第４章　対称性とその物理

自然界には様々な対称性があり，それらの対称性は物理学において極めて重要な役割を果たしている．対称性とは時間・空間の変数に対して何らかの変換を施した時に，その系がもつ不変性のことである．この場合，その変換の対称性は自然界の身近な言葉で表現された現象と関係つけられている．場の理論ではさらに進んで状態関数に対する変換を行い，この変換に対する系の不変性により様々な対称性が求められている．一般的に言って，系がある対称性をもっていると理論的に取扱いが簡単になり，また様々な観点から物理の理解を深めることができる．さらに、対称性は物理学の描像を描く上で非常に有用であり、その本質を理解するときの指針にもなっている．

物理学における基本的な対称性に関連して二つの原理がある．その一つとして相対性原理があり，これはローレンツ変換に対する対称性と関係している．この対称性はどの場の理論模型も必ず満たさなければならないものである．もう一つは Curie の原理として知られているものであり，どの模型もその運動力学の結果は必ず，Curie の原理を満たさなければならない．これは一種の因果律に関係していて，自然界では常に守られている．

4.1　変換と不変性

それでは「何らかの変換」とは具体的にはどのようなものであろうか？物理学を学んでいる読者にとって，この部分はよく知られていることと思われるが，対称性の問題を整理して考えるための一助にはなるものと考えている．

4.1.1　ラグランジアン

ここで現代物理学において，その記述のすべての基礎となっているラグランジアンについて少しだけ解説しておこう．まずは古典力学を取り扱うことになるが，この場合もっとも単純な系は２粒子系である．これは明らかで，１粒子

だけの状態は相互作用していないので自由粒子の状態であり，対称性を議論する場合の対象外となっている．２体系の場合のラグランジアン L は $L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1{}^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2{}^2 - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$ と書かれている．この右辺の第１，２項が運動エネルギーであり，第３項がポテンシャルである．

4.1.2　ラグランジアン密度

一方，場の理論はラグランジアン密度によって記述されていて，これは場の量 (場の変数，状態関数とも呼ばれる) で書かれている．実際，量子電磁力学，量子色力学，弱い相互作用そして重力のラグランジアン密度すべてが場の変数で書かれている．この場合，系の対称性は場の変数を変換した時，ラグランジアン密度が不変になることにより決定されている．そして様々な変換に対して，それぞれに対応した対称性が議論されている．その意味で古典力学と比べると，場の理論の対称性は非常に多様であることがわかっている．

4.1.3　変換と対称性 – 古典力学

まず，古典力学における対称性について簡単に議論しよう．古典力学ではラグランジアンが座標の関数になっているため，この系の対称性は座標の変換に関係している．

• **空間の平行移動：**　座標を $\boldsymbol{r} \Rightarrow \boldsymbol{r} + \boldsymbol{c}$ と定数ベクトル $\boldsymbol{c}$ だけ平行移動した場合，ラグランジアン L は不変である．これは明らかで運動エネルギーの部分は平行移動の変換に対して変わらないし，またポテンシャルの部分も定数部分が打ち消し合って不変である．ラグランジアン L が $\boldsymbol{r} \Rightarrow \boldsymbol{r} + \boldsymbol{c}$ の平行移動に不変だと、その系の運動量が保存量になっていることが証明される．

• **時間の平行移動：**　時間を $t \Rightarrow t + d$ と定数 d だけ平行移動した場合，ラグランジアン L は明らかに不変である．この場合，エネルギーが保存量になっていることが証明される．

• **空間回転：**　ある角度だけ座標を回転したとき，ポテンシャルが動径部分のみの関数の場合，ラグランジアン L が不変となっている．この時，角運動量が保存量になっていることが証明される．座標の回転を数式で書くことは簡単ではあるが，それ程意味があることではないのでここでは省略しよう．

4.1.4　変換と対称性 – 場の理論

場の理論においてはラグランジアン密度が場の量の関数になっているため，この場の量の変換に対する不変性を議論することになっている．

• **ローレンツ変換：**　すべてのラグランジアン密度はローレンツ変換に対して不変である必要がある．このため，どのラグランジアン密度もローレンツスカラーとなっている．これは絶対条件である．

• **時間・空間の平行移動：**　場の理論の場合，場の変数 $\psi(t, \boldsymbol{r})$ は時間と空間の関数である．ここで時間と空間を微小量だけ平行移動するとそれに応じて場の変数も変更を受ける．この変換に対してラグランジアン密度が不変であることを要請すると，エネルギー・運動量テンソルが保存量となることが証明される．これは後でもう少し詳しく議論しよう．

• **グローバルゲージ対称性：**　状態関数 ψ を $\psi' = e^{i\alpha}\psi$ と変換してその模型のラグランジアン密度が不変である場合，この模型はグローバルゲージ対称性があるという．この不変性はその系の電荷保存を保証している．

• **グローバルカイラルゲージ対称性：**　状態関数 ψ を $\psi' = e^{i\alpha\gamma_5}\psi$ と変換してその模型のラグランジアン密度が不変である場合，この模型はカイラル対称性があるという．もう少し詳しくは，後程，解説しよう．

• **SU(N) 変換対称性：**　状態関数が N 個の成分を持っているとして，それを SU(N) の群 U によって $\psi' = U\psi$ と変換することを考えよう．ここでラグランジアン密度がこの変換に対して不変である場合，この系は SU(N) 不変性を持っているという．この場合，群の表現論を用いて物理の状態を特定することができるため，この対称性は非常に有効であることが知られている．

4.2　相対性原理

相対性原理とそれに関連するローレンツ変換に対する対称性に関しては，どの場の理論の教科書にも詳しく議論されている．従って，ここではこの解説はしないが，基本的なことを少しだけ書いておこう．相対性原理の出発点は，座標系をどこに持って来たら良いかという問い掛けから始まっている．例えば，地球は太陽系の中にいるが，しかしその太陽は銀河系の外側を高速で周回している．その銀河系は宇宙の膨張にあわせて運動している．そうだとすると，どの座標系を取ったらよいのかわからなくなる．相対性原理とは慣性系である限り，どの慣性系をとっても物理的な観測量は同じであるという要請である．現在までのところ，これと矛盾する自然現象は観測されていないし，これは原理として十分意味があるものと考えてよい．

4.2.1　相対性理論

ある慣性系から他の慣性系へ移るときに，いかなる場の方程式も満たさなければならない変換則がある．それが相対性理論である．この変換則のことをローレンツ変換と呼ぶ．現在の場の理論はすべてこのローレンツ変換に対する不変性を持っている．ローレンツ不変な模型でないと，ある系で計算した結果が他の適当な慣性系で計算した結果と一致するとはかぎらなくなってしまうので模型として不適当である．その意味で，ローレンツ変換に対する不変性はどの模型も絶対に守らなければならないものである．

4.2.2　ローレンツ不変なラグランジアン密度

実例として具体的なラグランジアン密度を書いておこう．このラグランジアン密度は量子電磁力学のものである．

$$\mathcal{L} = i\bar{\psi}\partial_\mu\gamma^\mu\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

このラグランジアン密度の解説はしないが，すべての項がローレンツ変換に対してスカラーになっている．

4.3 Curie の原理

１９世紀の終わりにはCurieが自然界の対称性に注目して，非対称性があるとしたらどのようにそれが実現されるのかという問題を研究している．彼はCurie の原理を提唱しているが，それは「非対称な現象はその原因がない限り自然界では起こらない」というものである．この原理が信用されている主な理由は，その原理自体が観測に基づいて求められているからである．

4.3.1 圧電効果と逆圧電効果

例えば，ある種の誘電体に外から機械的圧力を加えると，それに応じて物質の対称性が壊れて非対称になる．このことが電束密度を誘起して電気的な力が誘発されることになり，これが圧電効果の現象である．これは非対称性が起こるのは外力が原因であるという Curie の原理そのものを表している．この圧電効果の重要性はよく知られており，実際，最近のタッチパネルによる情報の伝達は多種多様に利用されている．

さらに逆圧電効果も知られている．これは，電圧を掛けて対称性を少し変えると，それを戻そうとする力が働き，例えば，水晶の振動になるのである．この原理をうまく利用して実用化したのがクォーツ時計である．

4.3.2 場の理論における Curie の原理

場の理論の模型でも Curie の原理が成り立っているべきであり，実際，原因がないのに系の対称性が自然に破れると言う事はあってはならない．ところが自発的対称性の破れという理論模型は，原因がないのに対称性が破れると言う事を主張している．しかし直感的にも，自然に対称性が破れたとしたら，何か特別なこと (計算ミス) が起こったと考えざるを得ない．南部達が主張した自発的対称性の破れの概念は，現実にはその理論計算の途中で，ある「重要な近似」を使ったことが対称性が破れたように見えた原因であることが証明されている．従って，場の理論模型においても Curie の原理は例外なく成り立っている事が現在はわかっている．厳しい言い方をすれば，自発的対称性の破れを提唱した人達が Curie の原理をもう少し深く理解していたならば，あのような愚かな理論を提唱することはなかったことであろう．

4.4　場の理論の対称性

場の理論の模型には様々な対称性があり，そのうちローレンツ変換に対する対称性は最も基本的なものである．それ以外にも，時間・空間の平行移動に対するラグランジアン密度の不変性から，重要な対称性が知られている．時間も空間も場に対しては一様であるべきであることから，この平行移動の対称性はごく自然なものである．実際，この対称性は運動量とエネルギー保存則に関係していて自然界でよく成り立っていることがわかっている．

4.4.1　対称性と保存則

この平行移動の対称性から，エネルギーと運動量の保存則が導かれている．この保存則は エネルギー・運動量テンソルと呼ばれている量が保存されることから，証明されるものである．従って，この物理量は非常に重要な量になっている．式を書いてもあまり意味があるとは思えないが，重要なのでその定義だけを書いておこう．この エネルギー・運動量テンソル $T^{\mu\nu}$ はある場 ψ から作られているラグランジアン密度 $\mathcal{L}$ に対して次のように定義されている．

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\partial^\nu \psi^\dagger - \mathcal{L} g^{\mu\nu}$$

ここで $\partial_\mu T^{\mu\nu} = 0$ が成り立っていて，これがエネルギーと運動量の保存則そのものである．これは場の理論模型にとって最も重要な物理量となっている．

• **一般相対性理論のエネルギー・運動量テンソル：**　ここで一つコメントをしておこう．一般相対性理論の方程式においてはその右辺にはエネルギー・運動量テンソルが現れている．これはある意味で非常に不思議ではある．それは一般相対性理論は粒子描像であり、場の理論ではないためエネルギー・運動量テンソルのような場の量を作ることは基本的には不可能である．従って，この一般相対性理論におけるエネルギー・運動量テンソルとはどのようなものであるのかを検証する必要がある．これはその定義を見るとわかることであるが，実は，星などの分布関数から作られたものである．従って，これは到底，基本的な物理量ではない．さらに，分布関数がどのように作られるのかは一般相対性理論が関知しない事でもあり，この点からみても一般相対性理論が物理学の基礎方程式にはなり得ない事がよくわかるものである．

4.4.2 グローバルゲージ対称性

場の理論の模型における対称性のうちで，グローバルゲージ対称性と呼ばれているものがある．それは状態関数 ψ を定数位相だけ回転したとき，そのラグランジアン密度が不変であるときに生じる対称性である．

• **グローバルゲージ変換：** ここで状態関数 ψ を $\psi' = e^{i\alpha}\psi$ と変換してその模型のラグランジアンが不変である場合，この模型はグローバルゲージ対称性があるという．現在までによく議論されてきたすべての模型は，この対称性を持っている．その場合，この対称性に付随してその模型は電荷保存が成り立っている．さらに，この対称性が自然に破れると言う事はないし，自発的対称性の破れとしてこのグローバルゲージ対称性が議論されることもない．

• **カイラルゲージ変換：** 自発的対称性の破れと関連して議論されている変換がカイラルゲージ変換であり，状態関数 ψ は $\psi' = e^{i\alpha\gamma_5}\psi$ と変換される．この変換に対して、質量項があるとその系にはこの対称性はないので現実的な模型ではない。実際，フェルミオンの模型で質量項がないとしたら，その模型には定数スケールがないため，自然界の記述には適していない．例えば，量子電磁力学においてはフェルミオンの質量により，次元を持つすべての物理的観測量が表現されている．従って，質量項のないフェルミオン模型は自然界を記述しようとする模型としては不適当である．

4.4.3 対称性とその破れ

一方において，南部達は非常に特殊な模型を考えて，カイラル対称性について議論した．それは「質量項をなくすことによりカイラル対称性がある模型」のことである．これは上述したように，質量項がないため現実的な模型ではないが，しかしおもちゃの模型として議論を展開したのであろう．

• **自発的対称性の破れ：** この模型において南部達はこのカイラル対称性がその真空状態では「自然に」破れてしまうと言う事を主張したのである．これは物理的にどういう意味なのかが問題であるが，実際にはその模型の真空状態のカイラル対称性が自然に破れることはあり得ないことがわかっている．事実，破れていたわけではなく彼らが勝手にカイラル対称性自体が破れた状態になっていると思い込んでしまったのである．

• **間違いの原因：**　何故，このような勘違いを起こしてしまったのかという疑問に対しては，二つの理由が考えられる．一つには，新しく求められた真空状態のカイラル電荷が自由場が持っているカイラル電荷とは異なっていたことと関係している．自由場のカイラル電荷はゼロであるが，新しく求められた真空状態のカイラル電荷はゼロではなかったのである．しかしこれは何か特別なことが起こったというわけではなく，状態によってカイラル電荷の固有値が異なるのは当然の物理的な結果である．例えば，水素原子の問題において，その系は空間対称性があり，水素原子の基底状態は角運動量がゼロの状態が実現されている．そしてそれは励起状態の角運動量とは異なっているが，状態がそれぞれ異なった固有値を持つことは当然のことである．

二番目の理由としては，彼らが用いた近似法にその原因がある．Bogoliubov 変換と呼ばれているもので，これが近似法であることは周知の事実である．さらに，この近似法を採用すると，見かけ上，質量項が現れる場合があることはよく知られていた．これに対して，質量項が現れたからこれはカイラル対称性が破れた結果であると南部達は短絡して誤解してしまったのである [2]．このような非常に基本的なレベルの間違いが物理の世界で長い間 (５０年以上も) 通用してきた事実は，驚きを超えて，悲惨でさえある．

• **Weinberg-Salam 模型：**　南部達の理論だけが世の中に流通していたならば，それ程深刻な状況は生まれなかったことであろう．この理論模型が Higgs によって，Higgs 機構として新しい模型になり，さらにそれが Weinberg-Salam 達によって標準理論として使われたために，その混乱が大きくなってしまったのである．但し，Weinberg-Salam の模型はそれよりももっと基本的なところで深刻な問題点があることが知られている．それは彼らの模型は非可換ゲージ理論であり，その模型の構成粒子は観測量ではないことが証明されている．従って，彼らの模型は「厳密に解いたらその構成粒子であるウィークボソンは観測量ではない」のであるが，「Higgs 機構という近似法を採用したためにウィークボソンがうまい具合に観測量になった」と主張していることに対応している．これは明らかに科学の論理ではなく，その意味で Weinberg-Salam の模型は自発的対称性の破れ以前の深刻な問題をすでに抱えていたのである．

但し，この Weinberg-Salam の模型は弱い相互作用における最も重要な CVC 理論を再現するように作られている．この CVC 理論は弱い相互作用関連のほとんどすべての現象をよく記述している模型である．従って Higgs 粒子を除去して，さらに非可換ゲージ理論ではない等の修正をすれば，自然現象をよく記述している理論模型であることは間違いない．

4.5　カイラル・アノマリー方程式

場の理論の模型がカイラル変換に対して不変であると，軸性ベクトルカレントが保存している．式で書くと

$$\partial_\mu j_5^\mu = 0 \qquad (\text{但し} \quad j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi)$$

と書かれる．この保存則はフェルミオンの質量項が存在すると成立しない対称性から得られているため，現実の模型では見られない保存則ではある．

それではカイラルアノマリーとは，どのような物理現象であろうか？これは「１次発散を正則化したらアノマリー方程式が導出されたのだが，これが軸性ベクトルカレントの保存則を破ってしまった」というものである．Adler 達は，あるファインマン図 (三角形図) を計算した時に見かけ上現れた１次発散の無限大を正則化したのである．このように求められたアノマリー方程式は非常に斬新で独創的であり，このため人々を引き付けてしまったのであろう．普通，正則化という数学的な方法を採用して最も重要な物理の保存則を破ってしまった場合，これは正則化のどこかに間違いがあると考えるものである．ところが，その当時は「ゲージ条件」に対する理解が不十分であったこともあり，アノマリー方程式を人々が受け入れてしまったのである．物理学では，Noether の定理から導かれた保存則が単純な数学的手段などで破られることはなく，保存則はそれを破る相互作用がない限り，常に厳密に成り立っている．

物理学の理論体系からしたら，これは最も低いレベルのミステークであると言える．しかしこれが数十年に渡り人々に受け入れられてきた事も事実である．さらに悪いことに，超弦理論はこのアノマリー方程式をその理論構築の一つの根拠にしているのである．何故，このようなことが起こり得たのかという疑問に対する科学史的な考察は今後の研究に任せるしかないであろう．自発的対称性の破れの問題にしてもこのアノマリー方程式にしても，対称性について人々の理解がなかなか深まらなかったことが一つの原因でもあろうか？

さらに言えば，三角形図の計算に関しては，非常に正確な模型計算が Adler 達よりも前に西島先生によって提唱されている [1]．ところが，その論文 (実際には教科書における解説) がどういうわけか全く無視されて今日に至っている．このことがアノマリーの問題を野放しにしてしまった最も大きな原因であると言えよう．理論家各自が自分の手できちんと三角形図の計算を行っていれば，このような事態にはならなかったかも知れない．

4.6　Higgs 粒子の問題点

Higgs 機構の問題点を解説することは，それ程，大変なことではない．しかし同時にそれ程，面白い事でもない．実際，Higgs 粒子は存在しないことが，いずれ明らかにされることであろう．この Higgs 機構とはゲージ場が複素スカラー場と相互作用する場合，ラグランジアン密度の段階でゲージ固定してしまうことにより，ゲージ場に質量を与えようとする近似法である．これは物理学の論理としては到底，正当化できるものではない．

4.6.1　Higgs ポテンシャル

Higgs 機構を説明しようとすると，まずは Higgs ポテンシャルを解説する必要がある．これは不思議なポテンシャルである． Higgs 場はスカラー場なのでフェルミオンではなくボソン場であるが，そのボソン場が自己相互作用しているものが Higgs ポテンシャルである．しかし自己相互作用とは何か，と言うことが物理学では答えられていない．そのような自己相互作用は現実的なものとしては存在していないからである．

この「あり得ない相互作用」が提案されると，人々の反応は２つに分かれることになる．一つ目としては，これは非物理的であるとして排除する立場である．これは自然論学派としては当然取るべき方向であり，この形は自然界の記述には現れてこない相互作用であることが最大の理由である．ところが，第２番目の立場をとる人たちは，その考え方の独創性を評価するため，その模型に何か面白い物理があるかも知れないと期待するのであろう．そしてその期待が暴走すると，いつの間にか自然界との整合性の検証が二の次になってしまい，その模型が過大評価されてしまうのである．自己相互作用とはどのようなものなのかという，非常に基本的な概念をきちんと検証していれば，このような混乱は起こらなかったものと考えられる．

4.6.2　Higgs 機構

Higgs 機構自体は単純な模型計算である．ゲージ場が Higgs ポテンシャルと相互作用している系をまず考えることから始めている．この場合，スカラー場がゲージ場と相互作用する形はきちんと分っているわけではないため，その形を特定することはできない．しかし人々はミニマル変換という手法はゲージ不変だから，これがスカラー場とゲージ場が相互作用する形であろうと仮定して

相互作用の形を決めたのである．しかし実際問題としては，自然界に有限質量のスカラー粒子は存在しないため，これは実験から正当化しようがないものであり，よくわからない．

しかしながらここでは，このミニマル変換の手法をひとまず受け入れて，ゲージ場と Higgs ポテンシャルの相互作用の形を決めたとしよう．そこで Higgs はある種のゲージ固定をこの全体のラグランジアン密度に対して行ってしまうのである．通常のゲージ固定は運動方程式を解く時に変数の数と方程式の数を合わせるために行う物理過程であるが，Higgs はゲージ場に対するゲージ固定の条件式をラグランジアン密度に代入して，ゲージ不変性を破ってしまったのである．このことにより，ゲージ場が質量を獲得したと主張したのが Higgs 機構である．何故，このような奇妙な過程が容認されたのかという問題は確かに不思議ではある．しかしこれは自発的対称性の破れと関係していると考えられる．対称性が自発的に破れるのだから，このような荒っぽいことをしても構わないという雰囲気がその当時には存在していたのであろう．また非常に独創的な概念であることも、この奇妙な理論が容認された一因でもあろうか？

このように，ラグランジアン密度の段階でゲージ固定したため，勿論，ゲージ不変ではないラグランジアン密度になっている．従ってそれ以降は，ゲージ理論ではないので「非可換ゲージ理論ではその構成粒子が観測量ではない」という深刻な問題も回避されていると考えたのであろう．前述したように,「厳密に解いたら観測量ではないものが，近似をしたら観測量になってくれた」という論理は物理学では到底，受け入れられるものではない．この分野は，一体，どうしてこのような摩訶不思議な現象が横行していたのであろうか？

4.7　弱い相互作用の理論

弱い相互作用を歴史的に振り返ると学ぶところが多いものである．弱い相互作用とは，中性子がβ崩壊する過程を記述することができる理論体系である．この弱い相互作用における物理過程を理解しようとした最初の論文 (教科書での解説) はフェルミによって提案されている．

4.7.1　４点相互作用

フェルミは４点相互作用という模型を提案することにより，中性子崩壊の現象を理解する手掛かりを与えたのである．中性子の崩壊は中性子が陽子，電子そしてニュートリノへ崩壊するため，これは４個の粒子と関係している．従って，４点相互作用となっている．ここで粒子生成という非常に新しい概念が提唱されている．現在の場の理論ではこの粒子の生成・消滅を記述する場合，場の量子化を行うがフェルミはこれを行列要素で書いている．この表式はハイゼンベルグが行ったものであり，場の量子化と同じ結果を与えている．

4.7.2　パリティ非保存の相互作用

物理学における相互作用はいくつかの基本的対称性を持っている．その場合，それらのほとんどはその変換が連続変数による変換である．グローバルゲージ変換もその一つである．これら連続変数の変換から求められた対称性が基本的な相互作用の中で破れている現象は現在まで見つかってはいない．

一方，対称性の中には不連続な変数変換から導出された対称性も存在している．例えば，空間反転に対する対称性である．具体的には $\boldsymbol{r} \to -\boldsymbol{r}$ という変換 (パリティ変換) に対して，相互作用ラグランジアン密度が不変であるかどうかという問題である．この変換に対して，電磁的な相互作用は不変であり，さらに強い相互作用も重力もこのパリティ変換に対して不変である．

ところが，弱い相互作用はこのパリティ変換に対して不変ではないことが実験・理論両面からわかっている．その意味では，フェルミの相互作用は不十分であることが知られていた．このパリティ非保存の形式を最初に導入したのが Lee と Yang である．その後，実験的にも弱い相互作用はこのパリティを破っていることが証明され，物理学会に大きな衝撃を与えたものである．しかしこれは自然現象を理解するという立場からは，至極自然なものとも言えよう．

4.7.3 CVC 理論

弱い相互作用の理論的な体系はゲルマンやファインマンやその他の人達により CVC(Conserved Vector Current) 理論として完成された．この CVC 理論により，弱い相互作用に関連するほとんどすべての現象がうまく理解されることがわかっていた．ところが，この CVC 理論は 2 次の摂動論を展開すると 2 次発散が出てしまうことも知られていた．これは理論形式の問題であるが，CVC 理論には何かまだ不十分なところがあることを示していた．

4.7.4 Weinberg-Salam 模型

CVC 理論は SU(2) という群で記述すると簡単に書けることがわかっていたため，Weinberg-Salam 模型はそれに準拠している．さらに，この当時はゲージ理論のみが正しい理論体系であるという思い込みが物理屋のなかに蔓延していたのであろう．このため，Weinberg-Salam 模型は $SU(2) \otimes U(1)$ という非可換ゲージ理論の模型として提案されたのである．しかしゲージ粒子は質量がないため，実験的な観測とは合わないものであった．その当時，ウィークボソンの質量は少なくとも 10 GeV 以上であるという実験結果が報告されていたのである．

このため， Weinberg-Salam 模型ではこのゲージ不変性を何とか壊す必要があり，その時に採用された理論が Higgs 機構である．これは自発的に対称性が破れているわけだから，ある意味で何でもありの理論になってしまったのであろう．何度も強調することになるが，Weinberg-Salam 模型は非可換ゲージ理論である．このため，その基本粒子であるウィークベクトルボソンは「厳密に解いたら観測量ではないが Higgs 機構という近似法を採用したため，これらのボソンがうまく観測量になった」と主張しているのである．しかしながらこの論理が受け入れられないことは議論する必要もないであろう．

• **Weinberg-Salam 模型が実験を再現できる理由：** 何故，この Weinberg-Salam 模型が標準模型として人々に信用されてきたかと言う疑問に対してはそれなりの理由がある．それはこの模型の最終的なラグランジアン密度が CVC 理論を導出できるように調整されていたからである．従って，Weinberg-Salam 模型は理論的な整合性を別にしたら，実験をよく再現している模型と言うことができる．しかしだからと言って，この模型を評価することはできないのは至極当然のことでもあろう．

4.8　非対称性の物理

ある対称性を持つ系に外力を作用させると、一般的にはその対称性が壊れることになる。これが非対称な物理現象である。ここでは重要な具体例のうちで、いくつかの興味ある現象について簡単に解説しよう。

4.8.1　ゼーマン効果

原子系は空間回転対称性 (およびスピン空間回転対称性) を持っているため、水素原子の $1s_{\frac{1}{2}}$ 状態は縮退している。従ってスピンの状態関数 $\chi_{\frac{1}{2}}$ と $\chi_{-\frac{1}{2}}$ は同じエネルギーを持つ状態になっている。この状態に一様磁場 $\boldsymbol{B}$ を掛けるとこの磁場がスピン空間回転の対称性を破るため、縮退していたエネルギー状態に $\boxed{\Delta E = \pm\frac{e\hbar}{2m_e c}B}$ のような分裂が生じる。これはゼーマン効果と呼ばれていて、非対称な性質を示す現象のうちでも最も重要である。

• **MRI(核磁気共鳴)：**　物理学において、最も幅広く応用され様々な機器に実際使われている現象がこのゼーマン効果であろう。例えば、MRI(核磁気共鳴) は今や、日常的に使われているものである。これは磁場を掛けることにより水分子中の陽子状態がゼーマン分裂する機構を応用したものである。

• **偏光ゼーマン法：**　また、体内に蓄積された有機水銀の量を測定するため、その試料に磁場を掛けておき、偏光したフォトンの吸収率を測ることにより有機水銀の量を「絶対測定」する手法も開発されている。これは非対称現象をうまくとらえて、偏光したフォトンの磁気量子数の保存則を利用した極めて精巧な技術と言えよう。

4.8.2　電気双極子の物理

誘電体に外から電場 $\boldsymbol{E}$ を掛けると、誘電体においては電気双極子が誘発されてそのために $\boxed{\Delta E = -\boldsymbol{p}\cdot\boldsymbol{E}}$ というエネルギーシフトが生じる。これはもともとの結晶は中性なのだが、そこに電場を掛けたため対称性が壊れて偏極が生じたことに対応している。但しこの現象は物質の中で起こっているものであり、孤立系 (例えば水素原子）ではその電気双極子がゼロであるためこの現象は起こらない。孤立系の電気双極子が有限だとこれは「時間反転不変性」を壊すことになり、現在まで時間反転不変性を破る現象は発見されていない。

4.8.3 シュタルク効果

電場 $\boldsymbol{E}$ を $z-$ 軸方向に掛けると相互作用として $\boxed{H' = ezE}$ が現れる。この相互作用は電気双極子と電場の相互作用そのものである。これは空間回転の対称性を破り、さらに空間反転対称性も破っている。この相互作用の形を見ればわかるように、水素原子の基底状態での期待値はゼロでありこの効果の影響はない。しかし励起状態にはパリティの異なる縮退した状態 $(2s,\ 2p)$ がありこの場合、１次の摂動計算により確かにこの影響を確認することができる。

4.8.4 スピンー軌道相互作用

水素原子を勉強すると、必ず、スピン – 軌道相互作用と言うものが出てくる。これは Dirac 方程式の非相対論近似から得られる項であり $\boxed{H' = \xi(r)\,\boldsymbol{\ell}\cdot\boldsymbol{s}}$ の形である。この相互作用は軌道角運動量 $\boldsymbol{\ell}$ に比例しているため空間回転の対称性を破っている。この破れは勿論、近似により生じたものである。Dirac 方程式は相対論的な方程式であり、Lorentz 空間における回転対称性を持っている。しかし非相対論近似を実行するとこれは当然、Lorentz 対称性を破ることになり、従って、このようなスピン – 軌道相互作用が現れるのである。水素原子においてはこの非相対論近似は非常に良い近似であることが実際の計算で簡単に確かめることができる。

4.8.5 圧電効果

圧電効果とはある種の結晶体に機械的応力を掛けるとそれに応じて電気分極が起こり，電束密度が生じる現象である．これは Pierre Curie が１００年以上も前に発見しているもので、機械的な力が結晶構造の対称性を少し壊すため起こる現象である。この応用は非常に広範囲に渡っている．特に液晶画面に手で触れてそれを電気信号に変換する機構はこの圧電効果の応用そのものである．

4.9　繰り込み理論と対称性

繰り込み理論に関しては，Dirac の主張が何故，人々に無視されたのかと言うことが最も重要であろう．第１章では繰り込み理論の大雑把な解説をしているが，実はこれ以上に詳しく説明しようとすると，どうしても数式をふんだんに使わざるを得なくなる．それはこの本の趣旨ではないので，ここでは，繰り込み理論を対称性の観点から見直してみようと思う．また科学史的な観点からも繰り込み理論について少し考察してみよう．

4.9.1　局所的ゲージ対称性

繰り込み理論それ自体は特に対称性と関係しているわけではない．しかしその繰り込みの考え方は，実は，局所的ゲージ対称性と密接に関係している．何故,「局所的 (local)」と言う言葉を使うかは明らかであろう．このゲージ変換 χ が時間と空間の関数となっているからである．実際，ゲージ対称性とはラグランジアンが局所的ゲージ変換という非常に特殊な変換に対して不変であることと関係している．局所的ゲージ変換とは，ある場の変数，今の場合ベクトルポテンシャルを任意の関数の微分量だけ平行移動 ($\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi,\ \ A'_0 = A_0 - \frac{\partial\chi}{\partial t}$) してもラグランジアンは不変である (ラグランジアンが χ には依らない) と言うことが出発点である．量子電磁力学ではこれに加えて，フェルミオン場についても，任意関数の位相分 ($\psi' = e^{ie\chi}\psi$) だけ変換する操作を含んでいる．

局所的ゲージ対称性の重要性は明らかで，これによりフェルミオン場と電磁場の相互作用の形が一意的に決まってしまうのである．ゲージ理論で相互作用が決まる模型としてもう一つ，量子色力学 (QCD) が知られている．ところが，この QCD 模型の場合，相互作用の形が確かに決定されるのであるが，その後，摂動論が展開できないため事実上，観測量の計算は不可能である．その意味においてゲージ理論として計算可能な模型は実は QED だけである．

4.9.2　電子の自己エネルギーの発散

QED において摂動論を展開して電子の自己エネルギーを計算すると，その計算結果は無限大 (Log 発散) になってしまう事がわかる．この場合，電子の自己エネルギーの発散をどうとらえるかが問題である．古典電磁場においても，電子の作る自己電場を求めてその全エネルギーを計算すると，電子の自己エネルギーは発散している．しかし，どの教科書でもこの電子の自己エネルギーの

発散について何かを変えようという議論もなく，そのまま放置すればよいことになっている．それは当然で，この電子の自己電場のエネルギーは観測量ではないので，何も困ることはない．すなわち，電子が他の物質と相互作用をしない限り，その自己エネルギーを測定することもできないし，物理的な観測量になることもない．

4.9.3 電子の磁気能率補正

ところが，量子場の理論においては，電子の自己エネルギーが発散することは理論的に問題だとして，その無限大を打ち消すために，カウンター項をラグランジアン密度に足し算するのである．これにより，今度は自己エネルギーの発散が抑えられた形式ができたと思い，これが繰り込み理論の出発点になっている．この形式により観測量である電子の磁気能率補正を計算すると，やはり発散項がでてしまうのであるが，この発散項を電子の波動関数に押し込める形式を発展させたのである．何故，電子の波動関数にこの無限大を押し込めることが出来たのかと言う疑問に対しては，簡単に答えることができる．それは，自己エネルギーの無限大の形と電子の磁気能率補正の無限大の形とが全く同じものであったことに依っている．この手法はいかにも人工的ではあるが，この場合，有限量が電子の磁気能率の観測値と非常によく一致していたため，人々はこの繰り込み理論を受け入れたのであろう．実は、この磁気能率補正の計算にはもう一つ赤外発散の無限大が存在していることが知られている．しかしこれは波動関数に繰り込むことはできないので人々は単純に捨てているのである．計算結果に無限大があっても，その中の有限量が実験値とよく一致すればそれで良いとした理論計算を人々が受け入れてきたことは事実である．これはよく考えてみると非常に不思議な論理ではあるが，無限大に対して割合よく見られる現象でもある．しかし，科学史的に検討してみたら，これは他の分野でもよくあることなのであろうか？

4.9.4 フォトンの自己エネルギーの発散

フォトンの自己エネルギーは２次発散している．これは電子の自己エネルギーが Log 発散している事とは好対照である．ここでこのフォトンの自己エネルギーの無限大に対して，人々はこの２次発散を何とか処理する必要があると思い「ゲージ条件」を導入したのである．この「ゲージ条件」が問題で物理

的にも数学的にも間違っていることがわかっている．このような条件を付けることにより，フォトンの自己エネルギーを繰り込める形にしようと考えた動機がよくわからない．フォトンの自己エネルギーが発散しても何も問題を起こさないのに，何故，人々はこの「ゲージ条件」に固執したのか，今となっては謎ではある．この現象も科学史的に究明する必要があると思われる．

• **有限質量ベクトルボソンの自己エネルギー：**　簡単な計算ですぐに確かめられることであるが，実は，有限質量ベクトルボソンの自己エネルギーも同様に２次発散している．しかし不思議なことに，この２次発散に対しては誰も問題にはしていなく，議論さえもされていないのである．このことを理解すればわかるように，フォトンの自己エネルギーが２次発散していることはゲージ理論の特殊性ではない．つまりはフォトンを含めたベクトルボソンの自己エネルギーの２次発散はゲージ理論とは無関係な発散であったのである．

4.9.5　観測量の計算における発散の原因は？

通常の計算において，観測量の計算結果に無限大が出てきたら，その計算がどこかで間違っているか，あるいはその定式化のどこかに誤りがあるかと細心の注意を払って計算を検証するものである．確かに，人々は様々な形でこの検証を行っている．しかしながら，これはゲージ不変性の理解が不十分であったことが最も大きな原因であることが，今となっては明らかである．もし人々が，有限質量ベクトルボソンによる電子の磁気能率補正をきちんと計算していたら，話は全く別の展開になっていたことであろう．この有限質量ベクトルボソンによる電子の磁気能率補正の計算結果には，どこにも発散はない [3]．つまりは，ファインマンのフォトン伝搬関数を用いた電子の磁気能率補正にのみ，Log 発散が現れているのである．このことを認識していたら，ゲージ理論のみが特殊であることが明らかになっていたことであろう．将来，科学史の研究者はこの問題をどうとらえてゆくのであろうか？．

• **ファインマンの伝搬関数：**　フォトンによる電子の磁気能率補正の計算に無限大が出てきたのは，フォトンの伝搬関数としてファインマンの伝搬関数を用いたことに依っている．そしてこの伝搬関数が正しくないことは昔の場の理論の教科書でしっかり議論されている [6, 7]．しかし，いつの間にか，このファインマンの伝搬関数がフォトンの正しい伝搬関数であると人々が思い込んでしまったのである．この理由は一つには，繰り込み理論が過大評価されてしまったことに原因があると思われる．もう一つ，このファインマンの伝搬関数の取

扱いが非常に簡単であることが大きな原因でもあろうか？フォトンの正しい伝搬関数は勿論，よく知られていたがこの取扱いが専門家にもかなり複雑であり，人々はファインマンの伝搬関数を使うことにしたのであろう．さらに言えば，電子ー電子散乱の場合，どちらの伝搬関数を使っても散乱振幅は同じに求められることも知られていた．これらいくつかの偶然が重なって，現在の繰り込み理論に対して，ほとんど無防備なまでに人々がこの理論形式を受け入れてしまい，そしてそれが定着したのであろう．

現在，繰り込み理論の問題点を指摘しても，ほとんどの「いわゆる専門家」達はその批判内容を検討はしないで，単純に (少なくともしばらくの間は) 無視し続けることであろう．尤も今となっては，繰り込み理論においてその専門家と言えるような研究者はいなくなってしまったと言えるかも知れない．しかしながら，恐らく，繰り込み理論の提唱者達は (特に，朝永博士がもし生きておられたら) この理論に対する新しい知見と進展を喜んでくれると思うのだが，どうであろうか？

4.9.6　Dirac の主張が何故，無視されたのか？

繰り込み理論に関して，Dirac の最も最近の論文は１９８１年に出版された AIP 会議報告書であろう．その報告書以前からずっと，Dirac は繰り込み理論に対する問題点をかなり強く主張し，指摘している．ところが，歴史的にはこの繰り込み理論に対する批判は，物理屋にはほとんど見向きもされなかったのである．これは科学の考え方を教える科学教育の立場からするとかなり深刻な問題であり，科学史的にきちんと検証するべき事象である．１９８１年の AIP 会議報告書を読むと，その論理の確かさに仰天し，強い感銘を受けるものである．この論文を書いた時に Dirac は８０歳近い年齢であることを考えてみれば，なおさらに彼が繰り込み理論に対して非常に憂えていたことがよくわかるものである．それは，この繰り込み理論が現代物理学の基礎として君臨してしまったことに問題がある．電子の磁気能率補正の計算にのみ，この繰り込み理論が応用されたのならば，まだ，その被害は最小限になっていたことであろう．ところが現実の物理の世界では，繰り込みができる理論模型，できない理論模型とわけていて，そのため，ゲージ対称性を持つ理論のみが正しい理論であるという風潮が蔓延していたのである．実際には，ゲージ理論に対してのみ，観測量に奇妙な無限大が現れることが，今は証明されている．さらには，正しい伝搬関数を用いれば，恐らくフォトンの磁気能率補正の計算にも発散は存在しないことであろう．しかし，Dirac はこれらの事を知らない段階ですで

に，繰り込み理論の問題点を繰り返し指摘していたのである．この Dirac の主張に対して専門家が耳を傾けなかったことは，一体，何故なのであろうか？

4.9.7 今後の方向

繰り込みをする必要がなくなったと言うことは，現代物理学においてはどのような影響があるのであろうか？これは一言で表現することは難しいとは思うが，しかし場の理論の形式が非常に簡単になったことは確かである．現代物理学はすべて場の理論で書かれているが，これから物理的な観測量を計算する場合，摂動論の形式をきちんと実行すれば正しい答えが求められると言うことである．現実問題としては，強い相互作用の基礎理論である量子色力学は非可換ゲージ理論であるため，摂動計算が不可能である．このため，もともと繰り込み理論とは無関係であったので影響はほとんどないと考えられる．一方，重力理論は場の量子化をする必要がなかったため，繰り込み理論とは無関係であった．このため，繰り込みが必要かどうかの問いかけ自体が存在しないので，影響はもともとなかったのである．さらに弱い相互作用では観測量に対してそもそも発散項は存在しないため繰り込みは不要であった

4.9.8 繰り込み不要の影響

結局，繰り込みが不要になったことの影響は QED の計算に限られている．これはある意味では幸運でもあった．いずれにしても，現在，我々が手にしている場の理論の理論形式は非常にシンプルで信頼性の高いものである．従って，今後，場の理論の応用はより幅の広いものとして重要な役割を果たしてゆくものと期待できるものである．

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Chapter 2

Symmetry and Conservation Law

The Lagrangian density of fermions which is constructed in the previous chapter possesses various symmetries such as Lorentz invariance, time reversal symmetry and so on. These symmetries play a fundamental role for the determination of the vacuum state as well as the spectrum emerged from the model Hamiltonian. Therefore, one should be accustomed to these basic symmetries to understand the field theory.

In this chapter, we explain fundamental symmetry properties of Lorentz invariance, time reversal invariance, parity transformation, charge conjugation, translational invariance, global gauge symmetry, chiral symmetry and $SU(3)$ symmetry in field theory. In particular, the invariance of the Lagrangian density under these symmetries is discussed in detail since it is important to determine the vacuum structure.

If the Lagrangian density has a continuous symmetry, then there is a conserved current associated with this symmetry due to Noether's theorem. From the conservation of the current, one finds a conserved charge which plays an important role for the determination of physical states such as the vacuum state. All of the field theory models discussed here possess the translational invariance of the Lagrangian density, and this leads to the conserved quantity of the energy momentum tensor. From this energy momentum tensor, one can define the Hamiltonian density which is the energy density of the system. Clearly, the Hamiltonian is most important for the quantized field theory models since it can determine all of the physical states as the eigenstate of the Hamiltonian.

2.1 Introduction to Transformation Property

When one considers the transformation of the Lagrangian density or Hamiltonian density under the symmetry operator U, one first evaluates the transformation of the field ψ as

$$\psi' = U\psi.$$

Then, one calculates and sees how the Lagrangian density $\mathcal{L}$ should transform under the symmetry operator U

$$\mathcal{L}' \equiv \mathcal{L}(\psi', \partial_\mu \psi') = \mathcal{L}(U\psi, \partial_\mu (U\psi)).$$

If the Lagrangian density does not change its functional shape,

$$\mathcal{L}' = \mathcal{L}$$

then it is invariant under the transformation of U.

After the field quantization, the transformation procedure becomes somewhat different. The basic physical quantity in quantized field theory becomes the Hamiltonian $\hat{H}$, and the important point is that the Hamiltonian is now an operator and the problem becomes the eigenvalue equation for the field Hamiltonian

$$\hat{H}|\Psi\rangle = E|\Psi\rangle,$$

where $|\Psi\rangle$ is called *Fock state*. Since an operator $\mathcal{O}$ transforms under the symmetry operator U as

$$\mathcal{O}' = U\mathcal{O}U^{-1}$$

the Hamiltonian $\hat{H}$ transforms as

$$\hat{H}' = U\hat{H}U^{-1}.$$

In this case, the Fock state $|\Psi\rangle$ should transform as

$$|\Psi'\rangle = U|\Psi\rangle.$$

The transformation properties of the Lagrangian density should be kept for the quantized Hamiltonian after the field quantization. However, the vacuum state (the Fock state) may break the symmetry and indeed this can happen for the continuous global symmetry like the chiral symmetry. This physical phenomena are called *spontaneous symmetry breaking* which will be treated in Chapter 4.

2.2 Lorentz Invariance

The most important symmetry in physics must be the Lorentz invariance. The Lorentz invariance should hold in the theory of all the fundamental interactions. This is based on the observation that any physical observables should not depend on the systems one chooses if the systems $\mathcal{S}$ and $\mathcal{S}'$ are related to each other by the Lorentz transformation,

$$x'^{\mu} = \alpha^{\mu}{}_{\nu} x^{\nu}. \tag{2.1}$$

If the $\mathcal{S}'$ system is moving with its velocity of v along the x_1-axis, then the matrix $\alpha^{\mu}{}_{\nu}$ can be explicitly written as

$$\{\alpha^{\mu}{}_{\nu}\} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where

$$\cosh\omega = \frac{1}{\sqrt{1-v^2}}$$

is introduced. In this case, the Dirac wave function ψ should transform by the Lorentz transformation as

$$\psi'(x') = S\psi(x), \tag{2.3}$$

where S denotes a 4×4 matrix. Now, the Lagrangian density for free Dirac field is written both in $\mathcal{S}$ and $\mathcal{S}'$ systems

$$\mathcal{L} = \bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x) = \bar{\psi}'(x')(i\partial_\mu{}'\gamma^\mu - m)\psi'(x'). \tag{2.4}$$

From the equivalence between $\mathcal{S}$ and $\mathcal{S}'$ systems, one obtains

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}, \tag{2.5a}$$

$$S\gamma^\mu S^{-1}\alpha_\mu^{\ \nu} = \gamma^\nu. \tag{2.5b}$$

If one solves eq.(2.5b), then one can determine the shape of S explicitly when the $\mathcal{S}'$ system is moving along the x_1-axis

$$S = \exp\left(-\frac{i}{4}\,\omega\sigma_{\mu\nu}I_n^{\mu\nu}\right),$$

where $\sigma_{\mu\nu}$ and $I_n^{\mu\nu}$ are defined as

$$\sigma_{\mu\nu} = \frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right),$$

$$I_n^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.2.1 Lorentz Covariance

If physical quantities like Lagrangian density or equation of motions are written in a manifestly Lorentz invariant fashion, then they are called *Lorentz covariant*. The simplest case is that these equations are written as a Lorentz scalar. In this case, it is trivial to recognize that they are Lorentz invariant. For example, the continuity equation of the vector current reads

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

This is shown to be Lorentz invariant under the Lorentz transformation. However, it is not manifest, and therefore one defines the four vector current j^μ by

$$j^\mu = (\rho, \boldsymbol{j}).$$

In this case, one can rewrite the current conservation as

$$\partial_\mu j^\mu = 0$$

which is obviously Lorentz invariant since it is written in terms of the Lorentz scalar, and it is a Lorentz covariant expression.

However, one should not stress too much the importance of the Lorentz covariance. The Lorentz invariance is, of course, most important. However, as long as one starts from the Lorentz invariant theory, one does not have to worry about the violation of the Lorentz invariance since it can never be broken unless one makes mistakes in his calculations. In this respect, the Lorentz covariance may play an important role for avoiding careless mistakes if one carries out the perturbative calculation of the S-matrix in a covariant way.

2.3 Time Reversal Invariance

The world we live does not seem to be invariant under the time reversal transformation. Time flows always in the same direction. However, the physical law in the macroscopic world is quite different from the microscopic world, and time arrow defined by the entropy may not necessarily be related to the fundamental interactions.

Almost all of the fundamental interactions are invariant under the time reversal transformation. It is therefore important to understand the time reversal invariance (T-invariance) in field theory models.

2.3.1 T-invariance in Quantum Mechanics

Before going to field theory, we should first understand the definition of the T-invariance in quantum mechanics. When we make $t \to -t$, then the basic operators that appear in physics behave

$$t \to -t \;: \quad \begin{pmatrix} x_k \to x_k \\ p_k \to -p_k \\ \sigma_k \to -\sigma_k \\ E \to E \end{pmatrix} . \tag{2.6}$$

However, when the momentum p_k and the energy E are replaced by the differential operators as

$$\hat{p_k} = -i\,\frac{\partial}{\partial x_k}\,, \quad \hat{E} = i\,\frac{\partial}{\partial t} \tag{2.7}$$

then, the explicit t-dependences of the p_k and E become just opposite to eq.(2.6), and therefore one should recover them by hand. This can be realized when one makes complex conjugate of the operators $\hat{p_k}$ and $\hat{E}$

$$t \to -t \;:\; \hat{p_k} \to \hat{p_k}^* = i\frac{\partial}{\partial x_k} = -\hat{p_k}, \tag{2.8a}$$

$$t \to -t \ : \ \hat{E} \to \left(-i\frac{\partial}{\partial t}\right)^* = i\frac{\partial}{\partial t} = \hat{E} \tag{2.8b}$$

which can reproduce eq.(2.6). Therefore, the time reversal transformation in quantum mechanics means that the operator should be made complex conjugate as $A \to A^*$. This means that if the Hamiltonian contains an imaginary term, then this system violates the T-invariance. As one can see, the complex conjugate operation in accordance with the T-transformation should not be taken for the Pauli matrix σ_k as seen from eq.(2.6). For the Pauli matrix σ_k, one should make just the transformation of $\sigma_k \to -\sigma_k$ for $t \to -t$.

2.3.2 T-invariance in Field Theory

In field theory, momentum operators are all replaced by the differential operators, and therefore T-transformation of the field $\psi(x_k, t)$ means

$$t \to -t \ : \ \psi(x_k, t) \to \psi(x_k, -t)^* \ \text{ with } \ \sigma_k \to -\sigma_k. \tag{2.9}$$

As an example, one takes the plane wave solution of eq.(1.11) and makes the T-transformation. Then, one obtains

$$t \to -t \ : \ \psi(x_k, t) \to \left\{ \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \chi_s \\ \frac{\sigma_k p_k}{E_{\boldsymbol{p}} + m}\chi_s \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iE_{\boldsymbol{p}}t + ip_k x_k} \right\}^*$$
$$= \psi(x_k, t) \tag{2.10}$$

which is indeed invariant under the T-transformation. The γ_μ matrices transform under the T-transformation

$$t \to -t \ : \ \begin{pmatrix} \gamma_0 \to \gamma_0 \\ \gamma_k \to -\gamma_k \end{pmatrix}. \tag{2.11}$$

Therefore, the free Dirac Lagrangian density of eq.(1.7) transforms

$$t \to -t \ : \ \mathcal{L} \to \left\{ \psi_i^\dagger \left[\gamma_0(-i\partial_0\gamma^0 - i\partial_k\gamma^k - m) \right]_{ij} \psi_j \right\}^* = \mathcal{L} \tag{2.12}$$

which is, of course, T-invariant as expected.

2.3.3 T-violating Interactions (Imaginary Mass Term)

At present, it is most important and fundamental to discover any interactions which violate the T-invariance. The simplest way to introduce the T-violating interaction must be an imaginary mass term,

$$\mathcal{H}_T = i\eta\bar{\psi}\psi, \tag{2.13}$$

where η denotes a real constant. In fact, the CP violating phase is originated from this type of interaction.

2.3.4 T and P-violating Interactions (EDM)

The direct examination of the T-violating interaction is based on the measurement of electric dipole moments (EDM) in isolated systems. The fundamental interaction of the T and P-violations can be written

$$\mathcal{H}_{TP} = \frac{i}{2}\, d_f \bar{\psi}_f \sigma_{\mu\nu}\gamma_5 \psi_f F^{\mu\nu}, \tag{2.14}$$

where d_f denotes the intrinsic EDM of the f-fermion. $F_{\mu\nu}$ and $\sigma_{\mu\nu}$ denote the electromagnetic field strength and the anti-symmetric tensor, respectively and they are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$\sigma_{\mu\nu} = \frac{i}{2}\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).$$

γ_5 is defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3.$$

The Hamiltonian of eq.(2.14) can be obtained by integrating the Hamiltonian density over all space, and in the nonrelativistic limit, the particle Hamiltonian becomes

$$H_{TP} \simeq -d_f \boldsymbol{\sigma}\cdot \boldsymbol{E}. \tag{2.15}$$

The measurements of the neutron EDM have been carried out extensively, and we will see in near future whether the neutron EDM is finite or not. It may be worth quoting the recent experimental measurement on the neutron EDM d_n [64]

$$d_n \simeq (1.9 \pm 5.4)\times 10^{-26}\ \mathrm{e\cdot cm}.$$

2.4 Parity Transformation

The space reflection operation is called *parity* transformation $\hat{P}$, and it is defined as

$$\hat{P} x_k \hat{P}^{-1} = -x_k, \quad \hat{P} t \hat{P}^{-1} = t \tag{2.16a}$$

$$\hat{P}\gamma_k \hat{P}^{-1} = -\gamma_k, \quad \hat{P}\gamma_0 \hat{P}^{-1} = \gamma_0. \tag{2.16b}$$

In this case, ψ should also transform into ψ' as

$$\psi'(x_k,t) = \hat{P}\psi(x_k,t) = \gamma_0\psi(x_k,t). \tag{2.16c}$$

The strong and electromagnetic interactions are invariant under the parity transformation. For example, the fermion Lagrangian density with the electromagnetic interaction of eq.(1.20)

$$\mathcal{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}$$

can be seen under the parity transformation as follows.

$$\bar{\psi}\hat{P}^{-1}i\partial_0\gamma_0\hat{P}\psi = \bar{\psi}i\partial_0\gamma_0\psi, \quad \bar{\psi}\hat{P}^{-1}i\partial_k\gamma_k\hat{P}\psi = \bar{\psi}i\partial_k\gamma_k\psi,$$
$$\bar{\psi}\hat{P}^{-1}A_0\gamma_0\hat{P}\psi = \bar{\psi}A_0\gamma_0\psi, \quad \bar{\psi}\hat{P}^{-1}A_k\gamma_k\hat{P}\psi = \bar{\psi}A_k\gamma_k\psi,$$

where the following relations are employed

$$\hat{P}A_0\hat{P}^{-1} = A_0, \quad \hat{P}A_k\hat{P}^{-1} = -A_k. \tag{2.17}$$

Therefore, one sees that the fermion Lagrangian density with the electromagnetic interaction is invariant under the parity transformation.

Interaction with Parity Violation

For parity violating interactions, one takes for example

$$\mathcal{L}_I = g'\bar{\psi}\gamma_\mu\gamma_5 A^\mu\psi. \tag{2.18}$$

Under the parity transformation, one finds

$$\bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi$$

which shows that the Lagrangian density of $\mathcal{L}_I$ is odd under the parity transformation.

2.5 Charge Conjugation

The Lagrangian density for electrons interacting with the gauge field is invariant under the charge conjugation operation. The charge conjugate operation starts from the Maxwell equation which is invariant under the sign change of the vector potential.

2.5.1 Charge Conjugation in Maxwell Equation

The Maxwell equation is invariant under the sign change of the vector potential

$$\text{Charge Conjugation} \Longrightarrow A_c^\mu \equiv -A^\mu. \tag{2.19a}$$

This is clear since the Lagrangian density of the gauge field is written as

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

which is obviously invariant under the operation of eq.(2.19a)

$$\mathcal{L}_c = -\frac{1}{4}\left(\partial_\mu A_{c\nu} - \partial_\nu A_{c\mu}\right)(\partial^\mu {A_c}^\nu - \partial^\nu {A_c}^\mu) = \mathcal{L}.$$

When the gauge field interacts with the fermion current, then the Lagrangian density becomes

$$\mathcal{L} = -gj_\mu A^\mu - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu).$$

This Lagrangian density should be invariant under the charge conjugation operation and therefore gj_μ should change its sign

$$\text{Charge Conjugation} \Longrightarrow (gj_\mu)_c = -gj_\mu. \qquad (2.19b)$$

This is a constraint on the Dirac field and it is indeed realized in the Dirac equation.

2.5.2 Charge Conjugation in Dirac Field

The invariance of the charge conjugation on the Dirac field starts from the Dirac equation with the electromagnetic interaction

$$i(\partial_\mu \gamma^\mu)_{ij}\psi_j - g(A_\mu\gamma^\mu)_{ij}\psi_j - m\psi_i = 0. \qquad (2.20)$$

Now, one can make the complex conjugate of the above equation and multiply γ_0 from the left. This can be rewritten with the transposed representation of the gamma matrix γ_μ^T

$$-i(\partial^\mu \gamma_\mu^T)_{ij}\bar{\psi}_j - g(A^\mu\gamma_\mu^T)_{ij}\bar{\psi}_j - m\bar{\psi}_i = 0. \qquad (2.21)$$

Now, the $\bar{\psi}$ is transformed as

$$\psi^c \equiv C\gamma^0\psi^* = C\bar{\psi}^T, \qquad (2.22)$$

where C is a 4×4 matrix, and ψ^c denotes the state with charge conjugation and corresponds to an anti-particle state. Further, the operator C is assumed to satisfy the following equation

$$\gamma_\mu = -C(\gamma_\mu)^T C^{-1}. \qquad (2.23)$$

In this case, one obtains

$$i(\partial_\mu \gamma^\mu)_{ij}(\psi^c)_j + g(A_\mu\gamma^\mu)_{ij}(\psi^c)_j - m(\psi^c)_i = 0. \qquad (2.24)$$

This equation is just the same as eq.(2.20) if the sign of g is reversed, and indeed, the sign change of g is the requirement of the charge conjugation of eq.(2.19b). The operator C that satisfies eq.(2.23) is found to be

$$C = i\gamma_2\gamma_0 \qquad (2.25)$$

which is the charge conjugation operator in the Dirac field.

2.5.3 Charge Conjugation in Quantum Chromodynamics

The Lagrangian density of QCD

$$\mathcal{L} = \bar{\psi}\Big[i(\partial^\mu + ig_s A^{\mu,a}T^a)\gamma_\mu - m_0\Big]\psi - \frac{1}{4}\, G^a_{\mu\nu}G^{\mu\nu,a}$$

with

$$G^{\mu\nu,a} = \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} - g_s C^{abc} A^{\mu,b} A^{\nu,c}$$

is invariant under the charge conjugation operation

$$A_c^{\mu,a} \equiv -A^{\mu,a}, \quad (g_s)_c \equiv -g_s.$$

This can be easily seen since

$$(G^{\mu\nu,a})_c = \partial^\mu A_c^{\nu,a} - \partial^\nu A_c^{\mu,a} - (g_s)_c C^{abc} A_c^{\mu,b} A_c^{\nu,c} = -G^{\mu\nu,a}.$$

In addition, the Dirac field part of the Lagrangian density is invariant in the same way as the QED case, and therefore one sees that the Lagrangian density of QCD is invariant under the charge conjugation operation

$$\mathcal{L}_c = \bar{\psi}_c\Big[i\big(\partial^\mu + i(g_s)_c A_c^{\mu,a}T^a\big)\gamma_\mu - m_0\Big]\psi_c - \frac{1}{4}\, (G^a_{\mu\nu})_c (G^{\mu\nu,a})_c = \mathcal{L}.$$

2.6 Translational Invariance

When one transforms the coordinate x_k into $x_k + a_k$ with a_k a constant, then the wave function $\psi(x_k)$ becomes

$$\psi(x_k) \longrightarrow \psi(x_k + a_k). \tag{2.26}$$

This translation operation $\hat{R}_{a_k}$ can be written for a very small a as

$$\hat{R}_{a_k}\psi(x_k) = \psi(x_k + a_k) = \left(1 + a_k \frac{\partial}{\partial x_k}\right)\psi(x_k). \tag{2.27}$$

For the finite a_k, one can write

$$\psi(x_k + a_k) = \lim_{n\to\infty}\left(1 + \frac{a_k}{n}\frac{\partial}{\partial x_k}\right)^n \psi(x_k) = e^{ip_k a_k}\psi(x_k). \tag{2.28}$$

Therefore, one finds the translation operation $\hat{R}_{\boldsymbol{a}}$ in three dimensions as

$$\hat{R}_{\boldsymbol{a}} = e^{ip_k a_k} = e^{i\boldsymbol{p}\cdot\boldsymbol{a}}. \tag{2.29}$$

2.6.1 Energy Momentum Tensor

If the Lagrangian density is invariant under the translation, then there is a conserved quantity associated with this symmetry, which is called *energy momentum tensor* as will be defined below. Under the infinitesimal translation of a^ν, the field ψ transforms as

$$\psi' = \psi + \delta\psi, \quad \delta\psi = (\partial_\nu \psi) a^\nu,$$
$$\partial_\mu \psi' = \partial_\mu \psi + \delta(\partial_\mu \psi), \quad \delta(\partial_\mu \psi) = (\partial_\mu \partial_\nu \psi) a^\nu + (\partial_\nu \psi)(\partial_\mu a^\nu).$$

Since the Lagrangian density is invariant under the infinitesimal translation of a^ν, one has

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi', \partial_\mu \psi') - \mathcal{L}(\psi, \partial_\mu \psi) = \frac{\partial \mathcal{L}}{\partial \psi}\, \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \delta(\partial_\mu \psi)$$
$$= \frac{\partial \mathcal{L}}{\partial \psi}\, (\partial_\nu \psi) a^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, (\partial_\mu \partial_\nu \psi) a^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, (\partial_\nu \psi)(\partial_\mu a^\nu) = 0. \tag{2.30a}$$

By making use of the following equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial_\nu \psi a^\nu \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial_\nu \psi \right) a^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial_\nu \psi (\partial_\mu a^\nu)$$

and using the fact that the total divergence does not contribute to the action, one can obtain the following equation

$$\delta\mathcal{L} = \left[\frac{\partial \mathcal{L}}{\partial \psi}\, (\partial_\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, (\partial_\mu \partial_\nu \psi) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial_\nu \psi \right) \right] a^\nu. \tag{2.30b}$$

In addition, the following identity can be employed

$$\partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi}\, (\partial_\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, (\partial_\mu \partial_\nu \psi)$$

and one finds

$$\delta\mathcal{L} = \partial_\mu \left[\mathcal{L} g^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial^\nu \psi \right] a_\nu = 0. \tag{2.31}$$

The same thing should hold as well for the field $\psi^\dagger$ which is an independent functional variable in the Lagrangian density, and therefore eq.(2.31) should be modified

$$\delta\mathcal{L} = \partial_\mu \left[\mathcal{L} g^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial^\nu \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\, \partial^\nu \psi^\dagger \right] a_\nu = 0. \tag{2.31$'$}$$

This means that, if one defines the energy momentum tensor $\mathcal{T}^{\mu\nu}$ as

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\, \partial^\nu \psi^\dagger - \mathcal{L} g^{\mu\nu} \tag{2.32}$$

then $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0. \tag{2.33}$$

The reason why $\mathcal{T}^{\mu\nu}$ is called *energy momentum tensor* is because $\mathcal{T}^{0\nu}$ is related to the Hamiltonian and momentum densities.

2.6.2 Hamiltonian Density from Energy Momentum Tensor

Since the energy momentum tensor $\mathcal{T}^{0\nu}$ is a conserved quantity, one can define the Hamiltonian density $\mathcal{H}$ by $\mathcal{T}^{00}$

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\,\dot{\psi}^\dagger - \mathcal{L} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger}\,\dot{\psi}^\dagger - \mathcal{L}, \tag{2.34}$$

where the conjugate fields Π_ψ and $\Pi_{\psi^\dagger}$ are introduced as

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,, \quad \Pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\,.$$

The Hamiltonian H can be obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r$$

and it corresponds to the total energy of the field ψ.

2.7 Global Gauge Symmetry

If one transforms the field ψ into ψ' as

$$\psi' = e^{i\alpha}\psi \quad (\alpha \;\text{ is a real constant}) \tag{2.35}$$

then it is called *global gauge transformation* in which α does not depend on the coordinate x. This is a simple phase transformation which is also found in quantum mechanics since physical observables do not depend on the value of α.

Now, we discuss the invariance of the global gauge symmetry in the Lagrangian density. As examples, we consider the Lagrangian density of QED and the Thirring model

$$\mathcal{L} = \bar{\psi}\big(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m\big)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{2.36a}$$

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi - \frac{1}{2}\,gj^\mu j_\mu, \;\; \text{with} \;\; j_\mu = \bar{\psi}\gamma_\mu\psi. \tag{2.36b}$$

Obviously, the Lagrangian densities of eqs.(2.36) are invariant under the global gauge transformation

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi', \partial_\mu\psi', \psi'^\dagger, \partial_\mu\psi'^\dagger) - \mathcal{L}(\psi, \partial_\mu\psi, \psi^\dagger, \partial_\mu\psi^\dagger) = 0.$$

In this case, the Noether current associated with the global gauge symmetry is conserved as discussed in Appendix A.11

$$\delta\mathcal{L} = i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\psi^\dagger\right] = 0 \tag{2.37a}$$

which leads to the conservation of the vector current

$$\partial_\mu j^\mu = 0 \;\; \text{with} \;\; j^\mu = -i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \psi^\dagger \right]. \tag{2.37b}$$

For the Lagrangian densities of eqs.(2.36), the vector current j_μ in eq.(2.37b) just becomes

$$j_\mu = \bar{\psi}\gamma_\mu \psi.$$

In any field theory models, the conservation of the vector current is known to hold at any level of quantization or regularization, and therefore the charge Q associated with the vector current is always conserved.

2.8 Chiral Symmetry

In eqs.(2.36), if the fermion is massless ($m = 0$), then there is another symmetry which is called *chiral symmetry*. If one transforms the field ψ into ψ' as

$$\psi' = e^{i\alpha\gamma_5}\psi \;\; (\alpha \;\; \text{is a real constant}), \tag{2.38}$$

then one finds that the Lagrangian densities of the massless QED and the massless Thirring model are invariant under the chiral transformation. This is clear since the γ_5 anti-commutes with γ_μ

$$\{\gamma_5, \gamma_\mu\} = 0$$

and therefore one obtains for $\mu = 0, 1, 2, 3$

$$e^{-i\alpha\gamma_5}\gamma^\mu = \gamma^\mu e^{i\alpha\gamma_5}. \tag{2.39}$$

Thus, one sees that the $\bar{\psi}\gamma^\mu\psi$ transforms by the chiral symmetry as

$$\bar{\psi}'\gamma^\mu\psi' = \psi^\dagger e^{-i\alpha\gamma_5}\gamma_0\gamma^\mu e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^\mu\psi.$$

Since the Lagrangian density is invariant under the chiral transformation, the axial vector current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

is conserved, that is,

$$\partial_\mu j_5^\mu = 0.$$

2.8.1 Expression of Chiral Transformation in Two Dimensions

The chiral transformation of eq.(2.38) can be explicitly written in two dimensions for the field ψ.

Chiral Representation

In the chiral representation of the γ-matrix, the γ_5 and $e^{i\alpha\gamma_5}$ become

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^{i\alpha\gamma_5} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

Therefore, one has

$$\psi' = \begin{pmatrix} \psi'_a \\ \psi'_b \end{pmatrix} = e^{i\alpha\gamma_5} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\psi_a \\ e^{-i\alpha}\psi_b \end{pmatrix}. \tag{2.40}$$

Dirac Representation

In the Dirac representation of the γ-matrix, the γ_5 and $e^{i\alpha\gamma_5}$ become

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e^{i\alpha\gamma_5} = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}.$$

Therefore, one has

$$\psi' = \begin{pmatrix} \psi'_a \\ \psi'_b \end{pmatrix} = e^{i\alpha\gamma_5} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \psi_a\cos\alpha + i\psi_b\sin\alpha \\ \psi_b\cos\alpha + i\psi_a\sin\alpha \end{pmatrix}. \tag{2.41}$$

2.8.2 Mass Term

The mass term

$$m\bar{\psi}\psi$$

is not invariant under the chiral transformation

$$m\bar{\psi}'\psi' = m\bar{\psi}e^{2i\alpha\gamma_5}\psi \neq m\bar{\psi}\psi.$$

Therefore, if the system has a finite fermion mass, then the chiral symmetry is not preserved. In this respect, when one takes the massless limit

$$m \to 0$$

then the massless system may not necessarily be connected to the massive one if the chiral symmetry plays an important role for the determination of the vacuum. In fact, the massless limit is the singular point in the Thirring model, and the vacuum structures between the massive and massless Thirring models are completely different from each other. This is reasonable since the massless Thirring model has a vacuum which breaks the chiral symmetry while the massive Thirring model does not possesses the chiral symmetry and therefore its vacuum cannot be connected to the symmetry broken state. In addition, the massless Thirring model has no scaleful parameters, and thus physical observables should be measured in terms of the cutoff Λ, while, in the massive Thirring model, they are described by the mass m which cannot be set to zero after the system is solved.

Transformation of Mass Term in Two Dimensions

The mass term in two dimensions in the chiral representation transforms explicitly by the chiral transformation as

$$m\bar{\psi}'\psi' = m\left({\psi'}^\dagger_a {\psi'}_b + {\psi'}^\dagger_b {\psi'}_a\right) = m\left(e^{-2i\alpha}\psi^\dagger_a\psi_b + e^{2i\alpha}\psi^\dagger_b\psi_a\right) \neq m\bar{\psi}\psi$$

which shows again that the mass term is not invariant under the chiral transformation.

2.8.3 Chiral Anomaly

The conservation of the axial vector current is violated if there is a chiral anomaly. The chiral anomaly is closely related to the conflict between the local gauge invariance and the axial vector current conservation when the vacuum is regularized consistently with the local gauge invariance.

Four Dimensional QED

In four dimensional QED, the axial vector current is not conserved due to the anomaly and the conservation of the axial vector current is modified as

$$\partial_\mu j_5^\mu = \frac{g^2}{16\pi^2}\,\epsilon^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu}, \tag{2.42}$$

where $\epsilon^{\rho\sigma\mu\nu}$ denotes the anti-symmetric symbol in four dimensions. $F_{\mu\nu}$ denotes the electromagnetic field strength as given in eq.(1.44).

Two Dimensional QED

The same anomaly equation is found in the two dimensional QED and is written

$$\partial_\mu j_5^\mu = \frac{g}{2\pi}\,\epsilon_{\mu\nu}F^{\mu\nu}, \tag{2.43}$$

where $\epsilon_{\mu\nu}$ denotes the anti-symmetric symbol in two dimensions. The explicit derivation of the anomaly equation eq.(2.43) will be treated in detail in the context of the two dimensional QED in Chapter 5.

Two Dimensional QCD

There is no chiral anomaly in the two dimensional QCD. This can be easily understood when one writes a possible anomaly equation in QCD_2

$$\partial_\mu j_5^\mu \Longleftrightarrow \frac{g}{2\pi}\,\epsilon_{\mu\nu}G^{\mu\nu,a}.$$

However, the right hand side has the color index while the left hand side is a color singlet object, and there is no way to construct a color singlet object in the right hand side.

Four Dimensional QCD

Contrary to the two dimensional QCD, there is an anomaly in four dimensional QCD. The axial vector current conservation is modified as

$$\partial_\mu j_5^\mu = \frac{g^2}{32\pi^2}\,\epsilon^{\rho\sigma\mu\nu} G^a_{\rho\sigma} G^a_{\mu\nu}, \tag{2.44}$$

where $G^a_{\rho\sigma}$ is the chromomagnetic field strength as given in eq.(1.46).

2.8.4 Chiral Symmetry Breaking in Massless Thirring Model

The massless Thirring model has no local gauge invariance, and therefore there is no anomaly. Thus, the axial vector current is always conserved. Therefore, the axial charge is also a conserved quantity.

In the quantum field theory of the massless Thirring model, one quantizes the fermion fields and therefore the Hamiltonian becomes an operator. Thus, the eigenvalue equation for the Hamiltonian should be solved, and the lowest state is the vacuum where all the negative energy states are occupied by the negative energy particles. The construction of the vacuum state is very difficult since one has to solve infinite many body problems in the negative energy particles. Apart from the exactness of the vacuum state, one can discuss some properties of the vacuum state. One example is the symmetry of the Lagrangian density, and the vacuum can break the symmetry possessed in the Lagrangian density. When the symmetry broken vacuum is realized because it is the lowest energy, then it is called *spontaneous symmetry breaking* phenomenon if the current associated with the symmetry is conserved. We will discuss physics of the symmetry breaking in the vacuum state in detail in chapters 4 and 7.

2.9 $SU(3)$ Symmetry

In quantum mechanics, if the particle Hamiltonian H is invariant under the unitary transformation of $SU(3)$ group,

$$UHU^{-1} = H$$

then the eigenvalues of the particle Hamiltonian H are specified by the eigenvalues of $SU(3)$ group. The same transformation can be applied to the field theory models. Suppose ψ should have 3 degenerate states, and one transforms ψ as

$$\psi' = U\psi.$$

If the Lagrangian density is invariant under the $SU(3)$ transformation, then the Hamiltonian constructed from this Lagrangian density is specified by the eigenvalues of the $SU(3)$ group. In particular, hadron masses predicted by the Hamiltonian should be specified by the eigenvalues of the $SU(3)$ group. In the description of light baryons, one can assume that u, d and s quarks belong to the same multiplet. In this case, one can write ψ as

$$\psi(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix} . \tag{2.45}$$

By the unitary transformation of 3×3 matrix U, ψ transforms $\psi' = U\psi$ or explicitly

$$\begin{pmatrix} \psi'_u(x) \\ \psi'_d(x) \\ \psi'_s(x) \end{pmatrix} = U \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix} . \tag{2.46}$$

If the Hamiltonian $\hat{H}$ is invariant under the unitary transformation, then the hadron mass $\mathcal{M}$ can be described in terms of some function $G(a)$ as

$$\mathcal{M} = G([\lambda, \mu]), \tag{2.47}$$

where $[\lambda, \mu]$ denotes the quantum number of the symmetric group which specifies the representation of the $SU(3)$ group.

2.9.1 Dimension of Representation $[\lambda, \mu]$

The dimension $D_{[\lambda,\mu]}$ of the state represented by $[\lambda, \mu]$ becomes

$$D_{[\lambda,\mu]} = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2). \tag{2.48}$$

In fact, $[1, 0]$ or $[0, 1]$ are three dimensional representation which should just correspond to ψ or its anti-particle state. In this way, one sees that the $[1, 1]$ representation should have 8 states which are in fact found in nature as octet baryons

$$p, \ \ n, \ \ \Lambda, \ \ \Sigma^{\pm}, \ \ \Sigma^0, \ \ \Xi^{\pm}. \tag{2.49}$$

Indeed, their masses are found at around 1 GeV/c^2.

This success of the flavor $SU(3)$ is due to the fact that the interaction Hamiltonian is invariant under the $SU(3)$ transformation. Therefore, the flavor $SU(3)$ invariance of the Hamiltonian is broken by the mass term of the quarks. In particular, the mass of s-quark is assumed to be much larger than the masses of u- and d- quarks by one order of magnitude. However, the quark mass is still smaller than the hadron mass at least by an order of magnitude, and this is probably the main reason why the flavor $SU(3)$ works well. It is interesting to realize that, the fact that hadron masses are much larger than those of quarks indicates that the basic ingredients of generating hadron mass must come from the kinetic energy of quarks inside hadron which should give always positive energy contributions to the mass of hadron. Since the confinement of quarks must be due to the non-abelian character of the gauge fields, it should be most important to understand the properties of the non-abelian gauge field theory.

2.9.2 Useful Reduction Formula

Here, we summarize some examples of useful reduction formula of the $SU(3)$ product representations. First, we show the representation in terms of the dimension of the representation.

$$[1,0] = \mathbf{3}, \quad [0,1] = \mathbf{3}^*, \quad [0,0] = \mathbf{1}, \quad [1,1] = \mathbf{8}, \tag{2.50a}$$

$$[2,0] = \mathbf{6}, \quad [3,0] = \mathbf{10}, \quad [0,2] = \mathbf{6}^*, \quad [0,3] = \mathbf{10}^*. \tag{2.50b}$$

The following reduction formula may be useful

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus \mathbf{6}, \quad \mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}, \quad \mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10}. \tag{2.51}$$

For example, we can find the following results

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{3}^* \oplus \mathbf{6}) \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}, \tag{2.52}$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{27}. \tag{2.53}$$

Chapter 3

Quantization of Fields

In Chapter 1, we saw that the Lagrange equations for the fermion fields reproduce the Dirac equation which is the relativistic quantum mechanical equation of spin 1/2 fermions. In the Dirac equation, the field ψ would never disappear since the classical fields should always be present since they are c-number functions. The energy spectrum of the hydrogen atom can be described quite well by the Dirac equation.

However, if the hydrogen atom is in the excited state, then it naturally decays into the ground state at the final stage. During the process of transitions, the hydrogen atom emits photons. For example, when the hydrogen atom is in the $2p_{\frac{1}{2}}$-state, then it decays into the ground state of $1s_{\frac{1}{2}}$-state by emitting a photon. In this case, the photon is created during the transition. Therefore, one has to invent some scheme which takes into account the creation or annihilation of electromagnetic fields, and indeed the gauge fields should be quantized in terms of the commutation relations for the creation and annihilation operators.

The Dirac field should be always quantized because of the Pauli principle. The experimental observations in atoms show that one quantum state can be occupied only by one electron (Pauli principle). In order to accomodate the Pauli principle, one has to quantize the Dirac field in terms of anti-commutation relations for the creation and annihilation operators. The quantization of Dirac field with anti-commutation relations is also required from the presence of the negative energy states as the physical observables, and the negative energy states can be well fit into the theoretical framework in terms of the Pauli principle. The field quantization is also consistent with the observation that the pair of electron and positron can be created from virtual photons in the scattering process if some physical conditions are satisfied. In this sense, the field quantization should be made in terms of the creation and annihilation operators of fermion fields, and therefore, the field becomes an operator, and consequently the Hamiltonian becomes an operator.

$$H \Longrightarrow \hat{H} \quad (\text{operator after field quantization}).$$

In this case, one should solve an eigenvalue equation for the Hamiltonian with a corresponding eigenstate $|\Psi\rangle$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{3.1}$$

where E denotes the energy eigenvalue. The state $|\Psi\rangle$ is called *Fock state*, and in quantum field theory, the problem is now how one can solve the eigenvalue equation and determine the energy and eigenstate of the Hamiltonian. In general, it is extremely difficult to solve the eigenvalue equation in quantum field theory. The basic difficulty of the quantized field theory comes mainly from the fact that one has to construct the vacuum state $|\Omega\rangle$ which is composed of infinite numbers of negative energy particles interacting with each other. In addition, $|\Psi\rangle$ should be constructed on this vacuum state by creating particles and anti-particles, and it should satisfy eq.(3.1).

In this chapter, we first treat the quantization of free fermion fields. In most of the field theory models with interactions, the field quantization is done for free fields since one cannot directly quantize the interacting fields. Then, we discuss the quantization of the Thirring model since it can be solved exactly in terms of the Bethe ansatz method. Since the Thirring model gives a non-trivial field theory model, we can learn a lot from this field theory model. We also present the quantization of gauge fields so that we can calculate some scattering processes between electrons in the later chapter.

3.1 Quantization of Free Fermion Field

Classical fermion fields with the Dirac equation can describe the spectrum of the hydrogen atom quite well. However, the representation of the classical field has a limitation since experimental observations indicate that fermion and anti-fermion pairs can be created from the vacuum if the conditions of pair creations are satisfied. This means that the fermion fields cannot be taken as a c-number field. Instead, one should consider the fermion field as an operator. If the field becomes operator, then the value of the field should vary, depending on the state (Fock state) which should be prepared in accordance with the process one wishes to calculate in the perturbation theory.

3.1.1 Creation and Annihilation Operators

We start from the quantization of free Dirac fields. The Lagrangian density of free Dirac field is written as

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi = i\psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j .$$

In this case, the Hamiltonian can be obtained as given in eq.(1.19)

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi \, d^3r. \tag{1.19}$$

Now, we write the free Dirac field as

$$\psi(\boldsymbol{r}, t) = \sum_{\boldsymbol{n}, s} \frac{1}{\sqrt{L^3}} \left(a_{\boldsymbol{n}}^{(s)} u_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n} \cdot \boldsymbol{r} - iE_{\boldsymbol{n}} t} + b_{\boldsymbol{n}}^{(s)} v_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n} \cdot \boldsymbol{r} + iE_{\boldsymbol{n}} t} \right), \tag{3.2}$$

where $u_{\boldsymbol{n}}^{(s)}$ and $v_{\boldsymbol{n}}^{(s)}$ denote the spinor part of the plane wave solutions as given in Chapter 1, and can be written as

$$u_{\boldsymbol{n}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{n}}+m}{2E_{\boldsymbol{n}}}} \begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_n}}{E_{\boldsymbol{n}}+m}\chi_s \end{pmatrix}, \tag{3.3a}$$

$$v_{\boldsymbol{n}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{n}}+m}{2E_{\boldsymbol{n}}}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_n}}{E_{\boldsymbol{n}}+m}\chi_s \\ \chi_s \end{pmatrix}, \tag{3.3b}$$

where

$$\boldsymbol{p_n} = \frac{2\pi}{L}\boldsymbol{n}, \quad E_{\boldsymbol{n}} = \sqrt{\boldsymbol{p}_{\boldsymbol{n}}^2 + m^2}$$

and s denotes the spin index with $s = \pm\frac{1}{2}$. Inserting this field into eq.(1.19), one can express the Hamiltonian as

$$H = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger}{}_{\boldsymbol{n}}^{(s)} a_{\boldsymbol{n}}^{(s)} - b^{\dagger}{}_{\boldsymbol{n}}^{(s)} b_{\boldsymbol{n}}^{(s)} \right) + \text{some constants.}$$

The Hamiltonian is a conserved quantity, and therefore we can quantize it. Here, the basic method to quantize the fields is to require that the annihilation and creation operators $a_{\boldsymbol{n}}^{(s)}$ and $a^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for positive energy states and $b_{\boldsymbol{n}}^{(s)}$ and $b^{\dagger}{}_{\boldsymbol{n}'}^{(s')}$ for negative energy states become operators which satisfy the anti-commutation relations.

Anti-commutation Relations

The creation and annihilation operators for positive and negative energy states should satisfy the following anti-commutation relations,

$$\left\{ a_{\boldsymbol{n}}^{(s)}, a^{\dagger}{}_{\boldsymbol{n}'}^{(s')} \right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}, \quad \left\{ b_{\boldsymbol{n}}^{(s)}, b^{\dagger}{}_{\boldsymbol{n}'}^{(s')} \right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}. \tag{3.4}$$

All the other cases of the anti-commutations vanish, for examples,

$$\left\{ a_{\boldsymbol{n}}^{(s)}, a_{\boldsymbol{n}'}^{(s')} \right\} = 0, \quad \left\{ b_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')} \right\} = 0, \quad \left\{ a_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')} \right\} = 0. \tag{3.5}$$

This corresponds to the field quantization, and the quantization in terms of creation and annihilation operators should be the fundamental quantization procedure.

3.1.2 Equal Time Quantization of Field

The quantization of fields can also be written in terms of equal time anti-commutation relations for fields as,

$$\left\{ \psi_i(\boldsymbol{r},t), \pi_j(\boldsymbol{r'},t) \right\} = i\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r'}), \tag{3.6}$$

where the conjugate field π_i is given by the Lagrangian density as

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger .$$

Therefore, the quantization condition of eq.(3.6) becomes

$$\left\{ \psi_i(\boldsymbol{r}, t), \psi_j^\dagger(\boldsymbol{r'}, t) \right\} = \delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r'}). \tag{3.7}$$

It is important to note that the field quantization must be done always with equal time. Further, one may consider the quantization conditions of fields in terms of eqs.(3.4) and (3.5) as being more fundamental than eq.(3.6), and eq.(3.6) should be taken as the field quantization method which can be derived from eqs.(3.4) and (3.5) together with eq.(3.2).

3.1.3 Quantized Hamiltonian of Free Dirac Field

Now one finds the quantized Hamiltonian which is given as

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} - b^{\dagger(s)}_{\boldsymbol{n}} b^{(s)}_{\boldsymbol{n}} \right) + C_0, \tag{3.8}$$

where C_0 is a constant and normally it is discarded since it does not affect on physical observables. This Hamiltonian $\hat{H}$ is written in terms of the creation and annihilation operators.

It may be worthwhile noting that the Hamiltonian is obtained by integrating the Hamiltonian density over all space, and therefore it does not depend on space and also it is a conserved quantity. Instead it is not a c-number but the operator. Therefore, one should find Fock states which must be the eigenstates of the Hamiltonian. For free Dirac fields, one can easily find the eigenstates of the Hamiltonian, and indeed the vacuum state is constructed by filling out all the negative energy states by the negative energy particles as will be treated below.

Anti-particle Representation

The representation of $b^{(s)}_{\boldsymbol{n}}$ corresponds to the negative energy state. The anti-particle representation can be obtained by defining a new operator $\mathrm{b}^{(s)}_{\boldsymbol{n}}$ as

$$\mathrm{b}^{(s)}_{\boldsymbol{n}} = b^{\dagger(s)}_{-\boldsymbol{n}}.$$

In this case, the operator $\mathrm{b}^{(s)}_{\boldsymbol{n}}$ describes the annihilation of an anti-particle. In this representation, the Hamiltonian of free Dirac field becomes

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} + \mathrm{b}^{\dagger(s)}_{\boldsymbol{n}} \mathrm{b}^{(s)}_{\boldsymbol{n}} \right) + C_0{}'. \tag{3.9}$$

Perturbative Vacuum

This expression is employed in most of the field theory textbooks, and it is suitable for describing the processes of fermion creation and annihilation. However, one cannot treat the interacting vacuum state since it is assumed that the vacuum is a simple one which satisfies

$$a_{\boldsymbol{n}}^{(s)}|0\rangle = 0, \quad \mathrm{b}_{\boldsymbol{n}}^{(s)}|0\rangle = 0$$

which defines the vacuum state $|0\rangle$ in the perturbative sense. The construction of the vacuum state in interacting systems is quite difficult and mostly impossible in four dimensional field theory models.

3.1.4 Vacuum of Free Field Theory

When there is no interaction, one can easily construct the exact vacuum state. In this case, one considers the maximum number of freedom to be N, and the particles are put into the box with its length L. The momenta and energies of the negative energy particles can be written as

$$\boldsymbol{p}_n = \frac{2\pi}{L}\boldsymbol{n}, \quad E_{\boldsymbol{n}} = -\sqrt{m^2 + \boldsymbol{p}_{\boldsymbol{n}}^2}\,, \tag{3.10}$$

where n_k are integers and run

$$n_k = 0, \pm 1, \pm 2, \dots, \pm N.$$

Further, one defines the cut-off momentum Λ by

$$\Lambda = \frac{2\pi}{L} N \tag{3.11}$$

and one lets N and L as large as required, keeping Λ finite. If the model field theory has no scaleful parameter with massless fermions, then physical observables must be measured by the Λ. But they should not depend on either N nor L.

Fock Space Vacuum

The Fock space vacuum can be written as

$$|0\rangle = \prod_{n_k} b^{\dagger}{}_{\boldsymbol{n}}^{(s)}|0\rangle\rangle, \ \text{ with } \ E_{\boldsymbol{n}} = -\sqrt{m^2 + \boldsymbol{p}_{\boldsymbol{n}}^2}\,, \tag{3.12}$$

where $|0\rangle\rangle$ denotes a null vacuum state which is defined as

$$b_{\boldsymbol{n}}^{(s)}|0\rangle\rangle = 0, \quad a_{\boldsymbol{n}}^{(s)}|0\rangle\rangle = 0. \tag{3.13}$$

This exact vacuum state of the free Dirac field is called *perturbative vacuum state* since it is often employed for the quantum field theory with interactions.

It should be noted that the construction of the exact vacuum state is most important since from this vacuum one can create any physical states by applying creation operators. However, it is, at the same time, clear that the exact vacuum state cannot be normally obtained in the field theory models in four dimensions. There are two fermion field theory models which can be solved exactly. That is, the Schwinger model and Thirring model can be solved exactly, but they are two dimensional field theory models and will be treated in the later chapter.

3.2 Quantization of Thirring Model

Now, we present the quantized Hamiltonian of the Thirring model as an example. In the chiral representation of the γ matrices, the Hamiltonian of the Thirring model becomes

$$\hat{H}=\int dx \left[-i\left(\psi_a^\dagger \frac{\partial}{\partial x}\psi_a - \psi_b^\dagger \frac{\partial}{\partial x}\psi_b\right)+m_0(\psi_a^\dagger\psi_b+\psi_b^\dagger\psi_a)+2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b\right]. \quad (3.14)$$

Now, the fermion fields are quantized in one space dimension with a box length of L

$$\psi(x)=\begin{pmatrix}\psi_a(x)\\ \psi_b(x)\end{pmatrix}=\frac{1}{\sqrt{L}}\sum_n \begin{pmatrix}a_n\\ b_n\end{pmatrix} e^{ip_n x}, \quad (3.15)$$

where

$$p_n=\frac{2\pi}{L}n, \quad \text{with} \quad n=0,\pm 1,\dots .$$

The creation and annihilation operators satisfy the following anti-commutation relations

$$\{a_n,a_m^\dagger\}=\{b_n,b_m^\dagger\}=\delta_{nm}, \quad \{a_n,a_m\}=\{b_n,b_m\}=\{a_n,b_m\}=0. \quad (3.16)$$

Quantized Hamiltonian in Chiral Representation

In this case, the quantized Hamiltonian can be written

$$\hat{H}=\sum_n \left[p_n\left(a_n^\dagger a_n - b_n^\dagger b_n\right)+m_0\left(a_n^\dagger b_n + b_n^\dagger a_n\right)+\frac{2g}{L}\tilde{j}_a(p_n)\tilde{j}_b(p_n)\right], \quad (3.17)$$

where the currents $\tilde{j}_a(p_n)$ and $\tilde{j}_b(p_n)$ in the momentum representation are given by

$$\tilde{j}_a(p_n)=\sum_l a_l^\dagger a_{l+n}, \quad (3.18a)$$

$$\tilde{j}_b(p_n)=\sum_l b_l^\dagger b_{l+n}. \quad (3.18b)$$

3.2.1 Vacuum of Thirring Model

In general, it is very difficult to construct the vacuum state for interacting field theory models. One has to solve the eigenvalue equation of the Hamiltonian with the infinite number of the negative energy particles, and normally it is impossible.

Fortunately, however, the massless Thirring model can be solved exactly by the Bethe ansatz technique . Furthermore, the solution of the vacuum state is given analytically. Since detailed discussions will be given in Chapter 7, we give only the vacuum state which is constructed by operating creation operators.

Exact Vacuum

The exact vacuum state of the massless Thirring model $|\Omega\rangle$ can be written as

$$|\Omega\rangle = \prod_{k_i^\ell} a^\dagger_{k_i^\ell} \prod_{k_j^r} b^\dagger_{k_j^r} |0\rangle\rangle, \tag{3.19}$$

where $|0\rangle\rangle$ denotes the null vacuum state with

$$a_{k_i^\ell}|0\rangle\rangle = 0, \quad b_{k_j^r}|0\rangle\rangle = 0. \tag{3.20}$$

The momenta k_i^ℓ and k_j^r should satisfy the periodic boundary condition (PBC) equations which are solved analytically and the momenta k_i^ℓ for left mover and k_j^r for right mover are given as

$$k_1^r = \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \;\; n_1 = 0, \tag{3.21a}$$

$$k_j^r = \frac{2\pi n_j}{L} + \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \;\; n_j = 1, 2, \dots, N_0, \tag{3.21b}$$

$$k_i^\ell = \frac{2\pi n_i}{L} - \frac{2(N_0+1)}{L}\tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \;\; n_i = -1, -2, \dots, -N_0. \tag{3.21c}$$

In this case, the vacuum energy becomes

$$E_v^{\text{true}} = -\Lambda\left\{N_0 + 1 + \frac{2(N_0+1)}{\pi}\tan^{-1}\left(\frac{g}{2}\right)\right\},$$

where Λ denotes the cutoff momentum.

Cut-off momentum Λ

Here, g, L and N_0 denote the coupling constant, the box length and the particle number in the negative energy state, respectively. The cut-off Λ is defined as

$$\Lambda = \frac{2\pi N_0}{L}.$$

Since there is no mass scale in the massless Thirring model, all the observables must be measured in terms of the Λ. The number N_0 and the box length L can be set to any large number as required, and any physical observables should not depend on neither N_0 nor L.

3.3 Quantization of Gauge Fields in QED

In this section, we present the quantization of the gauge fields A_μ in QED. This can be found in any textbooks, and therefore we discuss it briefly so that we can calculate some of the scattering S-matrix in QED processes.

Here, we employ the quantization procedure with the Coulomb gauge fixing condition. The Hamiltonian of the electromagnetic fields is written as

$$\hat{H}_{\rm em} = \frac{1}{2}\int \left[\Pi_k^2 - \left(\frac{\partial A_0}{\partial x_k}\right)^2 + \left(\frac{\partial A_k}{\partial x_j}\frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}\right)\right] d^3r, \tag{3.22}$$

where Π_k is a conjugate field to A_k and is given as

$$\Pi_k = -\dot{A}_k.$$

It should be noted that there is no term corresponding to the $\dot{A}_0$ since there is no kinetic energy term arising from the A_0 term. In this sense, the A_0 is not a dynamical variable any more.

Coulomb Gauge Fixing

Therefore, one should quantize the gauge field $\boldsymbol{A}$. However, one should be careful for the number of the degree of freedom of the gauge fields since there is a gauge fixing condition. For example, if one takes the Coulomb gauge.

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{3.23}$$

then, the gauge field $\boldsymbol{A}$ should have two degrees of freedom. In this case, the gauge field $\boldsymbol{A}$ can be expanded in terms of the free field solutions

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\,\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c_{\boldsymbol{k},\lambda}^\dagger e^{ikx}\right], \tag{3.24}$$

where

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|.$$

The polarization vector $\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)$ should satisfy the following relations

$$\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{\epsilon}(\boldsymbol{k},\lambda') = \delta_{\lambda,\lambda'} \tag{3.25}$$

since the gauge field $\boldsymbol{A}$ should satisfy eq.(3.23).

Commutation Relations

Since the gauge fields are bosons, the quantization procedure must be done in the commutation relations, instead of anti-commutation relations. Therefore, the quantization can be done by requiring that $c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k},\lambda}$ should satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} \tag{3.26}$$

and all other commutation relations vanish.

In this case, the Hamiltonian $\hat{H}_{\rm em}$ of the electromagnetic fields is written in terms of the creation and annihilation operators as

$$\hat{H}_{\rm em} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}\left(c^\dagger_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right). \tag{3.27}$$

From eq.(3.27), one sees that there are two degrees of freedom for the quantized gauge fields. Since the gauge field $\boldsymbol{A}$ has always a gauge freedom, it may be the best to quantize the gauge field $\boldsymbol{A}$ in terms of the creation and annihilation operators $c_{\boldsymbol{k},\lambda},\ c^\dagger_{\boldsymbol{k},\lambda}$ after the gauge fixing is done.

Zero Point Energy

Eq.(3.27) contains a zero point energy. That is, the vacuum state where there is no electromagnetic field present has an infinite energy

$$E_{\rm vac} = 2\times\frac{1}{2}\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \sum|\boldsymbol{k}| \to \infty.$$

However, there is nothing serious since the vacuum state cannot be observed. Therefore, one should measure the energy of excited states from the vacuum, and thus

$$\Delta E_{\rm em} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}\left(c^\dagger_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right) - E_{\rm vac} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}c^\dagger_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda}$$

must be physical observables.

3.4 Quantization of Schrödinger Field

As we discussed in the first chapter, the non-relativistic fields do not have to be quantized since there are no creation and annihilation of particles in the non-relativistic kinematics.

Nevertheless, one can quantize the Schrödinger field and work out physical observables in the second quantized representation. It is, of course, the same as the classical field theory calculation, but sometime the second quantized representation is easier than the classical field version.

3.4.1 Creation and Annihilation Operators

We consider the Lagrangian density of eq.(1.2)

$$\mathcal{L} = i\psi^\dagger \frac{\partial\psi}{\partial t} - \frac{1}{2m}\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi - \psi^\dagger U \psi.$$

In this case, the Schrödinger field $\psi(\boldsymbol{r}, t)$ can be expanded in terms of the free field solutions

$$\psi(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - iEt}, \tag{3.28}$$

where $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ are required to satisfy the following anti-commutation relations

$$\{a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\} = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \quad \{a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}\} = \{a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\} = 0. \tag{3.29}$$

Since one assumes that the Schrödinger field corresponds to fermions, one can carry out the field quantization in the anti-commutation relations.

3.4.2 Fermi Gas Model

From eq.(3.28), one sees that there is no anti-particle present in this model. This is clear since one starts from the vacuum which has no particle at all. On the other hand, if one starts from the Fermi surface and identifies the vacuum state in which all the states are occupied up to the Fermi energy ϵ_F, then one can discuss the particle-hole states which have some similarity with the Dirac hole state. In fact, the formulation is just the same as the Dirac vacuum, but there is of course no anti-particle. The hole state is just a hole in the Fermi sea.

Fermi Momentum

In this picture, particles corresponding to the Schrödinger fields are assumed to obey the Pauli principle. In this respect, the Schrödinger field is considered as the fermion field which should satisfy the anti-commutation relations as shown in eq.(3.29). Therefore, in the Fermi gas model, particle states are occupied up to the Fermi momentum k_F, and when the system has the number of particle N, then one finds

$$N = \sum_{n_x, n_y, n_z} = \frac{L^3}{(2\pi)^3} \int_{|\boldsymbol{k}| \le k_F} d^3k = \frac{L^3}{6\pi^2} k_F^3. \tag{3.30}$$

In this case, the Fermi energy ϵ_F is written as

$$\epsilon_F = \frac{1}{2M} k_F^2, \tag{3.31}$$

where M denotes the mass of the Schrödinger particle. Eq.(3.30) indicates that the density ρ of the system becomes

$$\rho = \frac{N}{L^3} = \frac{1}{6\pi^2} k_F^3. \tag{3.32}$$

Spin and Isospin

If the particle is a nucleon, it has spin s and isospin t. In this case, eq.(3.32) becomes

$$\rho = \frac{N}{L^3} = (2s+1)(2t+1)\,\frac{1}{6\pi^2}\,k_F^3 = \frac{2}{3\pi^2}\,k_F^3, \tag{3.33}$$

where

$$s = t = \frac{1}{2}\,. \tag{3.34}$$

3.5 Quantized Hamiltonian of QED and Eigenstates

The Hamiltonian of fermions with electromagnetic fields is given in eq.(1.33)

$$H = \int \left\{\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi - g\boldsymbol{j}\cdot\boldsymbol{A}\right\} d^3r$$

$$+\frac{g^2}{8\pi}\int \frac{j_0(\boldsymbol{r'})j_0(\boldsymbol{r})}{|\boldsymbol{r'}-\boldsymbol{r}|}\,d^3r\,d^3r' + \frac{1}{2}\int \left(\boldsymbol{E}_T^2 + \boldsymbol{B}^2\right) d^3r.$$

Now, the fermion field ψ and the gauge field $\boldsymbol{A}$ are quantized as

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{n},s} \frac{1}{\sqrt{L^3}}\left(a_{\boldsymbol{n}}^{(s)} u_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}} + b_{\boldsymbol{n}}^{(s)} v_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}}\right), \tag{3.35}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\,\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + c_{\boldsymbol{k},\lambda}^{\dagger}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right]. \tag{3.36}$$

3.5.1 Quantized Hamiltonian

In this case, the quantized Hamiltonian of eq.(1.33) can be written as

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} - b^{\dagger(s)}_{\boldsymbol{n}} b^{(s)}_{\boldsymbol{n}}\right) + \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}\left(c_{\boldsymbol{k},\lambda}^{\dagger}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right)$$

$$-g\int \bar{\psi}(\boldsymbol{r})\boldsymbol{\gamma}\psi(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r})\,d^3r + \frac{g^2}{8\pi}\int \frac{(\bar{\psi}(\boldsymbol{r'})\gamma_0\psi(\boldsymbol{r'}))(\bar{\psi}(\boldsymbol{r})\gamma_0\psi(\boldsymbol{r}))}{|\boldsymbol{r'}-\boldsymbol{r}|}\,d^3r\,d^3r'. \tag{3.37}$$

The interaction terms are so complicated that we do not write them here in terms of the creation-annihilation operators in an explicit fashion. However, one notices that the interaction terms should induce particle-anti-particle creations or destructions.

3.5.2 Eigenvalue Equation

Now, one should solve eq.(3.1) with the above Hamiltonian eq.(3.37)

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{3.38}$$

where $|\Psi\rangle$ may be written even for simple bosonic excitation cases

$$|\Psi\rangle = \sum_{(\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n,t_n)} f\Big((\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n,t_n)\Big)$$
$$\times a^{\dagger(s_1)}_{\boldsymbol{p}_1}\cdots a^{\dagger(s_n)}_{\boldsymbol{p}_n} b^{(t_1)}_{\boldsymbol{q}_1}\cdots b^{(t_n)}_{\boldsymbol{q}_n}|\Omega\rangle, \tag{3.39}$$

where f is the wave function which should be determined so as to satisfy eq.(3.1). The energy eigenvalue E in eq.(3.38) may be calculated by the diagonalization procedure where the space is spanned in terms of $f\left((\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n),t_n)\right)$. For a practical evaluation, one has to truncate the space significantly so as to carry out any numerical calculations. Numerical calculations in two dimensional QED with finite fermion mass will be discussed in Chapter 5.

3.5.3 Vacuum State $|\Omega\rangle$

$|\Omega\rangle$ denotes the vacuum state which should satisfy the eigenvalue equation of eq.(3.1)

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle,$$

where E_Ω denotes the vacuum energy, and $|\Omega\rangle$ is full of negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger(s)}_{\boldsymbol{p}}|0\rangle\rangle,$$

where $|0\rangle\rangle$ denotes the null vacuum which satisfies

$$b^{(s)}_{\boldsymbol{p}}|0\rangle\rangle = 0, \quad a^{(s)}_{\boldsymbol{p}}|0\rangle\rangle = 0.$$

One sees clearly that it is practically impossible to solve the eigenvalue equation of the Hamiltonian in eq.(3.1).

Chapter 4

Goldstone Theorem and Spontaneous Symmetry Breaking

The continuous symmetry of the Lagrangian density leads to the conservation of currents and therefore the system should have a conserved charge associated with the symmetry. The best example must be a global gauge symmetry in which the Lagrangian density is invariant under the transformation of the Dirac field ψ as

$$\psi' = e^{i\alpha}\psi \Longrightarrow \mathcal{L}' = \mathcal{L},$$

where α is a real constant. In this case, the vector current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved. That is,

$$\partial_\mu j^\mu = 0. \tag{4.1}$$

In fermion field theory models, the global gauge symmetry is not broken at any level even though the vacuum state can, in principle, break its symmetry. In particular, the gauge invariance of the local gauge field theory should hold rigorously since the violation of the local gauge invariance should lead to the breakdown of defining physical observables.

On the other hand, the chiral symmetry behaves quite differently from the global gauge symmetry. When the Lagrangian density is invariant under the chiral symmetry transformation

$$\psi' = e^{i\alpha\gamma^5}\psi \Longrightarrow \mathcal{L}' = \mathcal{L}$$

the axial vector current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

is also conserved, that is,

$$\partial_\mu j_5^\mu = 0. \tag{4.2}$$

However, the chiral symmetry is broken not only because of the chiral anomaly but also in terms of the spontaneous symmetry breaking mechanism. In the former case, the conservation of the axial vector current is violated by the anomaly term while, in the spontaneous chiral symmetry breaking, the vacuum loses its symmetry since the vacuum state prefers the lowest energy state which does not have to keep its symmetry. The discussion of the anomaly term is given in Chapter 5 in the context of the Schwinger model where the basic mechanism of the chiral anomaly and the violation of the axial vector current are described in a transparent fashion.

In this chapter, we discuss the symmetry breaking phenomena in fermion field theory models. In particular, we clarify what is physics of the spontaneous symmetry breaking and the Goldstone theorem [60, 61]. Normally, the mathematics in the theorem can be understood in a straightforward way, but its physics in connection with the theorem is always difficult since one has to examine all the possible conditions in nature when the symmetry is broken spontaneously.

First, we explain the general feature of the symmetry and its preservation in quantum many body theory, before going to the discussion of the Goldstone theorem. Then, we explain the Goldstone theorem and the problem related to the proof of existence of the Goldstone boson. Also, we present a new interpretation of the Goldstone theorem. In particular, we give a good example of the chiral symmetry breaking in the Thirring model together with the exact solution of the vacuum state.

Further, we comment on the symmetry breaking of boson field theory models together with the Higgs mechanism. However, this part is presented with reservation since there are still some problems which are not yet clarified completely in this textbook. The difficulty of the symmetry breaking physics in boson field theory is partly because the scalar boson field itself has some serious problems as will be discussed in Appendix C and partly because there is no model which can present exact solutions of the boson field theory models.

4.1 Symmetry and Its Breaking in Vacuum

Various symmetries of Hamiltonian play an important role for determining the energy eigenvalues, and in quantum mechanics, one often sees that the lowest state (ground state) preserves the symmetry of the Hamiltonian. In fact, those states which break the symmetry are, in general, higher than the symmetry preserving state for the same quantum numbers or configurations of the wave function. This can be naturally realized in quantum many body theory as our experiences tell us.

However, the physics of the symmetry breaking in quantum field theory is quite different, and phenomena which seem to be in an apparent contradiction with the picture of quantum many body theory can indeed occur. This is called *spontaneous symmetry breaking* and it has been discussed extensively in many field theory textbooks. In the spontaneous symmetry breaking, the vacuum state of the field theory models is realized with the symmetry broken state. That is, the true vacuum prefers the symmetry broken state, contrary to the naive expectation in quantum many body theory.

A question may arise as to what should be an intuitive explanation why the vacuum of the field theory models has the symmetry broken state as the most favorable state, in contradiction with the experiences of quantum many body theory. Here, we show that the symmetry broken state of the vacuum in the field theory models is naturally realized in the context of quantum many body theory. In short, the vacuum of quantum field theory is constructed by the negative energy particles, and therefore, the symmetry preserving state should have the absolute magnitude of its total energy which is smaller than the symmetry broken state. This is just consistent with the prediction of quantum many body theory. However, energies of the vacuum are all negative, and thus the lower state is, of course, the one that breaks the symmetry since the absolute magnitude of its energy is larger than that of the symmetry preserving state. This is exactly what one observes in the Thirring model as will be discussed in Chapter 7.

4.1.1 Symmetry in Quantum Many Body Theory

In quantum mechanics, the ground state energy of the particle Hamiltonian H can be written

$$E_{\rm tot} = \langle 0|H|0\rangle, \tag{4.3}$$

where we denote the ground state by $|0\rangle$ which preserves the symmetry of the Hamiltonian H. In this case, the lowest state is normally the one that keeps the symmetry. The total energy of the states which do not keep the symmetry should be found in higher energies than the symmetry preserving state. This energy of the N particle state may be written more explicitly as

$$E_{\rm tot} = \sum_{i=1}^{N} \mathcal{E}_i(\boldsymbol{k}_i), \tag{4.4}$$

where we denote the energy of the i-th particle by $\mathcal{E}_i(\boldsymbol{k}_i)$. The momentum $\boldsymbol{k}_i$ should be determined by solving the many body equations of motion

$$F_i(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N) = 0, \quad i = 1, \ldots, N. \tag{4.5}$$

Now, suppose there is a symmetry in the Hamiltonian H. The solution of the above equations should be specified by the symmetric and the symmetry broken solutions. In quantum many body system, it is often the case that the symmetric solution $E_{\rm tot}^{sym}$ is lower than the symmetry broken solution $E_{\rm tot}^{sym.br}$.

$$E_{\rm tot}^{sym} < E_{\rm tot}^{sym.br}. \tag{4.6}$$

This, of course, depends on the interactions between particles in the system, and one can only claim that there should be some systems in which eq.(4.6) can hold.

4.1.2 Symmetry in Field Theory

In quantum field theory, the symmetry breaking phenomena occur in a completely different fashion. The lowest state which is called *vacuum* sometimes breaks the continuous symmetry since it is found to be lower than the symmetry preserving state. Since the continuous symmetry has the Noether current associated with its symmetry, the current conservation should hold true in the process of determining the lowest state of the model field theory as long as the model has no anomaly.

Why is it possible that the symmetry broken state becomes lower than the symmetry preserving state in an obvious contradiction with the picture of quantum many body theory? Here, we present a simple intuitive picture why it may occur. The basic point is that the vacuum in field theory models is constructed by particles with the negative energies which are solved in the many body Dirac equations. To be more specific, the vacuum energy $E_{\rm vac}$ can be written as

$$E_{\rm vac} = -\lim_{N\to\infty}\sum_{i=1}^{N}\mathcal{E}_i(\boldsymbol{k}_i), \tag{4.7}$$

where the energy of the i-th particle is denoted by $\mathcal{E}_i$. Since the system is infinite, one should make the number N infinity at the end of the calculation. It should be noted that one cannot make the system infinity from the beginning since in this case one cannot define the total energy of the system. This construction of the vacuum in terms of the finite N-particle system and then making the number N infinity must be well justified since the deep negative energy states in the vacuum should not have any effects on the physical properties of the vacuum state.

In order to determine the momenta of the negative energy particles, one should solve the equations of motion which may be similar to eq.(4.5)

$$G_i(\boldsymbol{k}_1,\dots,\boldsymbol{k}_N) = 0, \quad i = 1,\dots,N. \tag{4.8}$$

Again, one can assume that there is a symmetry in the Hamiltonian H. In this case, the solution of the above equations should be specified by the symmetric solution and the symmetry broken solution. Just in the same way as the positive energy case, if one defines the total energy $E_{\rm tot}$ by

$$E_{\rm tot} = \sum_{i=1}^{N}\mathcal{E}_i(\boldsymbol{k}_i) \tag{4.9}$$

then the symmetric solution $E_{\rm tot}^{sym}$ must be lower than the symmetry broken solution $E_{\rm tot}^{sym.br}$

$$E_{\rm tot}^{sym} < E_{\rm tot}^{sym.br}. \tag{4.10}$$

However, the energy of the vacuum is all negative, and therefore one sees that the symmetry broken vacuum state must be the lowest, that is

$$E_{\rm vac}^{sym.br} < E_{\rm vac}^{sym} \tag{4.11}$$

which is just opposite to the prediction of the quantum many body theory.

In this way, the vacuum in field theory models prefers the symmetry broken state. The symmetry preserving state has the lowest energy in magnitude, but due to the negative sign in front (eq.(4.7)), the lowest energy must be the one that breaks the symmetry. This is exactly what happens in the spontaneous symmetry breaking in fermion field theory models. This appearance of the two states, one that preserves the symmetry and the other that breaks the symmetry must depend on the dynamical properties of the models one considers. Up to now, the Thirring model exhibits the two states in the vacuum, and therefore it presents a good example of the spontaneous symmetry breaking physics. In the next section, we discuss the Goldstone theorem, keeping this fact in mind.

4.2 Goldstone Theorem

The physics of the spontaneous symmetry breaking started from the Goldstone theorem. The theorem states that there should appear a massless boson when the symmetry of the vacuum state is spontaneously broken. In this process of the spontaneous symmetry breaking, the current conservation should hold. This is important since the Goldstone theorem is entirely based on the current conservation, and without the current conservation the theorem cannot be proved.

Here, without loss of generality, we can restrict our discussion to the chiral symmetry breaking of the fermion field theory models. In this case, the chiral charge Q_5 must be a conserved quantity.

4.2.1 Conservation of Chiral Charge

When the Lagrangian density has the chiral symmetry which can be represented by the unitary operator $U(\alpha)$, there is a conserved current associated with the symmetry, which is eq.(4.2). In this case, there is a conserved chiral charge Q_5

$$Q_5 = \int j_5^0(x)\, d^3r. \tag{4.12}$$

The quantized Hamiltonian $\hat{H}$ of this system is invariant under the unitary transformation $U(\alpha)$,

$$U(\alpha)\hat{H}U(\alpha)^{-1} = \hat{H}. \tag{4.13}$$

Therefore, the chiral charge operator $\hat{Q}_5$ commutes with the Hamiltonian $\hat{H}$

$$\hat{Q}_5\hat{H} = \hat{H}\hat{Q}_5. \tag{4.14}$$

4.2.2 Symmetry of Vacuum

The symmetry of the vacuum is determined in terms of its energy, and when the lowest energy state is realized, the vacuum may break the symmetry which is possessed in the

Hamiltonian. Here, the symmetry of the vacuum can be defined in the following way. The symmetric vacuum is denoted by $|0\rangle$ while the symmetry broken vacuum is denoted by $|\Omega\rangle$. They satisfy the following equations,

$$U(\alpha)|0\rangle = |0\rangle, \tag{4.15a}$$

$$U(\alpha)|\Omega\rangle \neq |\Omega\rangle. \tag{4.15b}$$

These equations can be written in terms of the chiral charge operator $\hat{Q}_5$ as

$$\hat{Q}_5|0\rangle = 0, \tag{4.16a}$$

$$\hat{Q}_5|\Omega\rangle \neq 0. \tag{4.16b}$$

4.2.3 Commutation Relation

In the Goldstone theorem, one starts from the following commutation relation which is an identity equation,

$$\left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right] = -2\int \bar{\psi}(x)\psi(x)\, d^3r. \tag{4.17}$$

Now, one takes the expectation value of the above equation with the vacuum state $|\Omega\rangle$, and obtains

$$\langle\Omega|\left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right]|\Omega\rangle = -2\langle\Omega|\int \bar{\psi}(x)\psi(x)\, d^3r|\Omega\rangle. \tag{4.18}$$

If the right hand side (fermion condensate) has a finite value, then the vacuum state $|\Omega\rangle$ must be a symmetry broken state since it should at least satisfy eq.(4.16b) because of the finite value of the left hand side.

Now, one can rewrite eq.(4.18) and assume that the right hand side is nonzero because of the finite fermion condensate

$$\sum_n (2\pi)^3\delta(\boldsymbol{p}_n)\left[\langle\Omega|j_5^0|n\rangle\langle n|\bar{\psi}\gamma_5\psi|\Omega\rangle e^{-iE_nt} - \langle\Omega|\bar{\psi}\gamma_5\psi|n\rangle\langle n|j_5^0|\Omega\rangle e^{iE_nt}\right] \neq 0, \tag{4.19}$$

where $|n\rangle$ denotes the complete set of the fermion number zero states of the field theory model one considers. Therefore, bosonic states as well as the pair of massless free fermion and anti-fermion states should be included in the intermediate states. From eq.(4.19), one sees that the right hand side is nonzero and time-independent while the left hand side is time dependent unless there is a state $|n\rangle$ that satisfies

$$E_n = 0 \ \text{ for } \ \boldsymbol{p}_n = 0. \tag{4.20}$$

Eq.(4.20) is just consistent with the dispersion relation of a massless boson.

4.2.4 Momentum Zero State

However, one easily notices that the free massless fermion and anti-fermion pair can also satisfy eq.(4.20). To be more specific, one can write the energy and momentum of the state $|n\rangle$ as

$$E_n = E_f + E_{\bar{f}}, \tag{4.21a}$$

$$\boldsymbol{p}_n = \boldsymbol{p}_f + \boldsymbol{p}_{\bar{f}}, \tag{4.21b}$$

where $\boldsymbol{p}_f$ ($\boldsymbol{p}_{\bar{f}}$) and E_f ($E_{\bar{f}}$) denote the momentum and energy of the fermion (anti-fermion), respectively. For the free massless fermion and anti-fermion pair with

$$\boldsymbol{p}_f = 0 \text{ and } \boldsymbol{p}_{\bar{f}} = 0 \tag{4.22a}$$

one obtains [41, 42, 43, 69]

$$E_f = 0 \text{ and } E_{\bar{f}} = 0 \tag{4.22b}$$

and therefore eq.(4.20) is indeed satisfied

$$\boldsymbol{p}_n = \boldsymbol{p}_f + \boldsymbol{p}_{\bar{f}} = 0 \Longrightarrow E_n = E_f + E_{\bar{f}} = 0.$$

Dispersion Relation of Massless Boson

From one information of eq.(4.20), one could derive the dispersion relation of a massless boson if the state must be covariant

$$E_n^2 - \boldsymbol{p}_n^2 = (p_n)_\mu p_n^\mu = 0.$$

The requirement of the covariance for the state $|n\rangle$ may be justified when the state is an isolated system. However, it is difficult to show the covariance from only one information on the zero momentum state which is just eq.(4.20) since the vacuum is always in the momentum zero state.

On the other hand, the dispersion relation of a massless boson

$$E = |\boldsymbol{p}|$$

contains information which should be valid for arbitrary momentum $\boldsymbol{p}$. Intuitively, it is clear that one cannot obtain the dispersion relation of a massless boson from only one information which is at $\boldsymbol{p} = 0$. Therefore, one sees that the Goldstone theorem proves the existence of a free massless fermion and anti-fermion pair for the fermion field theory models, and this is, of course, a natural statement. But there exists no massless boson.

4.2.5 Pole in S-matrix

In the spontaneous symmetry breaking, a massless pole in the S-matrix calculations is sometimes found, and they claim that the pole should be related to a massless boson (Goldstone boson). However, the S-matrix in these calculations is evaluated in the trivial (perturbative) vacuum, and it has nothing to do with a physical massless boson. Furthermore, one has to be careful that a pole in the S-matrix may not have to correspond to a bound state, and if one wishes to find a bound state pole, then one should calculate poles in exact Green's function in which the evaluation should be based on the symmetry broken vacuum state. But this is just the same as solving the system exactly.

4.3 New Interpretation of Goldstone Theorem

Here, we present a new interpretation of the Goldstone theorem and clarify what is indeed the physics of the spontaneous symmetry breaking in fermion field theory models. Since the spontaneous symmetry breaking is connected with the structure of the vacuum, we should understand the physical feature of the vacuum. However, most of the difficulties of the field theory models are concentrated in the dynamical evaluation of the vacuum state, and therefore, we should first treat the spontaneous symmetry breaking physics in terms of finite number of freedoms. After that, we should examine whether the procedure can be justified when the number of the freedom is set to infinity.

4.3.1 Eigenstate of Hamiltonian and $\hat{Q}_5$

Since the operator $\hat{Q}_5$ commutes with the Hamiltonian $\hat{H}$ as discussed in eq.(4.14),

$$\hat{Q}_5 \hat{H} = \hat{H} \hat{Q}_5$$

the $\hat{Q}_5$ has the same eigenstate as the Hamiltonian. If one defines the symmetry broken vacuum state $|\Omega\rangle$ by the eigenstate of the Hamiltonian $\hat{H}$ with its energy eigenvalue E_Ω, then one can write

$$\hat{H}|\Omega\rangle = E_\Omega |\Omega\rangle. \tag{4.23}$$

In this case, one can also write the eigenvalue equation for the $\hat{Q}_5$

$$\hat{Q}_5 |\Omega\rangle = q_5 |\Omega\rangle \tag{4.24}$$

with its eigenvalue q_5. These equations should hold for the exact eigenstates of the Hamiltonian.

Now, if one takes the expectation value of eq.(4.17) with the symmetry broken vacuum $|\Omega\rangle$ which is the eigenstate of the Hamiltonian as well as $\hat{Q}_5$, then one obtains for the left hand side as

$$\langle\Omega| \left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r \right] |\Omega\rangle$$

$$= \langle\Omega|q_5 \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r - \left(\int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right) q_5|\Omega\rangle = 0 \qquad (4.25)$$

with the help of eq.(4.24). This means that the right hand side of eq.(4.18) must vanish, that is,

$$\langle\Omega| \int \bar{\psi}(x)\psi(x)\, d^3r|\Omega\rangle = 0. \qquad (4.26)$$

Therefore, the exact eigenstate of the vacuum has no fermion condensate even in the symmetry broken vacuum. The relation of eq.(4.18) has repeatedly been used, and if there is a finite fermion condensate, then the symmetry of the vacuum must be broken since the left hand side of eq.(4.18) vanishes due to eq.(4.16a) for the symmetric vacuum state. However, as seen above, the condensate must vanish even for the symmetry broken vacuum state if the vacuum is the eigenstate of the Hamiltonian, which is a natural consequence.

4.3.2 Index of Symmetry Breaking

The way out of this dilemma is simple. One should not take the expectation value of the vacuum state. Instead, the index of the symmetry breaking in connection with the condensate operator $\int \bar{\psi}(x)\psi(x)\, dx$ should be the following operator equation

$$\left(\int \bar{\psi}(x)\psi(x)\, d^3r\right)|\Omega\rangle = |\Omega'\rangle + C_1|\Omega\rangle, \qquad (4.27)$$

where $|\Omega'\rangle$ denotes an operator-induced state which is orthogonal to the $|\Omega\rangle$

$$\langle\Omega|\Omega'\rangle = 0.$$

C_1 is related to the condensate value. For the exact eigenstate which breaks the chiral symmetry, one finds

$$C_1 = 0. \qquad (4.28)$$

In this case, the identity equation of (4.17) can be applied to the state $|\Omega\rangle$ and one obtains

$$(\hat{Q}_5 - q_5) \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r|\Omega\rangle = -2|\Omega'\rangle \qquad (4.29)$$

with the help of eq.(4.24). Indeed, eq.(4.29) holds true for the exact eigenstate. It is now clear that one should not take the expectation value of eq.(4.17) by the vacuum state. It just gives a trivial equation of "0" $=$ "0".

4.4 Chiral Symmetry in Quantized Thirring Model

In this section, we show explicitly how the chiral symmetry in the quantized Thirring model Hamiltonian behaves in terms of the creation and annihilation operators.

4.4.1 Lagrangian Density

The Lagrangian density of the Thirring model is written as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{1}{2}\,g\bar{\psi}\gamma_\mu\psi\;\bar{\psi}\gamma^\mu\psi \tag{4.30}$$

which is invariant under the chiral transformation

$$\psi' = e^{i\alpha\gamma_5}\psi. \tag{4.31}$$

Therefore, the chiral charge

$$\hat{Q}_5 = \int \bar{\psi}(x)\gamma^0\gamma_5\psi(x)\,d^3r$$

is a conserved quantity. In fact, it commutes with the Hamiltonian $\hat{H}$

$$[\hat{H}, \hat{Q}_5] = 0.$$

4.4.2 Quantized Hamiltonian

The field ψ is quantized as

$$\psi(x) = \begin{pmatrix}\psi_a(x)\\ \psi_b(x)\end{pmatrix} = \frac{1}{\sqrt{L}}\sum_n \begin{pmatrix}a_n\\ b_n\end{pmatrix} e^{ip_n x}, \quad \text{with } \; p_n = \frac{2\pi}{L}\,n, \tag{4.32}$$

where the operators a_n and b_n should satisfy the anti-commutation relations

$$\{a_n, a_m^\dagger\} = \{b_n, b_m^\dagger\} = \delta_{nm}, \quad \{a_n, a_m\} = \{b_n, b_m\} = \{a_n, b_m\} = 0.$$

In this case, the quantized Hamiltonian of the Thirring model becomes

$$\hat{H} = \sum_n \left[p_n \left(a_n^\dagger a_n - b_n^\dagger b_n \right) + \frac{2g}{L}\left(\sum_l a_l^\dagger a_{l+n}\right)\left(\sum_m b_m^\dagger b_{m+n}\right)\right]. \tag{4.33}$$

4.4.3 Chiral Transformation for Operators

The chiral transformation of eq.(4.31) is written in terms of a_n and b_n as

$$U(\alpha)\begin{pmatrix}a_n\\ b_n\end{pmatrix}U^{-1}(\alpha) = \begin{pmatrix}e^{i\alpha}a_n\\ e^{-i\alpha}b_n\end{pmatrix}, \quad U(\alpha)\begin{pmatrix}a_n^\dagger\\ b_n^\dagger\end{pmatrix}U^{-1}(\alpha) = \begin{pmatrix}e^{-i\alpha}a_n^\dagger\\ e^{i\alpha}b_n^\dagger\end{pmatrix}. \tag{4.34}$$

In this case, one easily sees that the Hamiltonian is also invariant under the transformation of eq.(4.34)

$$U(\alpha)\hat{H}U^{-1}(\alpha) = \sum_n \Bigg[p_n U(\alpha)\left(a_n^\dagger a_n - b_n^\dagger b_n\right)U^{-1}(\alpha)$$

$$+\frac{2g}{L}\,U(\alpha)\left(\sum_l a_l^\dagger a_{l+n}\right)U^{-1}(\alpha)U(\alpha)\left(\sum_m b_m^\dagger b_{m+n}\right)U^{-1}(\alpha)\Bigg] = \hat{H}. \tag{4.35}$$

4.4.4 Unitary Operator with Chiral Charge $\hat{Q}_5$

Now, the unitary operator $U(\alpha)$ can be explicitly written as

$$U(\alpha) = e^{-i\alpha \hat{Q}_5}, \tag{4.36}$$

where the chiral charge operator $\hat{Q}_5$ is expressed in terms of the creation and annihilation operators as

$$\hat{Q}_5 = \int \bar{\psi}(x)\gamma^0\gamma_5\psi(x)\, d^3r = \sum_n \left[a_n^\dagger a_n - b_n^\dagger b_n \right].$$

In this case, one can confirm the following identities

$$U(\alpha) \begin{pmatrix} a_n \\ b_n \end{pmatrix} U^{-1}(\alpha) = e^{-i\alpha\hat{Q}_5} \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{i\alpha\hat{Q}_5} = \begin{pmatrix} e^{i\alpha} a_n \\ e^{-i\alpha} b_n \end{pmatrix}, \tag{4.37a}$$

$$U(\alpha) \begin{pmatrix} a_n^\dagger \\ b_n^\dagger \end{pmatrix} U^{-1}(\alpha) = e^{-i\alpha\hat{Q}_5} \begin{pmatrix} a_n^\dagger \\ b_n^\dagger \end{pmatrix} e^{i\alpha\hat{Q}_5} = \begin{pmatrix} e^{-i\alpha} a_n^\dagger \\ e^{i\alpha} b_n^\dagger \end{pmatrix} \tag{4.37b}$$

which are just the same as eq.(4.34).

4.4.5 Symmetric and Symmetry Broken Vacuum

In the Thirring model, the Bethe ansatz solutions show that there are symmetric vacuum $|0\rangle$ and symmetry broken vacuum $|\Omega\rangle$ states, and the energy of the symmetry broken vacuum is found to be lower than the symmetric vacuum energy. They are the eigenstate of the chiral charge $\hat{Q}_5$ and one finds

$$\hat{Q}_5|0\rangle = 0, \tag{4.38a}$$

$$\hat{Q}_5|\Omega\rangle = \pm|\Omega\rangle. \tag{4.38b}$$

Therefore, for the unitary operator $U(\alpha) = e^{-i\alpha\hat{Q}_5}$, the symmetric vacuum does not change

$$U(\alpha)|0\rangle = e^{-i\alpha\hat{Q}_5}|0\rangle = |0\rangle \tag{4.39a}$$

while the symmetry broken vacuum becomes

$$U(\alpha)|\Omega\rangle = e^{-i\alpha\hat{Q}_5}|\Omega\rangle = e^{\pm i\alpha}|\Omega\rangle \neq |\Omega\rangle \tag{4.39b}$$

which indeed satisfies the criteria of the symmetry broken vacuum state in eqs.(4.15).

4.5 Spontaneous Chiral Symmetry Breaking

There is one good example which perfectly satisfies the above requirements of the spontaneous chiral symmetry breaking and zero fermion condensate. That is the Bethe ansatz vacuum of the massless Thirring model which will be discussed in detail in Chapter 7. Here, we employ the results of the Bethe ansatz vacuum of the Thirring model and discuss the vacuum and its properties in the context of the spontaneous symmetry breaking.

4.5.1 Exact Vacuum of Thirring Model

Now, the left and right mover fermion creation operators can be denoted by $a_k^\dagger$, $b_k^\dagger$, respectively, and thus the vacuum state $|\Omega\rangle$ can be written as

$$|\Omega\rangle = \prod_{k_i^\ell} a_{k_i^\ell}^\dagger \prod_{k_j^r} b_{k_j^r}^\dagger |0\rangle\rangle, \tag{4.40}$$

where $|0\rangle\rangle$ denotes the null vacuum state with

$$a_{k_i^\ell}|0\rangle\rangle = 0, \quad b_{k_j^r}|0\rangle\rangle = 0. \tag{4.41}$$

The momenta k_j^ℓ for left mover and k_i^r for right mover should satisfy the periodic boundary condition (PBC) equations which are solved analytically, and therefore one can determine the momenta k_j^ℓ and k_i^r, as will be given in Chapter 7.

4.5.2 Condensate Operator

Now, the condensate operator $\int \bar{\psi}(x)\psi(x)\, dx$ can be written as

$$\int \bar{\psi}(x)\psi(x)\, dx = \sum_n (b_n^\dagger a_n + a_n^\dagger b_n). \tag{4.42}$$

Therefore, eq.(4.27) becomes

$$\begin{aligned} \int \bar{\psi}(x)\psi(x)\, dx|\Omega\rangle &= \sum_n (b_n^\dagger a_n + a_n^\dagger b_n)|\Omega\rangle \\ &= \sum_n \left\{ \prod_{k_i^\ell, k_i^\ell \neq n} a_{k_i^\ell}^\dagger \prod_{k_j^r} b_{k_j^r}^\dagger b_n^\dagger |0\rangle\rangle + \prod_{k_i^\ell} a_{k_i^\ell}^\dagger \prod_{k_j^r, k_j^r \neq n} b_{k_j^r}^\dagger a_n^\dagger |0\rangle\rangle \right\}. \end{aligned} \tag{4.43}$$

Clearly, the right hand side of eq.(4.43) is different from the vacuum state of the Bethe ansatz solution of eq.(4.40), and therefore denoting the right hand side of eq.(4.43) by $|\Omega'\rangle$, one obtains

$$\int \bar{\psi}(x)\psi(x)\, dx|\Omega\rangle = \sum_n (b_n^\dagger a_n + a_n^\dagger b_n)|\Omega\rangle = |\Omega'\rangle. \tag{4.44}$$

Obviously, the value of C_1 in eq.(4.27) is zero in the massless Thirring model, and indeed this confirms eq.(4.29).

It is now clear and most important to note that one cannot learn the basic dynamics of the symmetry breaking phenomena from the identity equation. If one wishes to study the symmetry breaking physics in depth, then one has to solve the dynamics of the vacuum in the field theory model properly even though it is extremely difficult to solve it exactly.

4.6 Symmetry Breaking in Two Dimensions

In two dimensional field theory models, it is well known that there should not exist any physical massless bosons because of the infra-red singularity of the propagator of the massless boson. Therefore, if one assumes the Goldstone theorem, then one finds that there should not occur any spontaneous symmetry breaking in two dimensional field theory models, which is known as Coleman's theorem [22, 31, 91].

4.6.1 Fermion Field Theory in Two Dimensions

However, as we will see in the later chapters, the vacuum states of the massless Thirring model as well as QCD in two dimensions prefer the symmetry broken vacuum states together with the current conservation. Therefore, the spontaneous symmetry breaking of the vacuum indeed takes place in two dimensional field theory models of Thirring and QCD_2. By now, this is not surprising since the Goldstone theorem does not hold in fermion field theory models. On the contrary, the spontaneous symmetry breaking in these models are consistent with the new picture of the symmetry breaking physics. As far as the spontaneous symmetry breaking physics is concerned, the two dimensional field theory is not at all special since there appears no massless boson after the symmetry is spontaneously broken in the vacuum.

4.6.2 Boson Field Theory in Two Dimensions

The spontaneous symmetry breaking should not occur in boson field theory models in two dimensions. This may be reasonable since the Goldstone theorem may hold for the boson field theory models where a massless boson should appear. However, there should not exist any physical massless boson in two dimensions, and therefore, the spontaneous symmetry breaking should be forbidden in two dimensional boson field theory models.

4.7 Symmetry Breaking in Boson Fields

In the subsequent two sections, we stray from the main stream of the spontaneous symmetry breaking physics in fermion field theory models, and come to discussions of the spontaneous symmetry breaking in boson field theory in four dimensions. In most of the field theory textbooks, the discussion of this subject can be found, and therefore, we discuss it briefly in this section.

4.7.1 Double Well Potential

Now, we discuss the spontaneous symmetry breaking in boson field theory models. This can be found in any field theory textbooks, and therefore we only sketch a simple picture why the massless boson appears in the spontaneous symmetry breaking. But it should be noted that the treatment here is approximate, and there is still some unsolved problem left when one wishes to understand the spontaneous symmetry breaking in boson field theory models in an exact fashion. The Hamiltonian density for complex boson fields can be written as

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{\nabla}\phi^\dagger)(\boldsymbol{\nabla}\phi) + U\left(|\phi|\right). \tag{4.45}$$

This has a $U(1)$ symmetry. However, the Hamiltonian density must be real, and therefore the $U(1)$ symmetry of the Hamiltonian density is a trivial constraint. Now, when one takes the potential as a double well type

$$U\left(|\phi|\right) = u_0\left(|\phi|^2 - \lambda^2\right)^2, \tag{4.46}$$

where u_0 and λ are constant, then the minimum of the potential $U\left(|\phi|\right)$ can be found at

$$|\phi(x)| = \lambda.$$

However, one must notice that this is a minimum of the potential, but not the minimum of the total energy.

4.7.2 Change of Field Variables

The minimum of the total energy must be found together with the kinetic energy term. Now, one rewrites the complex field as

$$\phi(x) = (\lambda + \eta(x))e^{i\frac{\xi(x)}{\lambda}}, \tag{4.47}$$

where η is assumed to be much smaller than the λ,

$$|\eta(x)| \ll \lambda.$$

In this case, one can rewrite eq.(4.45) as

$$\mathcal{H} = \frac{1}{2}\big[(\boldsymbol{\nabla}\xi)(\boldsymbol{\nabla}\xi) + (\boldsymbol{\nabla}\eta)(\boldsymbol{\nabla}\eta)\big] + U\big(|\lambda + \eta(x)|\big) + \cdots . \tag{4.48}$$

Here, one finds the massless boson ξ which is associated with the degeneracy of the vacuum energy. The important point is that this infinite degeneracy of the potential vacuum is converted into the massless boson degrees of freedom when the degeneracy of the potential vacuum is resolved by the kinetic energy term.

This is the spontaneous symmetry breaking which is indeed found by Goldstone, and he pointed out that there should appear a massless boson associated with the symmetry breaking. The degeneracy of the potential vacuum is converted into a massless boson degree of freedom. This looks plausible, and at least approximately there is nothing wrong with this treatment of the spontaneous symmetry breaking phenomena in contrast to the fermion field theory model. However, the treatment is still approximate, and one should confirm that the terms neglected in eq.(4.48) may not cause any troubles. At least, one cannot claim that the massless boson which appears after the spontaneous symmetry breaking is an isolated particle of the system. Also, the Goldstone theorem shows that there should be a boson state as given in eq.(4.20)

$$E_n = 0 \quad \text{for} \quad \boldsymbol{p}_n = 0$$

which is consistent with a massless boson. However, to be rigorous, one may still have to prove that the state with the above constraint is an isolated system.

In this respect, it should be most important to solve the boson field theory model with the double well potential in an exact fashion, and then one may learn the essence of the symmetry breaking physics in the boson field theory in depth, and this is indeed a future problem.

4.7.3 Current Density of Fields

In addition, the boson fields ξ and η are real fields, and therefore the current density $j_0(x)$ of the boson fields ξ and η must vanish as the classical field

$$j_0(x) = i\left(\xi^\dagger(x)\,\frac{\partial \xi(x)}{\partial t} - \frac{\partial \xi^\dagger(x)}{\partial t}\,\xi(x)\right) = 0 \ \text{ since } \ \xi^\dagger(x) = \xi(x)$$

and the same equation holds for the η field as well. However, they diverge when they are quantized as will be discussed in the Appendix C. Therefore, both of the fields cannot propagate as a physical particle. In this sense, one may say that the model field theory of eq.(4.45) is not realistic.

4.8 Breaking of Local Gauge Symmetry?

At present, all of the realistic field theory models have the local gauge symmetry. Quantum electrodynamics (QED), quantum chromodynamics (QCD) and Weinberg-Salam model have the local gauge invariance. The local gauge symmetry in QED and QCD should hold rigorously, and this is just what we observe from experiments.

However, this local gauge invariance seems to be broken in the Higgs mechanism, and therefore we should discuss the essence of the spontaneous symmetry breaking of the local gauge invariance [67]. This concept is employed in the electro-weak theory by Weinberg-Salam, and the $SU(2) \otimes U(1)$ gauge field model is quite successful in describing many experimental observations.

4.8.1 Higgs Mechanism

The Lagrangian density of the complex scalar field $\phi(x)$ which interacts with the $U(1)$ gauge field can be written as

$$\mathcal{L} = \frac{1}{2}\,(D_\mu\phi)^\dagger(D^\mu\phi) - u_0\left(|\phi|^2 - \lambda^2\right)^2 - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{4.49}$$

where

$$D_\mu = \partial_\mu + igA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Here, the same double well type potential as in eq.(4.46) is assumed for the complex scalar field. Now, one rewrites the complex scalar field as

$$\phi(x) = (\lambda + \eta(x))e^{i\frac{\xi(x)}{\lambda}}\ , \tag{4.50}$$

where $\eta(x)$ and $\xi(x)$ denote new fields, and therefore one can obtain a new Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \eta)(\partial^\mu \eta) - u_0\left(|\lambda + \eta|^2 - \lambda^2\right)^2$$
$$+\frac{1}{2}\,g^2\lambda^2\left(A_\mu + \frac{1}{g\lambda}\partial_\mu \xi\right)\left(A^\mu + \frac{1}{g\lambda}\,\partial^\mu \xi\right) - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \cdots\ . \tag{4.51}$$

This Lagrangian density is still invariant under the following gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \chi(x), \quad \xi'(x) = \xi(x) - g\lambda\chi(x),$$

where $\chi(x)$ is an arbitrary function of space and time.

4.8.2 Gauge Fixing

Now, one fixes the gauge such that

$$\xi(x) = 0$$

which is called *unitary gauge*. This means that one takes the following gauge fixing

$$\chi(x) = \frac{1}{g\lambda}\,\xi(x).$$

In this case, the Lagrangian density of eq.(4.51) becomes

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \eta)(\partial^\mu \eta) - u_0\left(|\lambda + \eta|^2 - \lambda^2\right)^2 + \frac{1}{2}\,g^2\lambda^2 A_\mu A^\mu - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \cdots\ . \tag{4.52}$$

This new Lagrangian density shows that the gauge field becomes massive and the complex scalar field has lost one degree of freedom and becomes a real scalar field $\eta(x)$. Since the new Lagrangian density is obtained by fixing the gauge, it does not have a gauge freedom any more. The peculiarity of this gauge fixing is that the $\xi(x) = 0$ has nothing to do with the redundancy of the gauge field A_μ itself. In fact, this gauge fixing does not reduce the number of freedoms of the gauge field. If all the physical observables can be reproduced by this gauge fixing, then this gauge fixing can be justified [53].

4.8.3 What Is Physics Behind Higgs Mechanism?

In the Higgs mechanism, the gauge field acquires the mass by the spontaneous symmetry breaking of the Higgs fields. This is mainly because one takes the double well type potential for a scalar field and the potential has a minimum at

$$|\phi(x)| = \lambda.$$

Therefore, the kinetic energy part of the Higgs field which couples with the gauge field A_μ becomes a constant λ. Therefore, this part has lost a coupling between the Higgs and the gauge fields. However, this mass-term-like interactions should be still gauge invariant and it cannot be considered as a mass term of the gauge field. The mass term of the gauge field is obtained by fixing the gauge at the Lagrangian density level. Therefore, after fixing the gauge, there is no gauge freedom any more in the Lagrangian density.

Normally, one fixes the gauge at the point where one calculates the physical observables. In terms of physics, the gauge fixing becomes necessary when one wishes to determine the gauge field solutions from the equations of motion. This is clear since the gauge field has redundant variables at the level of solving the equations of motion. By fixing the gauge, one can determine the gauge field, but of course physical observables should not depend on the choice of the gauge fixing.

At the same time, one should be careful for the choice of the gauge fixing. There is no guarantee that any kind of the gauge fixing can give the same physical observables. At least, one should examine whether the gauge choice one takes can indeed reproduce the right physical observables or not.

Here, the $U(1)$ gauge field is discussed, but it is straightforward to extend it to the non-abelian case. In the non-abelian gauge field theory, gauge fields themselves are not gauge invariant, and therefore they cannot be a physical observable. However, after the spontaneous symmetry breaking of the complex scalar fields, the non-abelian gauge fields become physical observables after the gauge fixing as Weinberg-Salam model shows. The physics behind this statement is difficult to understand, and in this respect, the Higgs mechanism should be understood more in depth in future.