal{L}\left(\psi, \dot{\psi}, \frac{\partial\psi}{\partial x_k}\right) d^3r\, dt = \int \left(\frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\,\delta\dot{\psi} + \frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial x_k})}\,\delta\left(\frac{\partial\psi}{\partial x_k}\right)\right) d^3r\, dt$$

$$= \int \left(\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \frac{\partial}{\partial x_k}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial x_k})}\right)\delta\psi\, d^3r\, dt = 0, \qquad (D.10.9)$$

**where the surface terms are assumed to vanish. Therefore, one obtains**

$$\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} + \frac{\partial}{\partial x_k}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial x_k})}\,, \qquad (D.10.10)$$

**which can be expressed in the relativistic covariant way as**

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right). \qquad (D.10.11)$$

# D.11 Noether Current

**If the Lagrangian density is invariant under the transformation of the field with a continuous variable, then there is always a conserved current associated with this symmetry. This is called** *Noether current* **and can be derived from the invariance of the Lagrangian density and the Lagrange equation.**

## D.11.1 Global Gauge Symmetry

**The Lagrangian density which is discussed in this textbook should have the following functional dependence in general**

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi + \mathcal{L}_I\left\{\bar{\psi}\psi, \bar{\psi}\gamma_5\psi, \bar{\psi}\gamma_\mu\psi\right\}$$

**which is obviously invariant under the global gauge transformation**

$$\psi' = e^{i\alpha}\psi, \quad \psi'^\dagger = e^{-i\alpha}\psi^\dagger, \qquad (D.11.1)$$

**where $\alpha$ ia a real constant. Therefore, the Noether current is conserved in this system. To derive the Noether current conservation for the global**

**gauge transformation, one can consider the infinitesimal global transformation, that is,** $|\alpha| \ll 1$

$$\psi' = \psi + \delta\psi, \quad \delta\psi = i\alpha\psi. \tag{D.11.2a}$$

$$\psi'^{\dagger} = \psi^{\dagger} + \delta\psi^{\dagger}, \quad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}. \tag{D.11.2b}$$

**Invariance of Lagrangian Density**

**Now, it is easy to find**

$$\delta\mathcal{L} = \mathcal{L}(\psi', \psi'^{\dagger}, \partial_\mu\psi', \partial_\mu\psi'^{\dagger}) - \mathcal{L}(\psi, \psi^{\dagger}, \partial_\mu\psi, \partial_\mu\psi^{\dagger}) = 0 \tag{D.11.3a}$$

**which becomes**

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\left(\partial_\mu\psi\right) + \frac{\partial\mathcal{L}}{\partial\psi^{\dagger}}\delta\psi^{\dagger} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\delta\left(\partial_\mu\psi^{\dagger}\right) \\ &= i\alpha\left[\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\mu\psi - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\right)\psi^{\dagger} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\partial_\mu\psi^{\dagger}\right] \\ &= i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\psi^{\dagger}\right] = 0\end{aligned} \tag{D.11.3b}$$

**where the equation of motion for** $\psi$ **is employed.**

**Current Conservation**

**Therefore, one defines the current** $j^\mu$ **as**

$$j^\mu \equiv -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{\dagger})}\,\psi^{\dagger}\right] \tag{D.11.4}$$

**and one has the current conservation**

$$\partial_\mu j^\mu = 0. \tag{D.11.5}$$

**For Dirac fields, one finds the conserved current**

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{D.11.6}$$

### D.11.2 Chiral Symmetry

When the Lagrangian density is invariant under the chiral transformation,

$$\psi' = e^{i\alpha\gamma_5}\psi \tag{D.11.7}$$

then there is another Noether current. Here, $\delta\psi$ as defined in eq.(D.11.2) becomes

$$\delta\psi = i\alpha\gamma_5\psi. \tag{D.11.8}$$

Therefore, a corresponding conserved current for massless Dirac fields becomes

$$j_5^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\gamma_5\psi = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{D.11.9}$$

and we have

$$\partial_\mu j_5^\mu = 0. \tag{D.11.10}$$

The conservation of the axial vector current holds for massless field theory models.

## D.12 Hamiltonian Density

The Hamiltonian density $\mathcal{H}$ is constructed from the Lagrangian density $\mathcal{L}$. If the Lagrangian density is invariant under the translation $a^\mu$, then there is a conserved quantity which is the energy momentum tensor $\mathcal{T}^{\mu\nu}$. The Hamiltonian density is constructed from the energy momentum tensor of $\mathcal{T}^{00}$.

### D.12.1 Hamiltonian Density from Energy Momentum Tensor

Now, the Lagrangian density is given as $\mathcal{L}\left(\psi_i, \partial_0\psi_i, \frac{\partial\psi_i}{\partial x_k}\right)$. If one considers the following infinitesimal translation $a^\mu$ of the field $\psi_i$ and $\psi_i^\dagger$

$$\psi_i' = \psi_i + \delta\psi_i, \quad \delta\psi_i = (\partial_\nu\psi_i)a^\nu,$$

$$\psi_i^{\dagger\prime} = \psi_i^\dagger + \delta\psi_i^\dagger, \quad \delta\psi_i^\dagger = (\partial_\nu\psi_i^\dagger)a^\nu,$$

**then the Lagrangian density should be invariant**

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi_i', \partial_\mu\psi_i') - \mathcal{L}(\psi_i, \partial_\mu\psi_i)$$

$$= \sum_i \left[ \frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) + \frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,\delta\psi_i^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\delta(\partial_\mu\psi_i^\dagger) \right] = 0. \quad (D.12.1)$$

**Making use of the Lagrange equation, one obtains**

$$\delta\mathcal{L} = \sum_i \left[ \frac{\partial\mathcal{L}}{\partial\psi_i}\,(\partial_\nu\psi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,(\partial_\mu\partial_\nu\psi_i) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial_\nu\psi_i\right)\right] a^\nu$$

$$+ \sum_i \left[ \frac{\partial\mathcal{L}}{\partial\psi_i^\dagger}\,(\partial_\nu\psi_i^\dagger) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,(\partial_\mu\partial_\nu\psi_i^\dagger) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial_\nu\psi_i^\dagger\right)\right] a^\nu$$

$$= \partial_\mu \left[ \mathcal{L}g^{\mu\nu} - \sum_i \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger \right)\right] a_\nu = 0. \quad (D.12.2)$$

**Energy Momentum Tensor $\mathcal{T}^{\mu\nu}$**

**Therefore, if one defines the energy momentum tensor $\mathcal{T}^{\mu\nu}$ by**

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\partial^\nu\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^\dagger)}\,\partial^\nu\psi_i^\dagger \right) - \mathcal{L}g^{\mu\nu} \quad (D.12.3)$$

**then, $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is**

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0.$$

**This leads to the definition of the Hamiltonian density $\mathcal{H}$ in terms of $\mathcal{T}^{00}$**

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left( \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i)}\,\partial^0\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i^\dagger)}\,\partial^0\psi_i^\dagger \right) - \mathcal{L}. \quad (D.12.4)$$

## D.12.2 Hamiltonian Density for Free Dirac Fields

**For a free Dirac field with its mass $m$, the Lagrangian density becomes**

$$\mathcal{L} = \psi_i^\dagger\dot{\psi}_i + \psi_i^\dagger\left[i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m\gamma^0\right]_{ij}\psi_j. \quad (D.12.5)$$

**Therefore, we find the Hamiltonian density as**

$$\mathcal{H} = \mathcal{T}^{00} = \bar{\psi}_i\left[-i\gamma_k\partial_k + m\right]_{ij}\psi_j = \bar{\psi}\left[-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right]\psi. \quad (D.12.6)$$

### Hamiltonian for Free Dirac Fields

The Hamiltonian $H$ is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi}\, [-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m]\, \psi\, d^3r. \qquad (D.12.7)$$

In classical field theory, this Hamiltonian is not an operator but is just the field energy itself. However, this field energy cannot be evaluated unless one knows the shape of the field $\psi(x)$ itself. Therefore, one should determine the shape of the field $\psi(x)$ by the equation of motion in the classical field theory.

## D.12.3 Role of Hamiltonian

The classical field Hamiltonian itself is not useful. This is similar to the classical mechanics case in which one has to derive the Hamilton equations in order to calculate physical properties of the system, and the Hamilton equations are equivalent to the Lagrange equations in classical mechanics.

### Classical Field Theory

In classical field theory, the situation is just the same as the classical mechanics case. If one stays in the classical field theory, then one should derive the field equation from the Hamiltonian by the functional variational principle.

### Quantized Field Theory

The Hamiltonian of the field theory becomes important when the fields are quantized. In this case, the Hamiltonian becomes an operator, and thus one has to solve the eigenvalue problem for the quantized Hamiltonian $\hat{H}$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \qquad (D.12.8)$$

where $|\Psi\rangle$ is called *Fock state* and should be written in terms of the creation and annihilation operators of fermion and anti-fermion. The space spanned by the Fock states is called *Fock space*. In normal circumstances of the field theory models such as QED and QCD, it is practically impossible to find the eigenstate of the quantized Hamiltonian. The difficulty of the quantized field theory comes mainly from two reasons. Firstly, one has to construct the vacuum state which is composed of infinite many negative energy particles interacting with each other. The vacuum state should be the eigenstate of the Hamiltonian

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle,$$

where $E_\Omega$ denotes the energy of the vacuum and it is in general infinity with the negative sign. The vacuum state $|\Omega\rangle$ is composed of infinitely many negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger}{}_{\boldsymbol{p}}^{(s)}|0\rangle\rangle,$$

where $|0\rangle\rangle$ denotes the null vacuum state. In the realistic calculations, the number of the negative energy particles must be set to a finite value, and this should be reasonable since physical observables should not depend on the deep negative energy particles.

## D.13 Variational Principle in Hamiltonian

Now, one can derive the equation of motion by requiring that the Hamiltonian should be minimized with respect to the functional variation of the state $\psi(\boldsymbol{r})$.

### D.13.1 Schrödinger Field

When one minimizes the Hamiltonian

$$H = \int \left[ -\frac{1}{2m}\psi^\dagger \boldsymbol{\nabla}^2\psi + \psi^\dagger U \psi \right] d^3r \qquad (D.13.1)$$

with respect to $\psi(\boldsymbol{r})$, then one can obtain the static Schrödinger equation.

**Functional Derivative**

**First, one defines the functional derivative for an arbitrary function $\psi_i(\boldsymbol{r})$ by**

$$\frac{\delta\psi_i(\boldsymbol{r}')}{\delta\psi_j(\boldsymbol{r})} = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'). \qquad (D.13.2)$$

**This is the most important equation for the functional derivative, and once one accepts this definition of the functional derivative, then one can evaluate the functional variation just in the same way as normal derivative of the function $\psi_i(\boldsymbol{r})$.**

**Functional Variation of Hamiltonian**

**For the condition on $\psi(\boldsymbol{r})$, one requires that it should be normalized according to**

$$\int \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, d^3r = 1. \qquad (D.13.3)$$

**In order to minimize the Hamiltonian with the above condition, one can make use of the Lagrange multiplier and make a functional derivative of the following quantity with respect to $\psi^\dagger(\boldsymbol{r})$**

$$\begin{aligned} H[\psi] = \int \left[-\frac{1}{2m}\,\psi^\dagger(\boldsymbol{r}')\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + \psi^\dagger(\boldsymbol{r}')U\psi(\boldsymbol{r}')\right] d^3r' \\ -E\left(\int \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}')\, d^3r' - 1\right), \end{aligned} \qquad (D.13.4)$$

**where $E$ denotes a Lagrange multiplier and just a constant. In this case, one obtains**

$$\frac{\delta H[\psi]}{\delta\psi^\dagger(\boldsymbol{r})} = \int \delta(\boldsymbol{r}-\boldsymbol{r}')\left[-\frac{1}{2m}\,\boldsymbol{\nabla}'^2\psi(\boldsymbol{r}') + U\psi(\boldsymbol{r}') - E\psi(\boldsymbol{r}')\right] d^3r' = 0. \qquad (D.13.5)$$

**Therefore, one finds**

$$-\frac{1}{2m}\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) + U\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \qquad (D.13.6)$$

**which is just the static Schrödinger equation.**

## D.13.2 Dirac Field

**The Dirac equation for free field can be obtained by the variational principle of the Hamiltonian eq.(D.12.7). Below, we derive the static Dirac equation in a concrete fashion by the functional variation of the Hamiltonian.**

### Functional Variation of Hamiltonian

**For the condition on $\psi_i(\boldsymbol{r})$, one requires that it should be normalized according to**

$$\int \psi_i^\dagger \psi_i(\boldsymbol{r})\, d^3r = 1. \qquad (D.13.7)$$

**Now, the Hamiltonian should be minimized with the condition of eq.(D.13.7)**

$$H[\psi_i] = \int \psi_i^\dagger(\boldsymbol{r}) \left[-i(\gamma^0 \boldsymbol{\gamma}\cdot\boldsymbol{\nabla})_{ij} + m(\gamma^0)_{ij}\right] \psi_j(\boldsymbol{r})\, d^3r$$
$$-E\left(\int \psi_i^\dagger(\boldsymbol{r})\psi_i(\boldsymbol{r})\, d^3r - 1\right), \qquad (D.13.8)$$

**where $E$ is just a constant of the Lagrange multiplier. By minimizing the Hamiltonian with respect to $\psi_i^\dagger(\boldsymbol{r})$, one obtains**

$$(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta)\,\psi(\boldsymbol{r}) - E\psi(\boldsymbol{r}) = 0 \qquad (D.13.9)$$

**which is just the static Dirac equation for free field.**

# 付 録E Wave Propagations in medium and vacuum

The classical wave such as sound can propagate through medium. However, it cannot propagate in vacuum as is well known. This is, of course, clear since the classical wave is the chain of the oscillations of the medium due to the pressure on the density.

On the other hand, quantum wave including photon can propagate in vacuum since it is a particle. Here, we clarify the difference in propagations between the classical wave and quantum wave. The most important point is that the classical wave should be always written in terms of real functions while photon or quantum wave should be described by the complex wave function of the shape $e^{ikx}$ since it should be an eigenstate of the momentum.

This part is written as Appendix to the field theory text book "Fundamental problems in quantum field theory" published in Bentham publishers in 2013.

## E.1 What is wave ?

The sound can propagate through medium such as air or water. The wave can be described in terms of the amplitude $\phi$ in one dimension

$$\phi(x,t) = A_0 \sin(\omega t - kx) \tag{E.1}$$

where $\omega$ and $k$ denote the frequency and wave number, respectively. The dispersion relation of this wave can be written as

$$\omega = vk. \tag{E.2}$$

Here, it is important to note that the amplitude is written as the real function, in contrast to the free wave function of electron in quantum

**mechanics. In fact, the free wave of electron can be described in one dimension as**

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - kx)} \tag{E.3}$$

**which is a complex function. The electron can propagate by itself and there is no medium necessary for the electron motion.**

**What is the difference between the real wave amplitude and the complex wave function? Here, we clarify this point in a simple way though this does not contain any new physics.**

### E.1.1 A real wave function: Classical wave

**If the amplitude is real such as (E.1), then it can only propagate in medium. This can be clearly seen since the energy of the wave can be transported in terms of the density oscillation which is a real as the physical quantity. In addition, the amplitude becomes zero at some point, and this is only possible when it corresponds to the oscillation of the medium. This means that the wave function of (E.1) has nothing to do with the probability of wave object. Instead, if it is the oscillation of the medium, then it is easy to understand why one finds the point where the amplitude vanishes to zero. The real amplitude is called a classical wave since it is indeed seen in the world of the classical physics.**

### E.1.2 A complex wave function: Quantum wave

**On the other hand, the free wave function of electron is a complex function, and there is no point where it can vanish to zero. Since this is just the wave function of electron, its probability of finding the wave is always a constant $\frac{1}{V}$ at any space point of volume $V$.**

## E.2 Classical wave

The sound propagates in the air, and its propagation should be transported in terms of density wave. The amplitude of this wave can be written in terms of the real function as given in eq.(E.1). This is quite reasonable since the density wave should be described by the real physical quantity. Instead, this requires the existence of the medium (air), and the wave can propagate as long as the air exists. Here, we first write the basic wave equation in one dimension

$$\frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} \tag{E.4}$$

which is similar to the wave equation in quantum mechanics, though it is a real differential equation. Here, $v$ denotes the speed of wave.

### E.2.1 Classical waves carry their energy ?

In this case, a question may arise as to what is a physical quantity which is carried by the classical wave like sound. It seems natural that the wave carries its energy (or wave length). In fact, the transportation of the energy should be carried out by the compression of the density and successive oscillations of the medium. Therefore this is called compression wave.

### E.2.2 Longitudinal and transverse waves

Here, we discuss the terminology of the longitudinal and transverse waves, even though one should not stress its physics too much since there is no special physical meaning.

• Longitudinal wave : The sound propagates as the compressional wave, and the oscillations should be always in the direction of the wave motion. In this case, it is called longitudinal wave. This wave can be easily understood since one can make a picture of the density wave.

• Transverse wave : On the other hand, if the motion of the oscillations is in the perpendicular to the direction of the wave motion, then it is called transverse wave. The tidal wave may be the transverse wave, but its description may not be very simple since the density change may not directly be related to the wave itself.

## E.3 Quantum wave

Photon and quantum wave are quite different from the classical wave, and the quantum wave is a particle motion itself. No medium oscillation is involved. For example, a free electron moves with the velocity $v$ in vacuum, and this motion is also called "wave". The reason why we call it wave is due to the fact that the equation of motion that describes electrons looks similar to the classical wave equation of motion. Further, the solution of the wave equation can be described as $e^{ikx}$, and thus it is the same as the wave behavior in terms of mathematics. But the physical meaning is completely different from the classical wave, and quantum wave is just the particle motion which behaves as the probabilistic motion.

### E.3.1 Quantum wave (electron motion)

The wave function of a free electron in one dimension can be described as

$$\psi(x,t) = \frac{1}{\sqrt{V}} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \tag{E.5}$$

which is a solution of the Schrödinger equation of a free electron,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\boldsymbol{\nabla}^2\psi \tag{E.6}$$

where $k = \sqrt{2m\omega}$, and $V$ denotes the corresponding volume. Since the Schrödinger equation is quite similar to the wave equation in a classical sense, one calls the solution of the Schrödinger equation as a wave. However, the physics of the quantum wave should be understood in terms of the quantum mechanics, and the relation to the classical wave should not be stressed. That is, the quantum wave is completely different from the classical wave, and one should treat the quantum wave as it is. In addition, the behavior and physics of the classical wave are very complicated and it is clear that we do not fully understand the behavior of the classical wave since it involves many body problems in physics.

### E.3.2 Photon

The electromagnetic wave is called photon which behaves like a particle and also like a wave. This photon can propagate in vacuum and thus it should be considered to be a particle. Photon can be described by the vector potential $\boldsymbol{A}$.

• $\boldsymbol{A}$ is real ! : However, this $\boldsymbol{A}$ is obviously a real function, and therefore, it cannot propagate like a particle. This can be easily seen since the free Hamiltonian of photon commutes with the momentum operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$, and therefore it can be a simultaneous eigenstate of the Hamiltonian. Thus, the $\boldsymbol{A}$ should be an eigenstate of the momentum operator since the free state must be an eigenstate of momentum. However, any real function cannot be an eigenstate of the momentum operator, and thus the vector field in its present shape cannot describe the free particle state.

• Free solution of vector field : What should we do ? The only way of solving this puzzle is to quantize a photon field. First, the solution of $\boldsymbol{A}$ can be written as

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k},\lambda} \frac{1}{\sqrt{2\omega_k V}} \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \left( c^\dagger_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda} e^{ikx} \right) \tag{E.7}$$

with $kx \equiv \omega_k t - \boldsymbol{k}\cdot\boldsymbol{r}$. Here, $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ denotes the polarization vector which will be discussed later more in detail. As one sees, the vector field is indeed a real function.

• Quantization of vector field : Now we impose the following quantization conditions on $c^\dagger_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}$

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'}, \tag{E.8}$$

$$[c_{\boldsymbol{k},\lambda},\ c_{\boldsymbol{k}',\lambda'}] = 0, \qquad [c^\dagger_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = 0. \tag{E.9}$$

In this case, $c^\dagger_{\boldsymbol{k},\lambda}$, $c_{\boldsymbol{k},\lambda}$ become operators. Therefore, one should now consider the Fock space on which they can operate. This can be defined as

$$c_{\boldsymbol{k},\lambda}|0\rangle = 0 \tag{E.10}$$

$$c^\dagger_{\boldsymbol{k},\lambda}|0\rangle = |\boldsymbol{k},\lambda\rangle \tag{E.11}$$

where $|0\rangle$ denotes the vacuum state of the photon field. Therefore, if one operates the vector field on the vacuum state, then one obtains

$$\langle \boldsymbol{k},\lambda|\boldsymbol{A}(x)|0\rangle = \frac{1}{\sqrt{2\omega_k V}}\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}e^{-ikx}. \tag{E.12}$$

As one sees, this new state is indeed the eigenstate of the momentum operator and should correspond to the observables. Therefore, photon can be described only after the vector field is quantized. Thus, photon is a particle whose dispersion relation becomes

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|. \tag{E.13}$$

## E.4 Polarization vector of photon

Until recently, there is a serious misunderstanding for the polarization vector $\epsilon^\mu_{\boldsymbol{k},\lambda}$. This is related to the fact that the equation of motion for the polarization vector is not solved, and thus there is one condition missing in the determination of the polarization vector.

### E.4.1 Equation of motion for polarization vector

Now the equation of motion for $A^\mu = (A^0, \boldsymbol{A})$ without any source terms can be written from the Lagrange equation as

$$\partial_\mu F^{\mu\nu} = 0 \tag{E.14}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This can be rewritten as

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0. \tag{E.15}$$

**Now, the shape of the solution of this equation can be given as**

$$A^\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \epsilon^\mu_{\boldsymbol{k},\lambda} \left[ c_{\boldsymbol{k},\lambda} e^{-ikx} + c^\dagger_{\boldsymbol{k},\lambda} e^{ikx} \right] \tag{E.16}$$

**and thus we insert it into eq.(E.15) and obtain**

$$k^2 \epsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0. \tag{E.17}$$

**Now the condition that there should exist non-zero solution of $\epsilon^\mu_{\boldsymbol{k},\lambda}$ is obviously that the determinant of the matrix in the above equation should vanish to zero, namely**

$$\det\{k^2 g^{\mu\nu} - k^\mu k^\nu\} = 0. \tag{E.18}$$

**This leads to $k^2 = 0$, which means $k_0 \equiv \omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. This is indeed a proper dispersion relation for photon.**

### E.4.2 Condition from equation of motion

**Now we insert the condition of $k^2 = 0$ into eq.(E.17), and obtain**

$$k_\mu \epsilon^\mu = 0 \tag{E.19}$$

**which is a new constraint equation obtained from the basic equation of motion. Therefore, this condition (we call it "Lorentz condition") is most fundamental. It should be noted that the Lorentz gauge fixing is just the same as eq.(E.19). This means that the Lorentz gauge fixing is improper and forbidden for the case of no source term. In this sense, the best gauge fixing should be the Coulomb gauge fixing**

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon} = 0 \tag{E.20}$$

**from which one finds $\epsilon_0 = 0$, and this is indeed consistent with experiment.**

**• Number of freedom of polarization vector : Now we can understand the number of degree of freedom of the polarization vector. The Lorentz condition $k_\mu \epsilon^\mu = 0$ should give one constraint on the polarization vector,**

and the Coulomb gauge fixing $\boldsymbol{k}\cdot\boldsymbol{\epsilon} = 0$ gives another constraint. Therefore, the polarization vector has only two degrees of freedom, which is indeed an experimental fact.

• State vector of photon : The state vector of photon is already discussed. But here we should rewrite it again. This is written as

$$\langle \boldsymbol{k}, \lambda | \boldsymbol{A}(x) | 0 \rangle = \frac{\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}}{\sqrt{2\omega_k V}} e^{-ikx}. \tag{E.21}$$

In this case, the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ has two components, and satisfies the following conditions

$$\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda'} = \delta_{\lambda,\lambda'}, \qquad \boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\lambda} = 0. \tag{E.22}$$

### E.4.3 Photon is a transverse wave ?

People often use the terminology of transverse photon. Is it a correct expression ? By now, one can understand that the quantum wave is a particle motion, and thus it has nothing to do with the oscillation of the medium. Therefore, it is meaningless to claim that photon is a transverse wave. The reason of this terminology may well come from the polarization vector $\boldsymbol{\epsilon}_{\boldsymbol{k},\lambda}$ which is orthogonal to the direction of photon momentum. However, as one can see, the polarization vector is an intrinsic property of photon, and it does not depend on space coordinates.

• No rest frame of photon ! : In addition, there is no rest frame of photon, and therefore, one cannot discuss its intrinsic property unless one fixes the frame. Even if one says that the polarization vector is orthogonal to the direction of the photon momentum, one has to be careful in which frame one discusses this property.

In this respect, it should be difficult to claim that photon behaves like a transverse wave. Therefore, one sees that photon should be described as a massless particle which has two degrees of freedom with the behavior of a boson. There is no correspondence between classical waves and photon, and even more, there is no necessity of making analogy of photon with the classical waves.

## E.5 Poynting vector and radiation

**We have clarified that the propagation of the real function requires some medium which can make oscillations. Here, we discuss the Poynting vector how it appears in physics, and show that it cannot propagate in vacuum at all. Also, we present a brief description of the basic radiation mechanism how photon can be emitted.**

### E.5.1 Field energy and radiation of photon

**Before discussing the propagation of the Poynting vector, we should first discuss the mechanism of the radiation of photon in terms of classical electrodynamics. The interaction Hamiltonian can be written as**

$$H_I = -\int \boldsymbol{j} \cdot \boldsymbol{A}\, d^3r \tag{E.23}$$

**which should be a starting point of all the discussions. Now, we make a time derivative of the interaction Hamiltonian and obtain**

$$W \equiv \frac{dH_I}{dt} = -\int \left[ \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A} + \boldsymbol{j} \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right] d^3r. \tag{E.24}$$

**Since we can safely set $A^0 = 0$ in this treatment, we find**

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}. \tag{E.25}$$

**Therefore, we can rewrite eq.(E.24) as**

$$W = \int \boldsymbol{j} \cdot \boldsymbol{E}\, d^3r - \int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A}\, d^3r. \tag{E.26}$$

**Defining the first term of eq.(E.24) as $W_E$, we can rewrite $W_E$ as**

$$W_E \equiv \int \boldsymbol{j} \cdot \boldsymbol{E}\, d^3r = -\frac{d}{dt}\left[ \int \left( \frac{1}{2\mu_0}|\boldsymbol{B}|^2 + \frac{\varepsilon_0}{2}|\boldsymbol{E}|^2 \right) d^3r \right] - \int \boldsymbol{\nabla} \cdot \boldsymbol{S}\, d^3r \tag{E.27}$$

**which is just the energy of electromagnetic fields.**

### E.5.2 Poynting vector

**Here, the last term of eq.(E.27) is Poynting vector $\boldsymbol{S}$ as defined by**

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} \tag{E.28}$$

**which is connected to the energy flow of the electromagnetic field. This Poynting vector is a conserved quantity, and thus it has nothing to do with the electromagnetic wave. In addition, it is a real quantity, and thus there is no way that it can propagate in vacuum. In addition, the Poynting vector cannot be a target of the field quantization, and thus it always remains classical since it is written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. However, there is still some misunderstanding in some of the textbooks on Electromagnetism, and therefore, one should be careful for the treatment of the Poynting vector.**

**• Exercise problem:　Here, we present a simple exercise problem of circuit with condenser with $C$ (disk radius of $a$ and distance of $d$) and resistance with $R$. The electric potential difference $V$ is set on the circuit. In this case, the equation for the circuit can be written as**

$$V = R\frac{dQ}{dt} + \frac{Q}{C}.$$

**This can be easily solved with the initial condition of $Q = 0$ at $t = 0$, and the solution becomes**

$$Q = CV\left(1 - e^{-\frac{t}{RC}}\right).$$

**Therefore, the electric current $J$ becomes**

$$J = \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}.$$

**In this case, we find the electric field $\boldsymbol{E}$ and the displacement current $\boldsymbol{j}_d$**

$$\boldsymbol{E} = \frac{Q}{\pi a^2}\boldsymbol{e}_z = \frac{VC}{\varepsilon_0 \pi a^2}\left(1 - e^{-\frac{t}{RC}}\right)\boldsymbol{e}_z \tag{E.29}$$

$$\boldsymbol{j}_d = \frac{\partial \boldsymbol{E}}{\partial t} = \frac{V}{R\pi a^2}e^{-\frac{t}{RC}}\boldsymbol{e}_z. \tag{E.30}$$

**Thus, the magnetic field $\boldsymbol{B}$ becomes**

$$\boldsymbol{B} = \frac{i_d\, r}{2}\boldsymbol{e}_\theta = \frac{r}{2\pi a^2 R}\ e^{-\frac{t}{RC}}\ \boldsymbol{e}_\theta$$

**where $\int_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 i_d \pi r^2$ is used. Therefore, the Poynting vector at the surface (with $r = a$ ) of the cylindrical space of the disk condenser becomes**

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} = -\frac{V^2}{2\pi a R d}\, e^{-\frac{t}{RC}} \left(1 - e^{-\frac{t}{RC}}\right) \boldsymbol{e}_r.$$

**It should be noted that the energy in the Poynting vector is always flowing into the cylindrical space. Therefore, the electric field energy is now accumlated in the cylindrical space. There is, of course, no electromagnetic wave radiation, and in fact, the Poynting vector is the flow of field energy, and has nothing to do with the electromagnetic wave.**

### E.5.3 Emission of photon

**The emission of photon should come from the second term of eq.(E.26) which can be defined as $W_R$ and thus**

$$W_R = -\int \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{A}\, d^3 r. \tag{E.31}$$

**In this case, we can calculate the $\frac{\partial \boldsymbol{j}}{\partial t}$ term by employing the Zeeman effect Hamiltonian with a uniform magnetic field of $\boldsymbol{B}_0$**

$$H_Z = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0. \tag{E.32}$$

**The relevant Schrödinger equation for electron with its mass $m_e$ becomes**

$$i\frac{\partial \psi}{\partial t} = -\frac{e}{2m_e}\boldsymbol{\sigma} \cdot \boldsymbol{B}_0\, \psi. \tag{E.33}$$

**Therefore, we find**

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e}{m_e}\left[\frac{\partial \psi^\dagger}{\partial t}\hat{\boldsymbol{p}}\psi + \psi^\dagger \hat{\boldsymbol{p}}\frac{\partial \psi}{\partial t}\right] = -\frac{e^2}{2m_e^2}\boldsymbol{\nabla} B_0(\boldsymbol{r}). \tag{E.34}$$

**In order to obtain the photon emission, one should quantize the field $\boldsymbol{A}$ in eq.(E.31).**

**• Field quantization : The field quantization in electromagnetic interactions can be done only for the vector potential $\boldsymbol{A}$. The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are classical quantities which are defined before the field quantization.**

## E.6 Gravitational wave

**People often discuss the gravitational wave which is supposed to come from the Einstein equation. In this case, one sees that the equation for the metric tensor is all real, and thus the solution of this equation must be also real. Therefore, the gravitational wave, if at all exists, is a real function, and thus it cannot propagate in vacuum unless one believes the**

aether hypothesis.

• No quantization of gravity : In addition, there is no physical meaning to quantize the metric tensor and therefore, there is no chance that the gravitational wave propagates in vacuum.

### E.6.1 General relativity

Since we treat the gravitational wave, we should make a comment on the general relativity. Einstein invented the general relativity which is the second order differential equation for the metric tensor $g^{\mu\nu}$. A question may arise as to why the general relativity can be related to the gravitational theory. This reason is simply because Einstein claimed that he had proved the gravitational Poisson equation should be derived from the general relativity at the weak gravitational limit. However, in his proof, he assumed the following strange equation

$$g^{00} \simeq 1 + 2\phi \tag{E.35}$$

where $\phi$ denotes the gravitational field. Because of this equation (E.35), he could derive the gravitational Poisson equation

$$\boldsymbol{\nabla}^2 \phi(\boldsymbol{r}) = 4\pi G \rho(\boldsymbol{r}) \tag{E.36}$$

where $G$ and $\rho$ denote the gravitational constant and the density, respectively.

• Eq.(E.35) is correct ? : Here, we show that eq.(E.35) is not only strange but simply incorrect. In order to do so, we should examine the physical meaning of the equation $g^{00} \simeq 1 + 2\phi$. We should notice that 1 (unity) in the right hand side of eq.(E.35) is a simple number. This is clear since the metric tensor is just the coordinate system itself. However, the gravitational field $\phi$ is a dynamical variable, and therefore this summation of two different categories is simply meaningless.

• No connection between general relativity and gravity : By now it should be clear that the general relativity has nothing to do with gravity. It is a theory for the coordinate system (metric tensor), but it is not a theory for nature.

**Note :**
**The new gravitational theory is explained in detail in Chapter 6 in the text book of**
**"Fundamental problems in quantum field theory" .**

## Reference :

**Fundamental Problems in Quantum Field Theory**
**T. Fujita and N. Kanda, Bentham Publishers, 2013**

# 関連図書


[1] K. Nishijima, “Fields and Particles”, (W.A. Benjamin, INC, 1969)

[2] S.L. Adler, “Axial-Vector Vertex in Spinor Electrodynamics,” Phys. Rev. vol. 177, pp. 2426−2438, Jan. 1969.

[3] T. Fujita and N. Kanda, “Fundamental Problems in Quantum Field Theory” (Bentham Publishers, 2013)

[4] M. Matsuda, “Exact Evaluation of Triangle Diagrams”, APIT-A-2023-001

[5] L. D. Landau, Dokl. Akad. Nawk., USSR vol. 60, pp. 207, 1948.

[6] C. N. Yang, “Selection Rules for the Dematerialization of a Particle into Two Photons,” Phys. Rev. vol. 77, pp. 242−245, Jan. 1950.

[7] A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie,” Annalen der Physik vol. 49, pp. 769−822, März. 1916.

[8] Dirac, AIP Conference Proceedings 74, 129 (1981)

[9] R.P. Feynman, “Space-Time Approach to Quantum Electrodynamics,” Phys. Rev. vol. 76, pp. 769−789, Sep. 1949.

[10] G. ’t Hooft and M. Veltman, “Regularization and renormalization of gauge fields,” Nucl. Phys. B. vol. 44, pp. 189−213, 1972.

[11] R. Feynman and M. Gell-Mann, “Theory of the Fermi Interaction,” Phys. Rev. vol. 109, pp. 193−198, Jan. 1958.

[12] A. Salam, In Elementary particle physics (Nobel Symposium No. 8), Ed. N. Svartholm; Almqvist and Wilsell, Stockholm (1968)

[13] S. Weinberg, “A Model of Leptons,” Phys. Rev. Lett. vol. 19, pp. 1264–1266, Nov. 1967.

[14] P.W. Higgs, “Broken symmetries, massless particles and gauge fields,” Phys. Lett. vol. 12, pp. 132–133, Sep. 1964.

[15] M. Matsuda, T. Sakamoto and T. Fujita
“Electromagnetic Interaction of Complex Scalar Fields”,
APIT-A-2022-001

[16] T. Fujita, “Symmetry and Its Breaking in Quantum Field Theory” (Nova Science Publishers, 2011, 2nd edition)

[17] Y. Nambu and G. Jona-Lasinio, “Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I,” Phys. Rev. vol. 122, pp. 345–358, Apr. 1961.

[18] N.N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947)

[19] W. Thirring, Ann. Phys. (N.Y) 3, 91 (1958)

[20] T. Fujita, M. Hiramoto, T. Homma and H. Takahashi, “New Vacuum of Bethe Ansatz Solutions in Thirring Model,” J. Phys. Soc. Japan vol. 74, pp. 1143–1149, Jan. 2005.

[21] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Mechanics”, (McGraw-Hill Book Company,1964)

[22] J.J. Sakurai, ”Advanced Quantum Mechanics”, (addison-Wesley,1967)

[23] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, ”Relativistic Quantum Theory”, (Pergamon Press, 1974)

[24] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Fields”, (McGraw-Hill Book Company,1965)

[25] J. Schwinger, “Quantum Electrodynamics. I. A Covariant Formulation,” Phys. Rev. vol. 74, pp. 1439–1461, Nov. 1948.

[26] S. Tomonaga, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields," Prog. Theor. Phys. vol. 1, pp. 27–42, Aug-Sept. 1946.

[27] G. 't Hooft and M. Veltman, "Combinatorics of gauge fields," Nucl. Phys. B. vol. 50, pp. 318–353, 1972.

[28] T. Fujita, N. Kanda, H. Kato, H. Kubo, Y. Munakata, S. Oshima and K. Tsuda, "New Renormalization Scheme of Vacuum Polarization in QED", arXiv:0901.3421

[29] W. Pauli and F. Villars, "On the Invariant Regularization in Relativistic Quantum Theory," Rev. Mod. Phys. vol. 21, pp. 434–444, Sep. 1949.

[30] J.C. Ward, "An Identity in Quantum Electrodynamics," Phys. Rev. vol. 78, pp. 182–182, Apr. 1950.

[31] N. Kanda, "Light-Light Scattering", hep-ph/1106.0592

[32] F. Mandl and G. Shaw, "Quantum Field Theory", (John Wiley & Sons, 1993)

[33] T. Fujita and N. Kanda, "A Proposal to Measure Photon-Photon Scattering", hep-ph/1106.0465

[34] H. Euler, " Über die Streuung von Licht an Licht nach der Diracshen Theorie," Annalen der Physik 1936; 26: 398–448.

[35] W. Heisenberg, "Bemerkungen zur Diracschen Theorie des Positrons," Zeits. f. Physik. vol. 90, pp. 209–231, Juni. 1934.

[36] W. Heisenberg and H. Euler, "Folgerungen aus der Diracschen Theorie des Positrons," Zeits. f. Physik vol. 98, pp. 714–732, Dezember 1935.

[37] R. Karplus and M. Neuman, "The Scattering of Light by Light," Phys. Rev. vol. 83, pp. 776–784, Aug. 1951.

[38] T. Fujita and N. Kanda, "Tomonaga's Conjecture on Photon Self-Energy", physics.gen-ph/1102.2974

[39] F. Moulin, D. Bernard, and F. Amiranoff, "Photon-photon elastic scattering in the visible domain," Z. Phys. C. vol. 72, pp. 607–611, 1996.

[40] D. Bernard, F. Moulin, F. Amirano, A. Braun4, J.P. Chambaret, G. Darpentigny, G. Grillon, S. Ranc, and F. Perrone, "Search for stimulated photon-photon scattering in vacuum," Eur. Phys. J. D. vol. 10, pp. 141–145, 2000.

[41] I. Larin et al. , "New Measurement of the $\pi^0$ Radiative Decay Width," Phys. Rev. Lett. vol. 106, 162303, Apr. 2011.

[42] C. Itzykson and J.B. Zuber, "Quantum Field Theory", (McGraw-Hill, 1980)

[43] N. Kanda, R. Abe, T. Fujita and H. Tsuda, "$Z^0$ decay into two photons", hep-ph/1109.0926

[44] C. Amsler et al. (Particle Data Group), Physics Letters B vol. 667, pp. 1, 2008.

[45] J. Schwinger, "Gauge Invariance and Mass. II," Phys. Rev. vol. 128, pp. 2425–2429, Dec. 1962.

[46] N. S. Manton, "The Schwinger Model and Its Axial Anomaly," Ann. Phys. vol. 159, pp. 220–251, 1985.

[47] C.N. Yang and R.L. Mills, "Conservation of Isotopic Spin and Isotopic Gauge Invariance," Phys. Rev. vol. 96, pp. 191–195, Oct. 1954.

[48] M. Lacombe, B. Loiseau, J.M. Richard, R. Vinh Mau, J. Cote, P. Pires, and R. de Tourreil, "Parametrization of the Paris N-N potential," Phys. Rev. C. vol. 21, pp. 861–873, Mar. 1980.

[49] M. Lacombe, B. Loiseau, J.M. Richard, R. Vinh Mau, J. Cote, P. Pires, and R. de Tourreil, "Parametrization of the deuteron wave function of the Paris N-N potential," Phys. Lett. B. vol. 101, pp. 139–140, May. 1981.

[50] R. Machleidt, K. Holinde, and Ch. Elster, "The bonn meson-exchange model for the nucleon?nucleon interaction," Phys. Rep. vol. 149, pp. 1−89, May. 1987.

[51] R. Machleidt, Adv. Nucl. Phys. 1989; 19: 189.

[52] A. Bohr and B.R. Mottelson, "Nuclear Structure" (Vol. 1), (World Scientific,1998)

[53] D. Kiang, M.A. Preston and P. Tip, "One-Boson-Exchange Potential and Nuclear Matter," Phys. Rev. vol. 170, pp. 907−915, Jun. 1968.

[54] R.D. Haracz and R.D. Sharma, "Two-Boson-Exchange Effects in Nucleon-Nucleon Scattering," Phys. Rev. vol. 176, pp. 2013−2018, Dec. 1968.

[55] M.H. Partovi and E.L. Lomon, "Field-Theoretical Nucleon-Nucleon Potential," Phys. Rev. D. vol. 2, pp. 1999−2032, Nov. 1970.

[56] W.R. Wortman, "Two-Pion-Exchange Contributions to Nucleon-Nucleon Scattering," Phys. Rev. vol. 176, pp. 1762−1768, Dec. 1968.

[57] F. Gross, "Relativistic quantum mechanics and field theory", (John Wiley & Sons, 1993)

[58] T. Fujita, N. Kanda and S. Oshima, "Nuclear Potential with Two Pion Exchange", arXiv:1209.3067

[59] T. Fujita, M. Hiramoto and H. Takahashi, "Boson after symmetry breaking in quantum field theory", in *Focus on Boson Research* (Nova Science Publisher, 2005)

[60] T. Fujita, N. Kanda and H. Tsuda, "Higgs Mechanism and New Propagator of Massive Vector Bosons," J. Mod. Phys. vol. 3, pp. 619−624, Jul. 2012.

[61] J. Goldstone, Nuovo Cimento vol. 19, pp. 154, 1961.

[62] J. Goldstone, A. Salam and S. Weinberg, "Broken Symmetries," Phys. Rev. vol. 127, pp. 965−970, Aug. 1962.

[63] T. Fujita, T. Kobayashi, M. Hiramoto, H. Takahashi, “Non-equivalence between Heisenberg XXZ spin chain and Thirring model,” Eur. Phys. J. C. vol. 39, pp. 511−518, Jan. 2005.

[64] S.L. Glashow, “Partial-symmetries of weak interactions,” Nucl. Phys. vol. 22, pp. 579−588, Feb. 1961.

[65] T. Fujita and N. Kanda, “No Anomaly and New Renormalization Scheme,” J. Mod. Phys. vol. 3, pp. 665−681, Aug. 2012.

[66] T. Fujita, A. Kusaka, K. Tsuda and S. Oshima, “Unphysical Gauge Fixing in Higgs Mechanism”, arXiv:0806.2957

[67] T. Fujita, “Physical observables in gauge field theory” in *New Fundamentals in Fields and Particles*, Transworld Research Network (2009), p.1−p.20

[68] E. Fermi, ”Elementary Particles”, (Yale University Press, New Haven, 1951)

[69] S.S Gershtein and J.B. Zeldovich, JETP vol. 2, pp. 576, 1955.

[70] T.D. Lee and C.N. Yang, “Question of Parity Conservation in Weak Interactions,” Phys. Rev. vol. 104, pp. 254−258, Oct. 1956.

[71] C.S. Wu, “Parity Experiments in Beta Decays,” Rev. Mod. Phys. vol. 31, pp. 783−790, 1959.

[72] T. Fujita, “Physical observables in path integral formulation”, in *New Fundamentals in Fields and Particles*, Transworld Research Network (2009), p.31−p.45

[73] T. Fujita, “Quantum Gravity without General Relativity”, arXiv:0804.2518

[74] S. Kanemaki, A. Kusaka, S. Oshima and T. Fujita, “Problems of scalar bosons”, in *New Fundamentals in Fields and Particles*, ed. by T. Fujita, Transworld Research Network (2009), p.47−p.60

[75] C.W. Misner, K.S. Thorne and J.A. Wheeler, “Gravitation” (Freeman, 1973)

[76] T. Fujita and N. Kanda, "Novel Solution of Mercury Perihelion Shift", physics.gen-ph/0911.2086

[77] T. B. Bahder, "Relativity of GPS measurement," Phys. Rev. D. vol. 68, 063005, Sep. 2003.

[78] B.W. Parkinson and J.J. Spilker, eds., Global Positioning System, Progress in Astronautics and Aeronautics, 163, 164 (1996)

[79] T. Fujita and N. Kanda, "Physics of Leap Second", physics.gen-ph/0911.2087

[80] T. Fujita and S. Oshima, "Electric dipole moments of neutron-odd nuclei ," J. Phys. G. vol. 39, pp. 095106(6pp), Jul. 2012.

[81] K. Muto, "Time reversal invariance and EDM in atomic system", in *New Fundamentals in Fields and Particles*,
ed. by T. Fujita, Transworld Research Network (2008), p.119−p.155

[82] S. Oshima, "Electric dipole moments (EDM) of ionic atoms," Phys. Rev. C. vol. 81, pp. 038501(4pp), Mar. 2010.

[83] P. G. Harris, C. A. Baker, K. Green, P. Iaydjiev, S. Ivanov, D. J. R. May, J. M. Pendlebury, D. Shiers, K. F. Smith, M. van der Grinten and P. Geltenbort, "New Experimental Limit on the Electric Dipole Moment of the Neutron," Phys. Rev. Lett. vol 82, pp. 904−907, Sep. 1999.

[84] K. F. Smith, N. Crampin, J. M. Pendlebury, D. J. Richardson, D. Shiers, K. Green, A. I. Kilvington, J. Moir, H. B. Prosper and, D. Thompson, N. F. Ramsey, B. R. Heckel and, S. K. Lamoreaux, P. Ageron and, W. Mampe and A. Steyerl, "A search for the electric dipole moment of the neutron," Phys. Lett. B. vol. 234 pp. 191−196, Jan. 1990.

[85] M. Kobayashi and T. Maskawa, "CP-Violation in the Renormalizable Theory of Weak Interaction," Prog. Theor. Phys. vol. 49 pp. 652−657, 1973.

[86] E.P. Shabalin, Sov. J. Nucl. Phys. vol. 28, pp. 75, 1978.

[87] E.P. Shabalin, Sov. J. Nucl. Phys. vol. 31, pp. 864, 1980.

[88] T.P. Cheng and L.F. Li, "Guage Theory of Elementary Particle Physics", (Oxford University Press, 1988)

[89] H.A. Bethe, "The Electromagnetic Shift of Energy Levels," Phys. Rev. vol. 72, pp. 339−341, Aug. 1947.

[90] M.I. Eides, H. Grotch and V.A. Shelyuto, "Theory of light hydrogen-like atoms," Phys. Rep. vol. 342, pp. 63−261, Feb. 2001.

[91] E.A. Uehling, "Polarization Effects in the Positron Theory," Phys. Rev., vol. 48, pp. 55−63, Jul. 1935.

[92] R. Pohl et al. , "The size of the proton ," Nature Letters vol. 466, pp. 213−217, Jul. 2010.

[93] T. Sakamoto, T. Fujita, N. Kanda, H. Kato and K. Tsuda, "A Puzzle of Lamb Shifts in Muonic Hydrogen", to be published

[94] K.A. Woodle, A. Badertscher, V.W. Hughes, D.C. Lu, M. W. Ritter, M. Gladisch, H. Orth, G. zu Putlitz, M. Eckhause, J. Kane and F.G. Mariam, "Measurement of the Lamb shift in the n = 2 state of muonium," Phys. Rev. A. vol. 41, pp. 93−105, Jan. 1990.