群論の初歩　：　物理院生用

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はじめに

物理学を深く理解するためには群論を自分のものにする事がかなり重要である。ところが、物理学科の授業で群論を教える事はほとんどないのが現状であると言えよう。そしてこれは１９７０年代の教育状況から変わらず続いてきたものと考えられる。しかしながら大学院に進学した途端に、理論物理においては群論をある程度、理解している事は当然であると言う事になっている。

このノートでは物理屋に取って最低限のレベルにおける群論とその表現を紹介している。数学的な厳密さはほとんど考慮していないが、しかし基本的な戦略は正しいものと考えている。このノートは大学院での講義ノートをベースにして書いたものである。この大学院講義自体は英語で行われたものであるが、このノートは主に日本語の記述になっている。但し、一部、必要に応じて英語で書かれたものはそのまま英語での記述になっている。

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第1章　群論とは？

群論を理論物理学者が解説する事はかなり難しいものと思われる。到底、厳密にはできないし、そうする事に興味もないものである。しかし物理学者にとって群論は非常に有用である。例えば、回転群の表現を正確に捉えていれば、電磁場のスピンが１であることを納得できるものである。偏極ベクトルはリー代数を満たさないがしかしランク１のテンソルであることから、ほとんどスピン１の粒子と考えて十分であることが良くわかるものである。後で解説するように、通常のベクトル、例えば空間座標 $\boldsymbol{r}$ はランク１のテンソルである。従ってベクトルポテンシャルも当然、ランク１のテンソルとなっている。

この章ではまず、群とは何なのかと言う事から始めて行こう。群の概念は非常に面白いものであり、自然界の様々な構造をうまく classify(分類) するのに役立つものである。まずは群の定義をきちんと理解することが重要である。

1.1　群の定義

群とは基本的に何かの集合体である。この場合、数学ではその集合体を構成する要素をまず考える事になる。今考えている群 $\mathcal{G}$ の要素をここでは 例えば

$$\mathcal{G} \ : \ \{a, b, c, \cdots, g\} \tag{1.1}$$

としよう。この場合はただ並べただけであり、特別な意味はない。群論においては、ここで何らかの演算をこの要素間に定義する事になる。例えば、普通の掛け算とか行列の掛け算とかである。以降、演算を代表して $\bullet$ と表記しよう。

1.1.1 群の形成

群を形成するためには次の 4 個の条件を満たす必要がある。

(1) 要素 a と b の演算結果 $(a \bullet b)$ が元の群 $\mathcal{G}$ に属している

(2) 単位要素が存在する

(3) 逆元が存在する

(4) 結合則 $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ が成り立つ

群 $\mathcal{G}$ がこの 4 個の条件を満たしている場合、これは群を形成していると言う。

- **可換群 (Abelian group) :**
  $a \bullet b = b \bullet a$ が成り立つ場合、これを可換群 (abelian group) と言う。

- **非可換群 (Non-abelian group) :**
  $a \bullet b \neq b \bullet a$ の場合、これを非可換群 (non-abelian group) と言う。Special Unitary 行列 SU(N) によって作られる群は典型的な非可換群である。

1.1.2 群の実例

群についていくつかの実例をあげて議論して行こう。まずは群をなさない集合体の例を挙げて、その中身をみてみよう。

[1] 群でない例： 集合体 F $\{1, 2, \frac{1}{2}\}$

今、例として集合体 F $\{1, 2, \frac{1}{2}\}$ を考えてみよう。この場合、演算は掛け算としよう。この例では単位要素は存在するし、逆元も存在している。また、結合則も成り立っている。ところが、(1) の規則である「要素間 $a \bullet b$ の演算結果が元の群 G に属している」を満たしていない。それは明らかで、$2 \bullet 2 = 2 \times 2 = 4$ は F の要素の中にはない。従って、これは群をなしていない事がわかる。

[2] I, E_1, E_2, E_3 の群

今、群 $\mathcal{G}$ として

$$\mathcal{G}\ \{I, E_1, E_2, E_3\} \tag{1.2}$$

を考えよう。ここで演算として次のようなものを考えよう。但し、この群は可換群と仮定されている。

$$E_1 \bullet E_1 = E_2 \bullet E_2 = E_3 \bullet E_3 = I \tag{1.3}$$

$$E_1 \bullet E_2 = E_3,\quad E_1 \bullet E_3 = E_2,\quad E_2 \bullet E_3 = E_1 \tag{1.4}$$

$$E_1 \bullet I = E_1,\quad E_2 \bullet I = E_2,\quad E_3 \bullet I = E_3 \tag{1.5}$$

この集合体 $\mathcal{G}$ が群をなしている事を示そう。まず (1) の規則である「要素間で $a \bullet b$ の演算結果が元の群 $\mathcal{G}$ に属している」に関してであるが、これは明らかであろう。次に、単位元であるが、これは I である。さらに逆元は $E_1^{-1} = E_1$ など自分自身である。また結合則は

$$(E_1 \bullet E_2) \bullet E_3 = E_1 \bullet (E_2 \bullet E_3) \tag{1.6}$$

を示す事であるが、これは

$$E_3 \bullet E_3 = E_1 \bullet E_1 \quad \Rightarrow \quad I = I \tag{1.7}$$

であり明らかに成り立っている。従ってこれは確かに群をなしている事がわかる。

[3] 巡回群

今、群 $\mathcal{G}$ として

$$\mathcal{G}\ \{C_n,\ C_n^2,\ \cdots, C_n^n = I\} \tag{1.8}$$

を考えよう。ここで n は整数である。この場合 $C_n^k = e^{\frac{2\pi k}{n} i}$ と書かれている。これが群を形成している事を示そう。単位オペレータは $I = 1$ である。C_n^k の逆行列は

$$\left(C_n^k\right)^{-1} = e^{\frac{2\pi(n-k)}{n} i} \tag{1.9}$$

となっている。これは

$$\left(C_n^k\right)^{-1} C_n^k = e^{\frac{2\pi(n-k)}{n}i + \frac{2\pi k}{n}i} = I \tag{1.10}$$

から明らかである。

1.2 Special Unitary 行列 SU(N)

行列の要素として Special Unitary 行列 SU(N) を考えてみよう。これは非可換群である。

1.2.1 行列 SU(2)

今、簡単な例として 2×2 の Special Unitary 行列 A を考えよう。この場合

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.11}$$

としよう。Unitary 行列であるとは

$$AA^\dagger = 1 \tag{1.12}$$

が成り立っている事である。ここで $A^\dagger$ は

$$A^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{1.13}$$

である。この 行列が Unitary である事の条件は

$$|a|^2 + |b|^2 = 1, \quad ac^* + bd^* = 0 \tag{1.14}$$

$$|c|^2 + |d|^2 = 1, \quad ca^* + db^* = 0 \tag{1.15}$$

となっている。また Special であるとは

$$\det A = ad - bc = 1 \tag{1.16}$$

が成り立っているものである。このため SU(2) の場合、最初、8 個の自由度が存在していたが 5 個の条件があるため、自由度は 3 個である。

1.2.2 SU(N) は群

ここで SU(N) は群を形成している事を証明しよう。この群の演算は行列の掛け算である。

• **(1)** $C = AB$ **も SU(N):**
この行列の掛け算の結果もやはり SU(N) であることを証明しよう。まずはユニタリーであることを示そう。

$$CC^\dagger = AB(AB)^\dagger = ABB^\dagger A^\dagger = AA^\dagger = 1 \tag{1.17}$$

により示された。ここで簡単な事ではあるが、$(AB)^\dagger = B^\dagger A^\dagger$ を示しておこう。これは成分で示すのが一番、楽である。

$$[(AB)^\dagger]_{ij} = \Big(\sum_{k=1}^{N} A_{ik}B_{kj}\Big)^\dagger = \sum_{k=1}^{N} A^*_{jk}B^*_{ki} = \sum_{k=1}^{N}(B^\dagger)_{ik}(A^\dagger)_{kj} = (B^\dagger A^\dagger)_{ij} \tag{1.18}$$

• **(2) 単位元:**
SU(N) の単位元とは単位行列の事であり、これは常に存在している。通常はこれを 1 とか E と表記している。この場合、$E_{ij} = \delta_{ij}$ である。

• **(3) 逆元:**
SU(N) の逆元 A^{-1} は $A^\dagger$ である。これは $AA^\dagger = 1$ の両辺に A^{-1} を掛けることにより

$$A^{-1}AA^\dagger = A^\dagger = A^{-1} \tag{1.19}$$

となる事から明らかである。

• **(4) 結合則 (Combination Rule):**
結合則 $(AB)C = A(BC)$ が成り立っているかどうかの検証をする場合、具体的に計算することが大切である。今の場合、

$$[(AB)C]_{ik} = \sum_{j=1}^{N}(AB)_{ij}C_{jk} = \sum_{j=1}^{N}\sum_{m=1}^{N} A_{im}B_{mj}C_{jk} = \sum_{m=1}^{N} A_{im}(BC)_{mk} = [A(BC)]_{ik}$$

となって、結合則は確かに成り立っている。この行列が群を形成しているためには掛け算された $C = AB$ の行列式が $\underline{\det\{AB\} = 1}$ である事を示す必要がある。この証明のために、まずは $\underline{\det\{A\} = \exp[\mathrm{Tr}\ln A]}$ を証明しておこう。

- $\det\{A\} = \exp[\mathrm{Tr}\ln A]$ **の証明：**
  行列式の定義式から

$$\det\{A\} = \sum_{i=1}^{N} A_{ij}\Delta_{ij} \tag{1.20}$$

と書かれている。ここで Δ_{ij} は小行列式である。ここで $A(x)$ として行列 A が x の関数になっていると仮定しよう。 ここで式 (1.20) を x で微分しよう。この時

$$\frac{d\det\{A\}}{dx} = \sum_{i,j=1}^{N} \frac{dA_{ij}}{dx}\,\Delta_{ij} \tag{1.21}$$

となる事が簡単に示される。ところが逆行列 A^{-1} は

$$(A^{-1})_{ij} \quad = \frac{\Delta_{ji}}{\det\{A\}} \tag{1.22}$$

と書かれているので、これより式 (1.21) は

$$\frac{d\det\{A\}}{dx} = \sum_{i,j=1}^{N} \frac{dA_{ij}}{dx}\,(A^{-1})_{ji}\det\{A\} = \mathrm{Tr}\left(A^{-1}\frac{dA}{dx}\right)\det\{A\} \tag{1.23}$$

となっている。ここで

$$A = e^{xB} \tag{1.24}$$

としよう。但し、B は定数行列である。この時、

$$\frac{dA}{dx} = Be^{xB}, \quad A^{-1} = e^{-xB} \tag{1.25}$$

である。よって式 (1.23) は

$$\frac{d\det\{e^{xB}\}}{dx} = (\mathrm{Tr}B)\det\{e^{xB}\} \tag{1.26}$$

となっている。この微分方程式は直ちに解けて

$$\ln \det\{e^{xB}\} = (\mathrm{Tr}B)x + C \tag{1.27}$$

となる。但し C は積分定数である。ここで $x = 0$ を入れると $C = 0$ が求まる。これより $x = 1$ と置いて $A = e^B$, $B = \ln A$ に注意すると式 (1.27) は

$$\det\{A\} = e^{\mathrm{Tr}\ln A} \tag{1.28}$$

となり、証明された。

• $\det\{C\} = \det\{AB\} = 1$ **の証明：**
本題に戻って $\det\{AB\} = 1$ を証明しよう。これは簡単で

$$\det\{AB\} = e^{\mathrm{Tr}\ln(AB)} = e^{\mathrm{Tr}(\ln A + \ln B)} = e^{\mathrm{Tr}\ln A} e^{\mathrm{Tr}\ln B} = \det\{A\}\det\{B\} = 1 \tag{1.29}$$

となり、証明された。これにより SU(N) は群をなしている事が示された。

1.3 対称群 (非可換群)

ここで対称群 S_n (Permutation group) について解説しよう。これは非可換群 (Non-abelian) である。今、群の要素を

$$S_n \equiv \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \tag{1.30}$$

と書く。この操作の意味は

$$1 \to i_1, \quad 2 \to i_2, \quad \cdots, \ n \to i_n \tag{1.31}$$

と変換しなさいと言う演算を意味している。従って、状態関数にこの S_n をオペレートした場合

$$S_n\psi(1, 2, \cdots, n) = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \psi(1, 2, \cdots, n) = \psi(i_1, i_2, \cdots, i_n) \tag{1.32}$$

となる。

1.3.1 対称群の構成要素

ここでは簡単のために S_3 を考えよう。 S_3 は SU(3) と準同形であることが示されているため、SU(3) の表現の一部を S_3 の表現で表す事が良く行われている。従って S_3 の表現論を理解しておくことは SU(3) の表現を求める時にかなり有用であると言える。この S_3 群の構成要素として6個の演算子がある。まずは単位元である。

- **単位要素** e **：**

単位要素を e と書くと

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tag{1.33}$$

となっている。

- **5個の要素** $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ **：**

これに加えて5個の要素がある。これらは具体的に

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{1.34}$$

$$\pi_2 = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{1.35}$$

$$\pi_3 = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.36}$$

$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{1.37}$$

$$\pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{1.38}$$

と書かれている。

1.3.2 対称群における演算の定義

S_3 における要素間に演算を定義しよう。2個の要素を

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & 2 & 3 \end{pmatrix} \tag{1.39}$$

としよう。この時、P_1 と P_2 の演算 $P_1 \bullet P_2$ を

$$P_1 \bullet P_2 = \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \bullet \begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{pmatrix} \tag{1.40}$$

と定義しよう。これより、例えば $\pi_4 \bullet \pi_5$ は

$$\pi_4 \bullet \pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \bullet \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = e \tag{1.41}$$

となっている。

1.3.3 対称群は群をなす

S_3 が群を作っている事を示そう。

• (1) $\pi_i \bullet \pi_j$ は群の要素：
実際の計算結果をまとめておこう。

$$\pi_1 \bullet \pi_1 = \pi_2 \bullet \pi_2 = \pi_3 \bullet \pi_3 = \pi_4 \bullet \pi_5 = \pi_5 \bullet \pi_4 = e, \tag{1.42}$$

$$\pi_1 \bullet \pi_2 = \pi_2 \bullet \pi_3 = \pi_3 \bullet \pi_1 = \pi_5, \tag{1.43}$$

$$\pi_2 \bullet \pi_1 = \pi_3 \bullet \pi_2 = \pi_1 \bullet \pi_3 = \pi_4, \tag{1.44}$$

$$\pi_1 \bullet \pi_4 = \pi_3, \quad \pi_1 \bullet \pi_5 = \pi_2, \quad \pi_4 \bullet \pi_1 = \pi_2, \quad \pi_5 \bullet \pi_1 = \pi_3 \tag{1.45}$$

$$\pi_2 \bullet \pi_4 = \pi_1, \quad \pi_2 \bullet \pi_5 = \pi_3, \quad \pi_4 \bullet \pi_2 = \pi_3, \quad \pi_5 \bullet \pi_2 = \pi_1 \tag{1.46}$$

$$\pi_3 \bullet \pi_4 = \pi_2, \quad \pi_3 \bullet \pi_5 = \pi_1, \quad \pi_4 \bullet \pi_3 = \pi_1, \quad \pi_5 \bullet \pi_3 = \pi_2 \tag{1.47}$$

• **(2) 単位元：**
対称群の単位元は e である。

• **(3) 逆元：**
対称群の逆元は確かに存在している。$\pi_1,\ \pi_2,\ \pi_3$ の逆元は自分自身である。一方、$\pi_4,\ \pi_5$ の逆元は

$$\pi_4^{-1} = \pi_5, \quad \pi_5^{-1} = \pi_4 \tag{1.48}$$

となっている。

• **(4) 結合則：**
結合則が成り立っている事は簡単に示すことができる。その一例を書いておこう。

$$(\pi_1 \bullet \pi_2) \bullet \pi_3 = \pi_5 \bullet \pi_3 = \pi_2 \tag{1.49}$$

$$\pi_1 \bullet (\pi_2 \bullet \pi_3) = \pi_1 \bullet \pi_5 = \pi_2 \tag{1.50}$$

で一致している。これらより対称群は確かに群を形成している事が示された。

第2章　群の表現

物理学においては群の表現論 (Group Representation) が重要になっている。群を表現するとは、その群 $\mathcal{G}$ の要素 g を基底ベクトル (状態ベクトル) を用意して行列表現する事に対応している。従ってどのような基底を用意するかと言う事が最も重要な問題となっている。表現の基底をうまく選ぶとその表現行列は単純なものとなると言う事に対応している。

2.1　基底ベクトルと表現行列

今、基底ベクトルとして N 個の状態ベクトルを用意しよう。これを

$$\Psi \ : \ \{\psi_1, \psi_2, \cdots, \psi_N\} \tag{2.1}$$

としよう。この基底ベクトルでは N 次元表現に対応している。この時、表現とは

$$g\psi_i = \sum_{j=1}^{N} D(g)_{ji}\, \psi_j \tag{2.2}$$

と書いた時の $D(g)$ に対応している。この $D(g)$ を表現行列と言う。この場合、添え字が行列の和とは逆になっている事に注意しよう。

2.2 積表現

群 $\mathcal{G}$ の 2 個の要素 $g_1 g_2$ の積表現がどうなっているのか見てみよう。積表現の表現行列を $D(g_1 g_2)$ としよう。この時

$$g_1 g_2 \psi_i = \sum_{j=1}^{N} D(g_1 g_2)_{ji} \psi_j \tag{2.3}$$

となっている。この時、次の式が成り立っている。

$$D(g_1 g_2) = D(g_1) D(g_2) \tag{2.4}$$

この証明は簡単であるが、ここに書いておこう。

- $D(g_1 g_2) = D(g_1) D(g_2)$ **の証明：**

これは $g_1 g_2$ を状態関数にオペレートして行けば示すことができる。すなわち

$$\begin{aligned} g_1 g_2 \psi_j &= g_1 \sum_{k=1} D(g_2)_{kj} \psi_k = \sum_{k,i=1} D(g_2)_{kj} D(g_1)_{ik} \psi_i \\ &= \sum_{i=1} [D(g_1) D(g_2)]_{ij} \psi_i = \sum_{i=1}^{N} D(g_1 g_2)_{ij} \psi_i \end{aligned} \tag{2.5}$$

となる。従って、これより

$$D(g_1 g_2) = D(g_1) D(g_2) \tag{2.6}$$

が証明された。

2.3 既約と可約

群 $\mathcal{G}$ の 2 つの表現 $D^{(1)}(g)$ と $D^{(2)}(g)$ からより大きな行列 $D(g)$ を

$$D(g) = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix} \tag{2.7}$$

と作った時、この $D(g)$ も群 $\mathcal{G}$ の表現となっている。この時、表現 $D(g)$ は表現 $D^{(1)}(g)$ と $D^{(2)}(g)$ の直和であると言い

$$D(g) = D^{(1)}(g) \oplus D^{(2)}(g) \tag{2.8}$$

と表記される。この時、表現 $D(g)$ は**可約** (reducible) であると言う。可約ではない表現を**既約表現** (irreducible) と言う。

2.4 群の表現の具体例

回転群などの表現行列を求める前に、まずは第 1 章で議論した群についてその表現を求めてみよう。これらは簡単な群のため、表現行列も簡単に計算する事ができる。

2.4.1 例題 1. $G\ \{I, E_1, E_2, E_3\}$ の表現

群 G として

$$G\ \{I, E_1, E_2, E_3\} \tag{2.9}$$

を考えて、その基本表現を求めておこう。状態ベクトルとして

$$\Psi\ :\ \{\psi_1 = I,\ \psi_2 = E_1,\ \psi_3 = E_2,\ \psi_4 = E_3\} \tag{2.10}$$

としよう。この時

$$E_1\psi_1 = E_1 = \psi_2 \tag{2.11}$$
$$E_1\psi_2 = E_1 \bullet E_1 = I = \psi_1 \tag{2.12}$$
$$E_1\psi_3 = E_1 \bullet E_2 = E_3 = \psi_4 \tag{2.13}$$
$$E_1\psi_4 = E_1 \bullet E_3 = E_2 = \psi_3 \tag{2.14}$$

と求まる。従って

$$E_1 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = (\psi_1, \psi_2, \psi_3, \psi_4) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.15}$$

であるから E_1 の表現行列 $D(E_1)$ は

$$D(E_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.16}$$

となっている。

2.4.2 例題 2. 対称群 S_2 $\{I, \pi_2\}$ の表現

簡単な例題として対称群 S_2 を考えよう。要素は 2 個であり I, π_2 である。今の場合

$$I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{2.17}$$

である。ここで状態ベクトルとして

$$\Psi \ : \ \{\psi_1 = u(1)v(2),\ \psi_2 = u(2)v(1)\} \tag{2.18}$$

としよう。この時

$$\begin{aligned} \pi_2\psi_1 &= u(2)v(1) = \psi_2 \\ \pi_2\psi_2 &= u(1)v(2) = \psi_1 \end{aligned} \tag{2.19}$$

と求まる。従って表現行列 $D(\pi_2)$ は

$$D(\pi_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20}$$

となっている。一方、状態ベクトルを

$$\Psi \ : \ \{\psi_1 = u(1)v(2) + u(2)v(1),\ \ \psi_2 = u(1)v(2) - u(2)v(1)\} \tag{2.21}$$

と選んでみよう。この時

$$\begin{aligned} \pi_2\psi_1 &= \psi_1 \\ \pi_2\psi_2 &= -\psi_2 \end{aligned} \tag{2.22}$$

なので表現行列 $D(\pi_2)$ は

$$D(\pi_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.23}$$

となっている。この場合は表現行列が対角的になっている。従って、この状態ベクトルは π_2 オペレータの固有関数になっている。

第3章　群論の応用

群論を量子力学へ応用する事は非常に重要である。特に Hamiltonian が空間回転に対して不変な場合がよくあるため、その対称性から得られる様々な性質は一般性があり応用範囲が広いものである。ここでは空間回転の対称性と関係している回転群やそれと同形の $SU(2)$ の群について簡単な解説をしよう。

3.1　状態関数の変換とオペレータの変換性

状態関数 ψ をユニタリーオペレータ $\hat{U}$ で変換するとは

$$\psi' = \hat{U}\psi \tag{3.1}$$

と状態関数にオペレータ $\hat{U}$ を掛けて演算を実行する事である。それでは、量子力学で良く出てくる一般のオペレータ $\hat{O}$ はどのように変換するのであろうか？これは変換する前と変換後におけるオペレータ $\hat{O}$ の期待値が同じであると言う要請から理解されることである。すなわち

$$\langle\psi'|\hat{O}'|\psi'\rangle = \langle\psi|\hat{O}|\psi\rangle \tag{3.2}$$

である。よってこれは

$$\langle\psi|\hat{U}^\dagger\hat{O}'\hat{U}|\psi\rangle = \langle\psi|\hat{O}|\psi\rangle \tag{3.3}$$

となる事から

$$\hat{U}^\dagger\hat{O}'\hat{U} = \hat{O} \quad \Longrightarrow \quad \hat{O}' = \hat{U}\hat{O}\hat{U}^{-1} \tag{3.4}$$

と求まる。例えば Hamiltonian $\hat{H}$ の場合、 $\hat{H}' = \hat{U}\hat{H}\hat{U}^{-1}$ となる。

3.2 Hamiltonian の対称性

今、Hamiltonian $\hat{H}$ がユニタリーオペレータ $\hat{U}$ の変換で不変であるとしよう。この時、数学的には

$$\hat{H}' = \hat{H} = \hat{U}\hat{H}\hat{U}^{-1} \tag{3.5}$$

と書かれている。ユニタリーオペレータ $\hat{U}$ を考える理由は簡単でノムルが

$$\langle U\psi|U\psi\rangle = \langle\psi|U^{\dagger}U\psi\rangle = 1 \tag{3.6}$$

と保存されているからである。この対称性の式 (3.5) があると、Hamiltonian $\hat{H}$ の固有値は $\hat{U}$ の固有値で指定される事が示される。

• **$\hat{H}$ 固有値が $\hat{U}$ の固有値で指定される事の証明：**
式 (3.5) から $\hat{H}$ と $\hat{U}$ は交換する。すなわち

$$\hat{H}\hat{U} = \hat{U}\hat{H} \tag{3.7}$$

である。今、$\hat{H}$ の固有値と固有関数を $E_a,\ \psi_a$ としよう。

$$\hat{H}\psi_a = E_a\psi_a \tag{3.8}$$

この式に左から $\hat{U}$ をオペレートして式 (3.7) を使うと

$$\begin{aligned} \hat{U}\hat{H}\psi_a &= E_a U\psi_a \\ \hat{H}(\hat{U}\psi_a) &= E_a(\hat{U}\psi_a) \end{aligned} \tag{3.9}$$

となり、これは $\hat{U}\psi_a$ も $\hat{H}$ の固有関数であることを示している。よって $\hat{U}\psi_a$ は ψ_a に比例している。従って

$$\hat{U}\psi_a = k\psi_a \tag{3.10}$$

と書くことができる。この式は ψ_a が $\hat{U}$ の固有関数となっている事を表している。

3.2.1 エネルギー固有値の $d-$ 重縮退

ここで Hamiltonian $\hat{H}$ の固有値が $d-$ 重に縮退している場合を考えよう。従って

$$\hat{H}\psi_n = E\psi_n \quad (n = 1, 2, \cdots, d) \tag{3.11}$$

である。ここで

$$\langle\psi_n|\psi_m\rangle = \delta_{nm} \tag{3.12}$$

は仮定されている。エネルギーは縮退していると言う事はそれに関連して、何かの対称性 $\hat{R}$ が必ず、存在しているはずである。すなわち

$$\hat{R}\hat{H}\hat{R}^{-1} = \hat{H} \tag{3.13}$$

となるようなオペレータ $\hat{R}$ が存在するはずである。この場合、

$$\hat{H}(\hat{R}\psi_n) = E(\hat{R}\psi_n) \quad (n = 1, 2, \cdots, d) \tag{3.14}$$

が成り立っている。この式から状態関数 $\hat{R}\psi_n$ も同じエネルギー E の状態に属している事がわかる。従ってこれは

$$\hat{R}\psi_n = \sum_{n=1}^{m} D_{mn}(R)\psi_m \tag{3.15}$$

と ψ_n の線形結合で書かれている。この式は $D_{mn}(R)$ が表現行列に対応している事を示している。

- **空間回転不変性**

特に、空間回転における Hamiltonian の不変性に関しては今後、詳しく議論して行く事になる。連続変換の対称性に関してはこの回転対称性が物理学では最も重要な問題である。

3.2.2 空間反転対称性

空間反転 (パリティ変換) $\boldsymbol{r} \Rightarrow -\boldsymbol{r}$ のオペレータを

$$\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r}) \tag{3.16}$$

で定義しよう。単位オペレータ I を

$$\hat{I}\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) \tag{3.17}$$

で定義する。この時 $\{\hat{I},\ \hat{P}\}$ は群を形成している。

- $\hat{P}$ **の表現行列：**

ここで状態ベクトルとして

$$\Psi\ :\ \{\psi_1 = u(\boldsymbol{r}),\ \psi_2 = u(-\boldsymbol{r})\} \tag{3.18}$$

としよう。この時

$$\begin{aligned} \hat{P}\psi_1 &= \psi_2 \\ \hat{P}\psi_2 &= \psi_1 \end{aligned} \tag{3.19}$$

と求まる。従って表現行列 $D(\hat{P})$ は

$$D(\hat{P}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.20}$$

となっている。これは対称群 S_2 と全く同じ表現行列である。

3.3 回転対称性 (3 次元空間)

空間回転における対称性について解説しよう。まず座標 $\boldsymbol{r}$ を $z-$ 軸周りに θ だけ回転した時、その回転したあとの座標を $\boldsymbol{r}'$ としよう。すなわち

$$\boldsymbol{r}' = \hat{R}_\theta \boldsymbol{r} \tag{3.21}$$

である。ここで $\hat{R}_\theta$ は回転のオペレータであり式 (3.21) が

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.22}$$

である事から

$$\hat{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

となっている。

3.3.1 $\hat{R}_\theta$ は群をなす

この $\hat{R}_\theta$ は群を形成している事を示そう。

• $\hat{R}_{\theta_1}\hat{R}_{\theta_2}$ も回転オペレータ :
今、 2 個の回転の積 $\hat{R}_{\theta_1}\hat{R}_{\theta_2}$ を考えてみよう。この時、

$$\begin{aligned} \hat{R}_{\theta_1}\hat{R}_{\theta_2} &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{R}_{\theta_1+\theta_2} \end{aligned} \tag{3.24}$$

となり確かに、これも回転演算子となっている。

• 単位オペレータ：
これは簡単で $\theta = 0$ の時で $\hat{R}_\theta = I$ に対応する。

• 逆元オペレータ：
この場合 $(\hat{R}_\theta)^{-1} = \hat{R}_{-\theta}$ に対応する。

• 結合則：
これは簡単なので省略しよう。従って、$\hat{R}_\theta$ は確かに群を形成している事が示された。この場合、$\hat{R}_\theta$ は直交行列であり、

$$\hat{R}_\theta(\hat{R}_\theta)^t = 1 \tag{3.25}$$

を満たしている。

3.3.2 状態関数に対する回転群演算子

これまで見てきた空間回転のオペレータは座標を回転するものである。それでは状態関数に対する回転のオペレータはどうなっているのであろうか？このオペレータを $\hat{R}_z(\theta)$ としよう。この場合

$$\psi'(\boldsymbol{r}) = \hat{R}_z(\theta)\psi(\boldsymbol{r}) \tag{3.26}$$

となる。これは $z-$ 軸の周りの θ 回転を意味している。座標と状態関数を同時に回転すれば元に戻るので

$$\psi'(\boldsymbol{r}') = \psi(\boldsymbol{r}) \tag{3.27}$$

となっている。これより

$$\psi'(\boldsymbol{r}) = \psi(\hat{R}_\theta^{-1}\boldsymbol{r}) \tag{3.28}$$

となっている。ここで θ が充分小さい場合　$\theta \ll 1$ を考えよう。この時

$$\hat{R}_\theta^{-1} \simeq \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.29}$$

なので $\psi(\hat{R}_\theta^{-1}\boldsymbol{r})$ は

$$\begin{aligned}\psi'(\boldsymbol{r}) &= \psi(\hat{R}_\theta^{-1}\boldsymbol{r}) = \psi(x+\theta y, y-\theta x, x) \\ &= \left[1+\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\theta+\cdots\right]\psi(\boldsymbol{r}) \end{aligned} \tag{3.30}$$

である。一方、角運動量 ℓ_z は

$$\ell_z = -i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) \tag{3.31}$$

である。但し $\hbar = 1$ としている。これより

$$\hat{R}_z(\theta) \simeq (1-i\ell_z\theta+\cdots) \tag{3.32}$$

となる。これは θ が充分小さい場合であるが、θ が有限の場合は以下のように計算する。$\frac{\theta}{n}$ を n 回だけ回転させると

$$\hat{R}_z(\theta) = \lim_{n\to\infty}\left(1-\frac{i\ell_z\theta}{n}\right)^n = e^{-i\ell_z\theta} \tag{3.33}$$

となる事がわかる。これが状態関数を θ だけ回転させるオペレータとなっている。

• $\hat{R}_z(\theta) = e^{-i\ell_z\theta}$ **は群を形成する：**
この場合、$z-$ 軸周りの回転だけなので単純である。このため この証明は省略しよう。実際の回転群はすべての角度によるものとなり、これはかなり難しい問題となっている。以下にできる限り平明に解説して行こう。

3.3.3 Euler 角と回転群演算子

任意の角度の回転を考えると 3 つの自由度が必要となる事が Euler により示されている。それは Euler 角と呼ばれるものであり、この Euler 角は $(\alpha,\ \beta,\ \gamma)$ で表されている。この Euler 角により 3 次元の回転は正確に記述される事が分かっている。Euler 角による回転の演算子 $\hat{R}(\alpha,\ \beta,\ \gamma)$ は次のように表す事ができる。

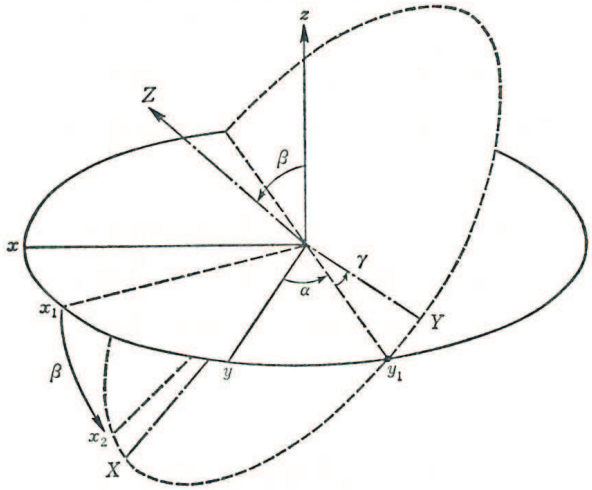

図 3.1: Euler 角

3.3.4 回転演算子 $\hat{R}(\alpha,\ \beta,\ \gamma)$ の性質

この回転演算子 $\hat{R}(\alpha,\ \beta,\ \gamma)$ は

$$\hat{R}(\alpha,\ \beta,\ \gamma) = e^{-i\alpha\ell_z} e^{-i\beta\ell_y} e^{-i\gamma\ell_z} \tag{3.34}$$

と書かれている。これが何故、このような形になったのかを簡単に解説しよう。これは図を見ながら考えて行くことが大切である。これは Euler 角の回転演算子についてこの式に至るまでの経過を見て行く事になるが、このためにはそれぞれのステップを検証する事が必要となる。しかしながらこの過程はかなり複雑であり、しっかり考えないと良くわからない事であると言える。

- **第一ステップ**

  最初に $z-$ 軸の周りに α だけ回転しよう。これは $\hat{R}_z(\alpha)$ である。

- **第二ステップ**

  次に y_1- 軸周りに β だけ回転しよう。 これは

$$\hat{R}_{y_1}(\beta) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z^{-1}(\alpha) \tag{3.35}$$

と書かれている。

- **第三ステップ**

  次に $Z-$ 軸周りに γ だけ回転しよう。 これは

$$\hat{R}_Z(\gamma) = \hat{R}_{y_1}(\beta)\hat{R}_z(\gamma)\hat{R}_{y_1}^{-1}(\beta) \tag{3.36}$$

と書かれている。

- **連続操作**

  従ってこれを連続して操作すると

$$\begin{aligned}
\hat{R}(\alpha,\ \beta,\ \gamma) &= \hat{R}_Z(\gamma)\hat{R}_{y_1}(\beta)\hat{R}_z(\alpha) \\
&= \hat{R}_{y_1}(\beta)\hat{R}_z(\gamma)\hat{R}_{y_1}^{-1}(\beta)\hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z^{-1}(\alpha)\hat{R}_z(\alpha) \\
&= \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z^{-1}(\alpha)\hat{R}_z(\gamma)\hat{R}_z(\alpha)\hat{R}_y^{-1}(\beta)\hat{R}_z^{-1}(\alpha)\hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z^{-1}(\alpha)\hat{R}_z(\alpha) \\
&= \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) = e^{-i\alpha\ell_z}e^{-i\beta\ell_y}e^{-i\gamma\ell_z}
\end{aligned}$$

となる。これが Euler 角 $(\alpha,\ \beta,\ \gamma)$ による回転演算子である。

3.4 角運動量 ℓ とその固有関数

回転群のオペレータ $\hat{R}(\alpha,\ \beta,\ \gamma) = e^{-i\alpha\ell_z}e^{-i\beta\ell_y}e^{-i\gamma\ell_z}$ の exp の肩に現われる $\ell_x,\ \ell_y,\ \ell_z$ の事を生成子 (generator) と言う。これは角運動量演算子である。ここでは角運動量演算子の性質とその固有関数について解説しよう。

- **Lie 代数**

この要素間の演算は交換関係で

$$[\ell_x,\ \ell_y] = i\ell_z,\quad [\ell_y,\ \ell_z] = i\ell_x,\quad [\ell_z,\ \ell_x] = i\ell_y \tag{3.37}$$

が成り立っている。これを Lie 代数と言う。但し、これは群を作ってはいない。単位オペレータが存在しないし、逆元もないからである。

- **Casimir オペレータ**

ここで $\boldsymbol{\ell}^2 = \ell_x^2 + \ell_y^2 + \ell_z^2$ を定義する。この場合、簡単な計算で

$$[\boldsymbol{\ell}^2,\ \ell_i] = 0,\qquad (i = x,\ y\ , z) \tag{3.38}$$

が証明される。この Casimir オペレータは $O(3)$ と $SU(2)$ に関しては 1 個存在している。一方、 $SU(3)$ の場合、 2 個の Casimir オペレータが存在している。

3.4.1 球面調和関数

角運動量演算子 $\boldsymbol{\ell}^2,\ \ell_z$ の固有関数が球面調和関数 $Y_{\ell m}(\theta,\varphi)$ である。従って

$$\boldsymbol{\ell}^2 Y_{\ell m}(\theta,\varphi) = \ell(\ell+1)Y_{\ell m}(\theta,\varphi) \tag{3.39}$$

$$\ell_z Y_{\ell m}(\theta,\varphi) = m Y_{\ell m}(\theta,\varphi) \tag{3.40}$$

となっている。ほとんどの場合、これは簡略化して

$$\boldsymbol{\ell}^2|\ell m\rangle = \ell(\ell+1)|\ell m\rangle \tag{3.41}$$

$$\ell_z|\ell m\rangle = m|\ell m\rangle \tag{3.42}$$

と書かれている。

● 球面調和関数の性質

球面調和関数は空間反転に対して

$$Y_{\ell m}(\pi-\theta,\varphi+\pi)=(-)^{\ell}Y_{\ell m}(\theta,\varphi) \tag{3.43}$$

となる。また直交性

$$\int Y^{*}_{\ell m}(\theta,\varphi)Y_{\ell m'}(\theta,\varphi)\sin\theta d\theta d\varphi=\delta_{m,m'} \tag{3.44}$$

そして加法定理

$$\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\theta,\varphi)Y_{\ell m}(\theta,\varphi)=\frac{2\ell+1}{4\pi} \tag{3.45}$$

が成り立っている。また $Y_{\ell m}(\theta,\varphi)$ の具体的な表現として

$$\begin{aligned} Y_{\ell m}(\theta,\varphi) &= \frac{(-)^{\ell+m}}{(2\ell)!!}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}(\sin\theta)^{m} \\ &\quad\times\frac{d^{\ell+m}}{(d\,\cos\theta)^{\ell+m}}(\sin\theta)^{2\ell}e^{im\varphi} \end{aligned} \tag{3.46}$$

と書かれる。

3.4.2 合成角運動量の固有関数

ここで 2 個の角運動量 $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$ を考えて、その和である $\boldsymbol{L}=\boldsymbol{\ell}_1+\boldsymbol{\ell}_2$ の固有関数を求めよう。 $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$ それぞれの固有関数は

$$\boldsymbol{\ell}_1^2|\ell_1m_1\rangle=\ell_1(\ell_1+1)|\ell_1m_1\rangle,\quad \ell_{1z}|\ell_1\rangle=m_1|\ell_1m_1\rangle \tag{3.47}$$

$$\boldsymbol{\ell}_2^2|\ell_2m_2\rangle=\ell_2(\ell_2+1)|\ell_2m_2\rangle,\quad \ell_{2z}|\ell_2\rangle=m_2|\ell_2m_2\rangle \tag{3.48}$$

としよう。この時、$\boldsymbol{L}^2$ の固有関数 $|LM\rangle$ は

$$|LM\rangle=\sum_{m_1,\,m_2}(\ell_1m_1\ell_2m_2|LM)\;|\ell_1m_1\rangle|\ell_2m_2\rangle,\quad \text{with}\quad (M=m_1+m_2) \tag{3.49}$$

と書かれている。ここで $(\ell_1 m_1 \ell_2 m_2|LM)$ は Clebsch-Gordan 係数と呼ばれている。これは状態 $|LM\rangle$ が $\boldsymbol{L}^2,\ L_z$ の固有関数

$$\boldsymbol{L}^2|LM\rangle = L(L+1)|LM\rangle \tag{3.50}$$

$$L_z|LM\rangle = M|LM\rangle \tag{3.51}$$

になるように決められたものである。この Clebsch-Gordan 係数の決め方は少し面倒なのでここでは省略しよう。この式 (3.49) を逆に解いた式も成り立っている。

$$|\ell_1 m_1\rangle|\ell_2 m_2\rangle = \sum_{LM} (\ell_1 m_1 \ell_2 m_2|LM)\ |LM\rangle \tag{3.52}$$

- **Clebsch-Gordan 係数の性質**

ここで Clebsch-Gordan 係数の最も基本的な性質を書いて置こう。

$$\sum_{m_1,\,m_2} (\ell_1 m_1\, \ell_2 m_2|LM)\,(\ell_1 m_1\, \ell_2 m_2|LM') \;=\; \delta_{M,\,M'} \tag{3.53}$$

$$\sum_{JM} (\ell_1 m_1\, \ell_2 m_2|LM)\,(\ell_1 m'_1\, \ell_2 m'_2|LM) \;=\; \delta_{m_1,\,m'_1}\delta_{m_2,\,m'_2} \tag{3.54}$$

これは Clebsch-Gordan 係数の直交性である。他の基本的な性質として

$$(J_1 M_1\, J_2 M_2|JM) \;=\; (-)^{J_1+J_2-J}(J_2 M_2\, J_1 M_1|JM) \tag{3.55}$$

$$\;=\; (-)^{J_1+J_2-J}(J_1\ -M_1\, J_2\ -M_2|J\ -M) \tag{3.56}$$

が重要であろう。この CG 係数は 3j シンボル

$$(J_1 M_1\, J_2 M_2|JM) = (-)^{J_1-J_2+M}\sqrt{2J+1}\begin{pmatrix} J_1 & J_2 & J \\ M_1 & M_2 & -M \end{pmatrix} \tag{3.57}$$

と結びついている。この 3j シンボルには次のような簡単な対称性がある。

$$\begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} = \begin{pmatrix} J_3 & J_1 & J_2 \\ M_3 & M_1 & M_2 \end{pmatrix} = (-)^{J_1+J_2+J_3}\begin{pmatrix} J_2 & J_1 & J_3 \\ M_2 & M_1 & M_3 \end{pmatrix} \tag{3.58}$$

3.4.3 全角運動量 $\boldsymbol{J} = \boldsymbol{\ell} + \boldsymbol{s}$ とスピン $\boldsymbol{s}$

電子を考えるとその粒子の全角運動量は $\boldsymbol{J} = \boldsymbol{\ell} + \boldsymbol{s}$ となりスピン部分も足される事になる。但し $s = \frac{1}{2}$ である。この場合、全角運動量 $\boldsymbol{J}^2,\ J_z$ の固有関数を作るために Clebsch-Gordan 係数が必要であった。実際、その固有状態は

$$|JM\rangle = \sum_{m,\, m_s} \left(\ell m \, \frac{1}{2} m_s | LM\right)\ Y_{\ell m}\ \chi_{m_s} \tag{3.59}$$

となっている。この場合、J は必ず、半整数となっている。ここで χ_{m_s} はスピンの状態関数であり、今の場合 $\boldsymbol{s}^2,\ s_z$ の固有関数として

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.60}$$

をとっている。実際

$$\boldsymbol{s}^2 \chi_{m_s} = \frac{3}{4}\,\chi_{m_s}, \quad s_z \chi_{m_s} = m_s \chi_{m_s} \tag{3.61}$$

である。この $\boldsymbol{J}^2$ についてその固有値 J が半整数である。しかし $\boldsymbol{J}$ はリー代数を満たしているため、整数の場合と同じように取り扱っても特に問題となる事はない。

- **$\boldsymbol{J}^2$ の固有関数**

ここで $\boldsymbol{J}^2,\ J_z$ の固有関数を書いて置こう。

$$|jm\rangle = \sum_{m,\, m_s} (\ell m \, \frac{1}{2}\, m_s | jm)\, Y_{\ell m}(\theta, \varphi)\, \chi_{m_s} \tag{3.62}$$

今の場合、Clebsch-Gordan 係数が解析的に書かれているので $j = \ell \pm \frac{1}{2}$ に対して

$$|\ell \pm \frac{1}{2},\ m\rangle = \pm\sqrt{\frac{\ell \pm m + \frac{1}{2}}{2\ell + 1}}\, Y_{\ell,\, m-\frac{1}{2}}\, \chi_{\frac{1}{2}} + \sqrt{\frac{\ell \mp m + \frac{1}{2}}{2\ell + 1}}\, Y_{\ell,\, m+\frac{1}{2}}\, \chi_{-\frac{1}{2}} \tag{3.63}$$

となる。ここで $j = \ell \pm \frac{1}{2}$ に対して複合同順とする。

3.4.4 回転オペレータ $\hat{R}$ と表現行列 $D^{(\ell)}_{m'm}$

回転オペレータ $\hat{R}$ を $Y_{\ell m}(\theta,\varphi)$ にオペレートすると

$$\hat{R}Y_{\ell m}(\theta,\varphi) = \sum_{m'=-\ell}^{\ell} D^{(\ell)}_{m'm} Y_{\ell m'}(\theta,\varphi) \tag{3.64}$$

となる。この $D^{(\ell)}_{m'm}$ が回転オペレータ $\hat{R}$ の表現行列となっている。具体的な形は第４章で求める事になる。

- **既約テンソルオペレータ $T^{(k)}_{\kappa}$**

ここで既約テンソルオペレータ $T^{(k)}_{\kappa}$ を

$$\hat{R}T^{(k)}_{\kappa}\hat{R}^{-1} = \sum_{m'=-k}^{k} D^{(k)}_{m'\kappa} T^{(k)}_{m'} \tag{3.65}$$

により定義しよう。この $T^{(k)}_{\kappa}$ は基本的には空間回転における性質が $Y_{k\kappa}$ と同じとなっている。従って回転における性質は $Y_{k\kappa}$ に比例しているので

$$T^{(k)}_{\kappa} \simeq C_0 Y_{k\kappa}, \quad (C_0 \text{ 定数}) \tag{3.66}$$

と考えてめったに間違える事はない。直感的にはこれで十分と言える。

3.4.5 Wigner-Eckart の定理

回転群の応用で重要な定理が Wigner-Eckart の定理である。これは既約テンソルオペレータ $T^{(k)}_{\kappa}$ を状態 $|jm\rangle$ と $|j'\,m'\rangle$ で期待値を取った場合、その $z-$ 成分の影響はすべて Clebsch-Gordan 係数で決められると言うものである。式で書くと

$$\langle jm|T^{(k)}_{\kappa}|j'\,m'\rangle = \frac{1}{\sqrt{2j+1}}\,(j'\,m'k\kappa|jm)\,\langle j\,||T^{(k)}||\,j'\rangle \tag{3.67}$$

となると言う定理である。ここで $\langle j\,||T^{(k)}||\,j'\rangle$ はダブルバー行列要素と呼ばれるもので、これは磁気量子数 $m,\ m',\ \kappa$ には依らない。磁気量子数の依存性はすべて Clebsch-Gordan 係数の中に入っている。

• Wigner-Eckart の定理の証明

この定理の証明は比較的簡単である。まずは $T^{(k)}_\kappa$ と $|j'\,m'\rangle$ の合成状態を作ろう。この場合、式 (3.52) より

$$T^{(k)}_\kappa\,|j'\,m'\rangle = C(T^{(k)})\sum_{JM}(j'\,m'k\kappa|JM)\;|J\,M\rangle \tag{3.68}$$

と書くことができる。ここで $C(T^{(k)})$ は $T^{(k)}_\kappa$ の大きさにのみ関係する量である。また合成された角運動量は JM である。この式 (3.68) を用いて式 (3.67) の左辺を計算すると

$$\begin{aligned}\langle jm|T^{(k)}_\kappa|j'\,m'\rangle &= C(T^{(k)})\sum_{JM}(j'\,m'\,k\kappa|JM)\;\langle jm|J\,M\rangle &(3.69)\\ &= C(T^{(k)})\sum_{JM}(j'\,m'\,k\kappa|JM)\,\delta_{jJ}\,\delta_{mM} &(3.70)\\ &= C(T^{(k)})(j'\,m'\,k\kappa|jm) &(3.71)\end{aligned}$$

となる。よって

$$C(T^{(k)}) = \frac{\langle j\,||T^{(k)}||\,j'\rangle}{\sqrt{2j+1}} \tag{3.72}$$

とおけば Wigner-Eckart の定理が証明された事になっている。

$$\langle jm|T^{(k)}_\kappa|j'\,m'\rangle = (j'\,m'\,k\kappa|jm)\frac{\langle j\,||T^{(k)}||\,j'\rangle}{\sqrt{2j+1}}$$

3.5 Isospin 空間

陽子と中性子の質量はほとんど同じである。実際、

$$\begin{aligned} M_p &= 938.3 \text{ MeV} \\ M_n &= 939.6 \text{ MeV} \end{aligned}$$

となっている。また強い相互作用をすると言う点でも同じである。このため、この二つを核子と総称している。そしてその違いである電荷は Isospin の量子数の違いにより分類して行く事になる。

3.5.1 Isospin 空間

Isospin 空間は次の Pauli 行列によって記述されている。この場合、Isospin オペレータ $\boldsymbol{t}$ は

$$\boldsymbol{t} = \frac{1}{2}\boldsymbol{\tau} \tag{3.73}$$

であり、 $\boldsymbol{\tau}$ は Pauli 行列である。これは下記のように書かれている。

$$\begin{aligned} &\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z) \\ &\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{3.74}$$

また陽子と中性子の状態を

$$\begin{aligned} |p\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |n\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{3.75}$$

と定義している。この時、陽子と中性子は τ_z の固有状態となっていて

$$\begin{aligned} \tau_z|p\rangle &= |p\rangle \\ \tau_z|n\rangle &= -|n\rangle \end{aligned} \tag{3.76}$$

である。従ってこれらの isospin は

$$\begin{aligned} t_z|p\rangle &= \frac{1}{2}\,|p\rangle \\ t_z|n\rangle &= -\frac{1}{2}\,|n\rangle \end{aligned} \tag{3.77}$$

である。電荷の演算子 Q は $Q = \frac{1}{2}(1+\tau_z)$ と定義されている。よって

$$\begin{aligned} Q|p\rangle &= |p\rangle \\ Q|n\rangle &= 0 \end{aligned} \tag{3.78}$$

となる。

3.5.2 Isospin 空間における回転

3 次元空間での回転 ($z-$ 軸周り) は

$$\hat{R}_z(\theta) = e^{-i\ell_z\theta} \tag{3.79}$$

であった。これと同様に Isospin 空間における回転を考えよう。Isospin 空間において $z-$ 軸周りに α 回転するオペレータは

$$\hat{R}_z(\alpha) = e^{-i\alpha t_z} \tag{3.80}$$

と定義される。この式は

$$\hat{R}_z(\alpha) = \cos\frac{\alpha}{2} - i\tau_z \sin\frac{\alpha}{2} \tag{3.81}$$

と書く事ができる。この証明は簡単で exp を Taylor 展開すれば

$$\begin{aligned} \hat{R}_z(\alpha) &= e^{-i\alpha t_z} = 1 - \frac{i\alpha}{2}\tau_z - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 + \frac{1}{3!}\left(\frac{-i\alpha}{2}\right)^3\tau_z + \frac{1}{4!}\left(\frac{-i\alpha}{2}\right)^4 + \cdots \\ &= \cos\frac{\alpha}{2} - i\tau_z\sin\frac{\alpha}{2} \end{aligned} \tag{3.82}$$

となるからである。

3.5.3 Isospin 空間のスカラー

核力を考える時に、それは Isospin 空間でスカラーとなっている必要がある。実際、2 体の核力 V_{NN} は

$$V_{NN} = [a + b(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)]\ V_0(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{3.83}$$

と書かれている。ここで $a,\ b$ は定数である。

• $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ はアイソスカラー：
この $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ が Isospin 空間でスカラーである事を示そう。まず全アイソスピン $\boldsymbol{T}$ を

$$\boldsymbol{T} = \frac{1}{2}(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2) \tag{3.84}$$

で定義しよう。ここで Isospin 空間でスカラーであるとは、 Isospin 空間での回転

$$R_\alpha(T_i) = e^{-i\alpha T_i} \tag{3.85}$$

に対して不変となっている事である。但し、 T_i は $T_x,\ T_y,\ T_z$ を表している。ここで $\boldsymbol{T}^2$ を計算すると

$$\boldsymbol{T}^2 = \frac{1}{4}\Big(\boldsymbol{\tau}_1^2 + \boldsymbol{\tau}_2^2 + 2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\Big) = \frac{1}{2}\Big(3 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\Big) \tag{3.86}$$

となる。一方、

$$[\boldsymbol{T}^2, T_i] = 0, \quad (i = x,\ y,\ z) \tag{3.87}$$

が簡単に証明される。従って、微小の α に対して $\boldsymbol{T}^2$ を回転すると

$$R_\alpha(T_i)\, \boldsymbol{T}^2\, R_\alpha^{-1}(T_i) = \boldsymbol{T}^2 + i\alpha[\boldsymbol{T}^2, T_i] + \cdots = \boldsymbol{T}^2 \tag{3.88}$$

となり、回転に対して不変であることが示された。これより $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ が Isospin 空間でスカラーである事が示された。

3.5.4 $\boldsymbol{T}^2$ の固有値

2 体の核力を考える時、その状態のアイソスピンが重要になる事がある。簡単な計算ではあるが、結果だけでも書いておこう。$\boldsymbol{T}^2$ の固有値はその固有関数を Φ_T とすると

$$\boldsymbol{T}^2\Phi_T = T(T+1)\Phi_T \tag{3.89}$$

となる。核子のアイソスピンは $t = \frac{1}{2}$ なので $T = 0,\ 1$ が可能な量子数となる。

• $T = 0$ **の場合 (singlet) :**
その状態関数は

$$\Phi_0^{(0)} = \frac{1}{\sqrt{2}}\Big(|p\rangle_1|n\rangle_2 - |n\rangle_1|p\rangle_2\Big) \tag{3.90}$$

となる。この場合

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\, \Phi_0^{(0)} = -3\Phi_0^{(0)} \tag{3.91}$$

である。

• $T = 1$ **の場合 (triplet) :**
その状態関数は

$$\Phi_1^{(0)} = \frac{1}{\sqrt{2}}\Big(|p\rangle_1|n\rangle_2 + |n\rangle_1|p\rangle_2\Big), \quad \Phi_1^{(1)} = |p\rangle_1|p\rangle_2, \quad \Phi_1^{(-1)} = |n\rangle_1|n\rangle_2 \tag{3.92}$$

と 3 個の状態が存在している。この時

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\, \Phi_1^{(n)} = \Phi_1^{(n)}, \quad (n = -1,\ 0.\ 1) \tag{3.93}$$

となる。

3.5.5 アイソベクトル T の回転

ここで例えば T_z は回転 $R_\alpha(T_x) = e^{-i\alpha T_x}$ に対して不変とはなっていない。これは簡単な計算で

$$R_\alpha(T_x)\, T_z\, R_\alpha^{-1}(T_x) = e^{-i\alpha T_x}\, T_z\, e^{i\alpha T_x} = T_z \cos\alpha - T_y \sin\alpha \tag{3.94}$$

となる事から不変ではないことがわかる。この事は T_i がアイソベクトルであるため、回転に対して動いてしまう事に対応している。不変量はスカラーだけである。

3.5.6 Pauli 行列と SU(2)

SU(2) は special unitary の 2 行 2 列の行列である。この群の要素 u は

$$u = e^{iH} \tag{3.95}$$

と書くことができる。この場合、$u\, u^\dagger = 1$ より

$$uu^\dagger = e^{iH}\, e^{-iH^\dagger} = e^{i(H - H^\dagger)} = 1 \tag{3.96}$$

これより

$$H = H^\dagger, \qquad (H : \text{ Hermite matrix}) \tag{3.97}$$

すなわち、 H は Hermite 行列であることがわかる。従って

$$H = \begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix} \tag{3.98}$$

と書くことができる。ここで a, b, c, d は実数である。この式は

$$H = \begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.99}$$

と変形される。これは Pauli 行列 $\sigma_x,\ \sigma_y,\ \sigma_z$ により

$$H = a' + b\,\sigma_x + c\,\sigma_y + d'\,\sigma_z \tag{3.100}$$

と書かれている。ここで定数は任意である。

第4章　回転群 O(3) の表現

回転群は３次元空間の回転に関係して構成されている。これは O(3) とも呼ばれているが、この群は３行３列の直交行列で構成されている。ここではこの回転群の表現行列を求めて行こう。

4.1　回転群 O(3)

回転群のオペレータは Euler 角 $\alpha,\ \beta,\ \gamma$ により指定され

$$\hat{R}(\alpha,\ \beta,\ \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{4.1}$$

と書かれている。ここで $J_x,\ J_y,\ J_z$ は generator であり、リー代数を満たしている。すなわち

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y \tag{4.2}$$

が成り立つ。

4.1.1　$\boldsymbol{J}^2$ の固有値

ここで $\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ と定義しよう。これは Casimir オペレータであり

$$[\boldsymbol{J}^2, J_i] = 0, \qquad (i = x,\ y,\ z) \tag{4.3}$$

が成立している。ここでは J_z が対角的になるような基底をとろう。すなわち

$$\boldsymbol{J}^2 |JM\rangle = J(J+1)|JM\rangle \tag{4.4}$$

$$J_z |JM\rangle = M|JM\rangle \tag{4.5}$$

と取る。但し、以降は $|M\rangle \equiv |JM\rangle$ と表記しよう。この場合

$$\langle M|J_z|M'\rangle = M\delta_{MM'} \tag{4.6}$$

となる。ここで $J_\pm$ を

$$J_\pm = J_x \pm iJ_y \tag{4.7}$$

で定義しよう。この時、簡単な計算で

$$[J_z, J_\pm] = \pm J_\pm \tag{4.8}$$

が示される。ここでこの式 $\boxed{J_zJ_\pm - J_\pm J_z = \pm J_\pm}$ を状態 $|M\rangle,\ |M'\rangle$ で挟むと

$$\begin{aligned} & \langle M|J_zJ_\pm - J_\pm J_z|M'\rangle = \pm\langle M|J_\pm|M'\rangle \\ \text{書き直して} \qquad & (M - M' \mp 1)\langle M|J_\pm|M'\rangle = 0 \end{aligned} \tag{4.9}$$

となる。これより $\boxed{M' = M \mp 1}$ 以外では $\boxed{\langle M|J_\pm|M'\rangle = 0}$ であることがわかる。よってゼロでない行列要素を

$$\langle M|J_+|M-1\rangle = A \tag{4.10}$$

$$\langle M|J_-|M+1\rangle = B \tag{4.11}$$

として $A,\ B$ を求めて行く。一方、

$$J_-J_+ = \boldsymbol{J}^2 - J_z^2 - J_z = |J_+|^2 \geq 0 \tag{4.12}$$

である。ここで $-J \leq M \leq J$ を考慮すると、この式から

$$\langle J|J_-J_+|J\rangle = \langle J|\boldsymbol{J}^2 - J_z^2 - J_z|J\rangle = \langle J|\boldsymbol{J}^2|J\rangle - J(J+1) = 0 \tag{4.13}$$

が求まる。これより

$$\langle J|\boldsymbol{J}^2|J\rangle = J(J+1) \tag{4.14}$$

が得られる。

4.1.2 $J_\pm$ の行列要素

次に $J_\pm$ の行列要素 ($\langle M|J_+|M-1\rangle$ と $\langle M|J_-|M+1\rangle$) を求めてみよう。簡単な計算で

$$[J_+, J_-] = 2J_z \tag{4.15}$$

を示すことができる。この式を $|M\rangle$ の状態で挟むと

$$|\langle M-1|J_-|M\rangle|^2 - |\langle M|J_-|M+1\rangle|^2 = 2M \tag{4.16}$$

が得られる。ここで $\boxed{F(M) \equiv |\langle M-1|J_-|M\rangle|^2}$ と定義すると式 (4.16) は

$$F(M) - F(M+1) = 2M \tag{4.17}$$

となる。この式から任意の k に対して

$$F(M) - F(M+k) = 2Mk + k(k+1) \tag{4.18}$$

が成り立つ事が示される。これは数列の問題を解くのと同じような手法により求めることができる。ここで $\boxed{k = J - M + 1}$ とおくと

$$F(M) - F(J+1) = (J-M+1)(J+M) \tag{4.19}$$

となる。今、 $F(J+1) = 0$ なので、結局

$$\langle M-1|J_-|M\rangle = \sqrt{(J-M+1)(J+M)} \tag{4.20}$$

が求まる。同様に

$$\langle M+1|J_+|M\rangle = \sqrt{(J-M)(J+M+1)} \tag{4.21}$$

となっている。但し、このルートの前に現れる位相は無視している。

4.2 D 関数の計算

回転群のオペレータは

$$\hat{R}(\alpha,\ \beta,\ \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{4.22}$$

であった。ここでこの行列要素を

$$D^{(J)}_{MK}(R) \equiv \langle M | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | K \rangle \tag{4.23}$$

と定義しよう。これからこの $D^{(J)}_{MK}(R)$ を求めて行こう。この計算はかなり大変ではあるが、しかし何とか計算して結果を得る事が可能である。後で SU(2) による計算と比較して見るとわかるが、 SU(2) による手法の方がはるかに簡単である。これから 期待値を取る前の D 関数

$$D^{(J)}(R^{-1}) = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z} \tag{4.24}$$

に対する微分方程式を立てて、それを解いて形を決める作業をして行こう。

4.2.1 $D^{(J)}(R^{-1})$ に対する微分方程式

まずは $D^{(J)}(R^{-1})$ を α で微分しよう。これは簡単に計算できて

$$-i\frac{\partial}{\partial\alpha} D^{(J)}(R^{-1}) = D^{(J)}(R^{-1}) J_z \tag{4.25}$$

となる。次に β で微分しよう。この場合

$$-i\frac{\partial}{\partial\beta} D^{(J)}(R^{-1}) = e^{i\gamma J_z} e^{i\beta J_y} J_y e^{i\alpha J_z} = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z}\ e^{-i\alpha J_z} J_y e^{i\alpha J_z}$$

と変形できる。ここで $\boxed{e^{-i\alpha J_z} J_y e^{i\alpha J_z} = J_{y_1}}$ に注意すると

$$-i\frac{\partial}{\partial\beta} D^{(J)}(R^{-1}) = D^{(J)}(R^{-1}) J_{y_1} \tag{4.26}$$

と求まる。さらにγで微分しよう。ここで$\boxed{\left(D^{(J)}(R^{-1})\right)^{-1}J_zD^{(J)}(R^{-1})=J_Z}$に注意すると

$$-i\frac{\partial}{\partial\gamma}D^{(J)}(R^{-1})=J_zD^{(J)}(R^{-1})=D^{(J)}(R^{-1})J_Z \tag{4.27}$$

と求まる。

- **$J_x,\ J_y,\ J_z$ の座標系回転**

  ここで $J_x,\ J_y$ の座標系の回転を行おう。$z-$ 軸周りの α 回転は

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} J_{x_1} \\ J_{y_1} \end{pmatrix} \tag{4.28}$$

なので

$$J_\pm=J_x\pm iJ_y=e^{\pm i\alpha}(J_{x_1}\pm iJ_{y_1}) \tag{4.29}$$

が示される。同様にして $J_X,\ J_Z$ に対して β 回転すると

$$\begin{pmatrix} J_X \\ J_Z \end{pmatrix}=\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} J_{x_1} \\ J_{z_1} \end{pmatrix} \tag{4.30}$$

となる。従って

$$J_Z=J_{x_1}\sin\beta+J_z\cos\beta \tag{4.31}$$

となっている。ここで $J_{z_1}=J_z$ を使っている。

- **変数 β の微分方程式**

  $J_\pm$ を変形すると

$$J_\pm=e^{\pm i\alpha}\left(\frac{1}{\sin\beta}(J_Z-J_z\cos\beta)\pm iJ_{y_1}\right) \tag{4.32}$$

と求まる。従って

$$D^{(J)}(R^{-1})J_{\pm} = e^{\pm i\alpha}D^{(J)}(R^{-1})\left(\pm iJ_{y_1} + \frac{1}{\sin\beta}(J_Z - J_z\cos\beta)\right) \tag{4.33}$$

となる。ここで 式 (4.25), (4.26), (4.27) を用いると上式は

$$e^{\pm i\alpha}\left[\pm\frac{\partial}{\partial\beta} + \frac{1}{\sin\beta}\left(-\frac{\partial}{\partial\gamma} + \cos\beta\,\frac{\partial}{\partial\alpha}\right)\right]D^{(J)}(R^{-1}) = D^{(J)}(R^{-1})J_{\pm} \tag{4.34}$$

と求まる。この式を $|K\rangle$ と $|M\rangle$ 状態で期待値を取ると

$$\left[\pm\frac{\partial}{\partial\beta} + \frac{1}{\sin\beta}\,(K - M\cos\beta)\right]D^{(J)}_{MK}(\beta)^* \tag{4.35}$$

$$= D^{(J)}_{M\pm1,K}(\beta)^*\sqrt{(J\mp M)(J\pm M+1)} \tag{4.36}$$

となる。但し

$$D^{(J)}_{MK}(\beta) \equiv \langle M|e^{-i\beta J_y}|K\rangle \tag{4.37}$$

と定義している。

4.2.2 微分方程式の解

この微分方程式を解いて行こう。まず特別な場合を考えよう。

- $M = J$ **の場合：**

  この場合、微分方程式は

$$\left[\frac{\partial}{\partial\beta} + \frac{1}{\sin\beta}\,(K - J\cos\beta)\right]D^{(J)}_{MK}(\beta)^* = 0 \tag{4.38}$$

となる。この $D^{(J)}_{MK}(\beta)$ は実数なので＊を外すと微分方程式は

$$\frac{dD^{(J)}_{JK}(\beta)}{d\beta} = -\frac{1}{\sin\beta}\,(K - J\cos\beta)\,D^{(J)}_{JK}(\beta) \tag{4.39}$$

となる。これは直ちに解けて

$$\ln D^{(J)}_{JK}(\beta) = -K \ln \tan \frac{\beta}{2} + J \ln \sin \beta + C_0 \tag{4.40}$$

と求まる。C_0 は積分定数である。よって

$$D^{(J)}_{JK}(\beta) = A_0 \left(\sin \frac{\beta}{2}\right)^{J-K} \left(\cos \frac{\beta}{2}\right)^{J+K} \tag{4.41}$$

となる事がわかる。ここで A_0 は定数である。

- **一般の M の場合：**

M が一般の場合、逐次法によって解く事ができる。ここではその結果だけを書いておこう。

$$D^{(J)}_{MK}(\beta) = (-)^{M-K} \sqrt{\frac{(J+M)!}{(J-M)!(J+K)!(J-K)!}} \times \zeta^{-\frac{(M-K)}{2}} (1-\zeta)^{-\frac{(M+K)}{2}} \left(\frac{d}{d\zeta}\right)^{J-M} \zeta^{J-K} (1-\zeta)^{J+K} \tag{4.42}$$

但し、$\zeta = \sin^2 \frac{\beta}{2}$ である。これより、一般的な D 関数 $D^{(J)}_{MK}(R)$ は

$$D^{(J)}_{MK}(R) = e^{-i\alpha M - i\gamma K} D^{(J)}_{MK}(\beta) \tag{4.43}$$

と書くことができる。

4.2.3 D 関数の例題

具体的な例をいくつか挙げておこう。

- $J = \frac{1}{2}$ の D 関数

$J = \frac{1}{2}$ の場合の D 関数は直接、計算する事ができる。これは D 関数の定義より

$$D^{(\frac{1}{2})}_{MK}(\beta) = \langle M|e^{-\frac{i}{2}\sigma_y}|K\rangle \tag{4.44}$$

である。これは

$$D^{(\frac{1}{2})}_{MK}(\beta) = \langle M|\cos\frac{\beta}{2} - i\sigma_y \sin\frac{\beta}{2}|K\rangle \tag{4.45}$$

となる。これより $D^{(\frac{1}{2})}_{MK}(\alpha,\beta,\gamma)$ は

$$D^{(\frac{1}{2})}_{MK}(\alpha,\beta,\gamma) = e^{-i\alpha M - i\gamma K}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \tag{4.46}$$

と求まる。ここで $M,\ K$ は $\frac{1}{2},\ -\frac{1}{2}$ と行列の上から下へ、左から右へ動くとする。

- $J = 1$ の D 関数

$J = 1$ の場合、まず J_y の表現行列を求める。これは

$$\langle M+1|J_+|M\rangle = \sqrt{(J-M)(J+M+1)} \tag{4.47}$$

$$\langle M-1|J_+|M\rangle = \sqrt{(J+M)(J-M+1)} \tag{4.48}$$

より求める事ができる。その結果

$$J_y = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \tag{4.49}$$

となる。これは $J_y^3 = J_y$ を満たしている事から

$$D^{(1)}_{MK}(\beta) = \langle M|e^{-i\beta J_y}|K\rangle = \langle M|1 - iJ_y \sin\beta + (\cos\beta - 1)J_y^2|K\rangle \tag{4.50}$$

となる。ここで J_y の表現を具体的に代入すると

$$D^{(1)}_{MK}(\alpha,\beta,\gamma) = e^{-i\alpha M - i\gamma K} \begin{pmatrix} \cos^2\frac{\beta}{2} & -\frac{1}{\sqrt{2}}\sin\beta & \sin^2\frac{\beta}{2} \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \sin^2\frac{\beta}{2} & \frac{1}{\sqrt{2}}\sin\beta & \cos^2\frac{\beta}{2} \end{pmatrix} \tag{4.51}$$

と求める事ができる。ここで $M,\ K$ は $1,\ 0,\ -1$ と行列の上から下へ、左から右へ動くとする。

- **計算結果の比較：**

以上の例は微分方程式を解いて求めた式 (4.42) による D 関数の計算結果と一致している。次章で SU(2) の場合の計算を実行するが、その結果も式 (4.42) による D 関数と一致している。

第5章　SU(2) の表現

この章では SU(2) の表現について議論しよう。 SU(2) は３次元空間の回転群 O(3) と同形である。このため、ここで求めた D 関数は前章で求めた O(3) の D 関数と一致している。しかし SU(2) の場合、その求め方は非常に簡単である。以下に SU(2) の表現行列を求めて行こう。

5.1　SU(2) の基本表現

SU(2) が群を作っている事はすでに証明ずみである。ここではその表現を求めて行こう。ここで、SU(2) の群の要素を

$$u = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}, \quad u^\dagger = \begin{pmatrix} a^* & b \\ -b^* & a \end{pmatrix} \tag{5.1}$$

と書こう。この場合、special unitary の条件は $uu^\dagger = 1$ であり、これから

$$|a|^2 + |b|^2 = 1 \tag{5.2}$$

が求まる。また $\det\{\mathrm{u}\} = 1$ も同じ条件となっている。従って自由度は３である。

• 表現の基底

SU(2) の表現の基底として $\boxed{[\xi,\ \eta]}$ を選ぶ事にしよう。この時 $\hat{U}$ を SU(2) 群のオペレータとして

$$\begin{aligned} \hat{U}\xi &= a\xi + b^*\eta \\ \hat{U}\eta &= -b\xi + a^*\eta \end{aligned} \tag{5.3}$$

としよう。これから SU(2) 群の表現を求めて行こう。まずは群のオペレータを基底に演算しよう。この時

$$\begin{aligned} \hat{U}[\xi,\ \eta] &= [\hat{U}\xi,\ \hat{U}\eta] = [a\xi + b^*\eta, -b\xi + a^*\eta] \\ &= [\xi,\ \eta] \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} \end{aligned} \tag{5.4}$$

となり、この表現行列 $D(\hat{U})$ は

$$D(\hat{U}) = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} = u \tag{5.5}$$

となって、もとの SU(2) の行列と一致している。これは [**2**] 次元表現とも呼ばれているもので基本表現である。ここで $a,\ b$ を

$$\begin{aligned} a &= e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \\ b &= e^{-\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \end{aligned} \tag{5.6}$$

と選ぶと D 関数は

$$D^{(\frac{1}{2})}(\hat{U}) = \begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} & -e^{-\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \\ e^{\ \frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} & e^{\ \frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \end{pmatrix} \tag{5.7}$$

となる。これは前章で求めた D 関数である式 (4.46) と一致している。

5.2 $J = 1$ の表現

次に $J = 1$ の表現を求めてみよう。基本になる式は

$$\begin{aligned} \hat{U}\xi &= a\xi + b^*\eta \\ \hat{U}\eta &= -b\xi + a^*\eta \end{aligned} \tag{5.8}$$

である。ここで $J = 1$ の場合の基底 $(f_1,\ f_0,\ f_{-1})$ を以下のように選ぶ。

$$f_1 = \frac{1}{\sqrt{2}}\xi^2, \qquad f_0 = \xi\eta, \qquad f_{-1} = \frac{1}{\sqrt{2}}\eta^2 \tag{5.9}$$

ここで

$$\hat{U}[f_1,\ f_0,\ f_{-1}] = [\hat{U}f_1,\ \hat{U}f_0,\ \hat{U}f_{-1}] \tag{5.10}$$

を計算して行こう。

- **$\hat{U}f_1$ の計算**

  まず、$\hat{U}f_1$ を計算する。この場合、

$$\hat{U}f_1 = \frac{1}{\sqrt{2}}(\hat{U}\xi)(\hat{U}\xi) = \frac{1}{\sqrt{2}}(a\xi + b^*\eta)^2 = \frac{1}{\sqrt{2}}(a^2\xi^2 + (b^*)^2\eta^2 + 2ab^*\xi\eta \tag{5.11}$$

となる。よって

$$\hat{U}f_1 = a^2 f_1 + \sqrt{2}ab^* f_0 + (b^*)^2 f_{-1} \tag{5.12}$$

と求まる。

- **$\hat{U}f_0$ と $\hat{U}f_{-1}$ の計算**

  同様にして $\hat{U}f_0$ と $\hat{U}f_{-1}$ を計算すると

$$\hat{U}f_0 = -\sqrt{2}abf_1 + (|a|^2 - |b|^2)f_0 + \sqrt{2}a^*b^*f_{-1} \tag{5.13}$$

$$\hat{U}f_{-1} = b^2 f_1 - \sqrt{2}a^*bf_0 + (a^*)^2 f_{-1} \tag{5.14}$$

と求まる。これを表現行列の形に直すと

$$\hat{U}[f_1,\ f_0,\ f_{-1}] = [f_1,\ f_0,\ f_{-1}] \begin{pmatrix} a^2 & -\sqrt{2}ab & b^2 \\ \sqrt{2}ab^* & |a|^2 - |b|^2 & -\sqrt{2}a^*b \\ (b^*)^2 & \sqrt{2}a^*b^* & (a^*)^2 \end{pmatrix} \tag{5.15}$$

となる。これより右辺の行列が D 関数である。ここで

$$\begin{aligned} a &= e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \\ b &= e^{-\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \end{aligned} \tag{5.16}$$

と選ぶと D 関数は

$$D^{(1)}(\hat{U}) = \begin{pmatrix} e^{-i(\alpha+\gamma)} \cos^2\frac{\beta}{2} & -\frac{1}{\sqrt{2}} e^{-i\alpha} \sin\beta & e^{-i(\alpha-\gamma)} \sin^2\frac{\beta}{2} \\ \frac{1}{\sqrt{2}} e^{-i\gamma} \sin\beta & \cos\beta & -\frac{1}{\sqrt{2}} e^{i\gamma} \sin\beta \\ e^{i(\alpha-\gamma)} \sin^2\frac{\beta}{2} & \frac{1}{\sqrt{2}} e^{i\alpha} \sin\beta & e^{i(\alpha+\gamma)} \cos^2\frac{\beta}{2} \end{pmatrix} \tag{5.17}$$

となる。これは前章で求めた $J = 1$ の D 関数である式 (4.51) と一致している。

5.3 一般の場合の表現行列

ここで一般の次元についてその表現行列を求めよう。計算の仕方はこれまでと全く同じである。これは $N+1$ 次元の表現を求める事に対応している。まず基底として

$$f_n^{(N)} = \frac{\xi^{N-n}\eta^n}{\sqrt{(N-n)!\,n!}}, \qquad (n = 0,\ 1,\ \cdots, N) \tag{5.18}$$

と選ぼう。ここで $\boxed{N = 2J,\ n = J-K}$ と置くと

$$f_K^{(J)} = \frac{\xi^{J+K}\eta^{J-K}}{\sqrt{(J+K)!\,(J-K)!}} \tag{5.19}$$

となる。これまでと同様に基本になる式は

$$\begin{aligned} \hat{U}\xi &= a\xi + b^*\eta \\ \hat{U}\eta &= -b\xi + a^*\eta \end{aligned} \tag{5.20}$$

である。よって

$$\hat{U}f_K^{(J)} = \frac{(a\xi + b^*\eta)^{J+K}(-b\xi + a^*\eta)^{J-K}}{\sqrt{(J+K)!\,(J-K)!}} \tag{5.21}$$

となる。これを具体的に計算して行くと

$$\hat{U}f_K^{(J)} = \frac{1}{\sqrt{(J+K)!\,(J-K)!}} \sum_{s=0}^{J+K}\sum_{r=0}^{J-K} \frac{(J+K)!}{(J+K-s)!s!}(a\xi)^{J+K-s}(b^*\eta)^s \tag{5.22}$$

$$\times \frac{(J-K)!}{(J-K-r)!r!}(a^*\eta)^r(-b\xi)^{J-K-r} \tag{5.23}$$

となる。ここで $r+s = J-M$ により M を導入して和を s から M に変換する。この時、D 関数は

$$\hat{U}f_K^{(J)} = \sum_{M=-J}^{J} D_{MK}^{(J)}(U) f_M^{(J)} \tag{5.24}$$

と定義されているので、式 (5.27) と比較すれば求まる。その結果

$$D^{(J)}_{MK}(U) = \sum_{r=0}^{J-K} \frac{(a)^{M+K+r}(b^*)^{J-M-r}(-b)^{J-K-r}(a^*)^r}{(M+K+r)!(J-M-r)!(J-K-r)!r!} \tag{5.25}$$

$$\times\sqrt{(J+M)!(J-M)!(J+K)!(J-K)!} \tag{5.26}$$

となる。ここで

$$\begin{aligned} a &= e^{-\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} \\ b &= e^{-\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} \end{aligned} \tag{5.27}$$

と選ぶと D 関数は

$$D^{(J)}_{MK}(U) = \sum_{r=0}^{J-K} \frac{(-)^{J-K-r}\left(\cos\frac{\beta}{2}\right)^{M+K+2r}\left(\sin\frac{\beta}{2}\right)^{2J-M-K-2r}}{(M+K+r)!(J-M-r)!(J-K-r)!r!} \tag{5.28}$$

$$\times e^{-iM\alpha-iK\gamma}\sqrt{(J+M)!(J-M)!(J+K)!(J-K)!} \tag{5.29}$$

と求まった。これは回転群 O(3) で求めた D 関数と一致している。

第6章　SU(3) の群と表現

SU(3) の群とは３行３列の special unitary 行列の事である。この群は SU(2) の群と比べると格段に難しくなり、その表現を解析的に求める事は基本的にはできていない。しかし素粒子論における粒子の分類には非常に大きな役割をはたしてきた。さらに QCD はカラー自由度として SU(3) のゲージ理論となっている。その意味においても SU(3) の群は物理では良く使われてきたが、その取り扱いはそう簡単とは言えない。ここでは易しい部分だけ取り出して解説しよう。

6.1　SU(3) と素粒子

SU(3) の群が素粒子物理学に応用されたのは半世紀以上も昔の事である。今では $SU_f(3)$ (フレーバー物理) と呼ばれているもので素粒子の記述に応用されて大きな成功を収めたのである。これはアイソスピンを拡張して $u,\ d,\ s$ と言う３個のクォークを triplet と見なす事が基本となっている。ここで $s-$ クォークは strange quark を意味している。従って簡単な基底として

$$\psi_1 = u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_2 = d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_3 = s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.1}$$

を取ることになる。これは **3** 次元表現の基底に対応している。この **3** 次元表現と言う表記は対称群から来ている。SU(3) と S_3 が準同形であるため、SU(3) の表現を S_3 で表す事が良く行われている。

6.1.1 8 次元表現のバリオン

ここでこの **3** 表現による 3 個の直積から **8** 次元表現を作ることができる。**3** 表現を 3 個かけると 2 7 次元の表現が作られるが、この状態は可約である。これを既約状態の和に直すとそこから **8** 次元表現が現れるのである。これはクォーク 3 個からバリオンを構成する事に対応している。SU(2) の場合、2 個の粒子 (陽子と中性子) から構成されていて、この 2 個から作られる状態に singlet と triplet 状態があった。これは SU(2) なので $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$ と書かれる。SU(3) の場合、これがかなり複雑になる。基本表現を **3** 次元表現で表しているので 3 個の積は

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{6.2}$$

と言う既約表現の直和となる。この計算は Young diagram の手法に従えば簡単にできるので後の節でその方法を解説しよう。

- **8 個のバリオン**

もし **8** 次元表現 (8 重項状態) が自然界に対応しているとしたら 8 個のバリオンが見つかる事になっている。これは素粒子を記述する Hamiltonian の相互作用が $\mathrm{SU}_f(3)$ の回転に対して不変であるとバリオンの質量が縮退して観測されることになっているからである。そして実験的にも核子の仲間として 8 個のバリオンが見つかっている。それらは

$$P, \quad N, \quad \Lambda, \quad \Sigma^{\pm}, \quad \Sigma^0, \quad \Xi^{\pm} \tag{6.3}$$

の素粒子であり、これらの質量は 1 GeV 付近にある。これらの素粒子の量子数をみるとそれぞれの量子数では最低エネルギーの質量を持っていることが分かっている。従ってこれらの素粒子はすべて強い相互作用に関しては安定な粒子である。しかしながら、陽子以外はどれも弱い相互作用により崩壊するため不安定な粒子である。但し Σ^0 に関しては電磁的な相互作用で崩壊する事が知られている。これは Σ^0 と Λ でアイソスピン以外はその量子数が同じなので電磁気的な崩壊が可能となっているからである。

6.1.2 複素共役の表現 $\mathbf{3}^*$

SU(3) の場合、複素共役の表現 $\mathbf{3}^*$ は $\mathbf{3}$ 次元表現とは異なっている。一方、これは後程、議論することであるが、SU(2) の場合は複素共役の表現 $\mathbf{2}^*$ が $\mathbf{2}$ 次元表現と同値である。このため SU(2) においては $\mathbf{2}^*$ 表現を $\mathbf{2}$ 表現から区別する事はできない。しかしながら SU(3) では $\mathbf{3}^*$ 表現を $\mathbf{3}$ 表現から区別する必要があり、これは物理においては $\mathbf{3}^*$ 表現を反クォークに対応させることができるのでむしろ有効である。従って、クォーク・反クォークによりメソンを作る場合、$\mathbf{3}$ 表現と $\mathbf{3}^*$ 表現の直積を作ることにより

$$\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8} \tag{6.4}$$

として $\mathbf{8}$ 次元表現を作ることができるのである。

- **8 個のメソン**

これはメソンの状態に対応している。実際、メソンでは質量が低いところに 8 個の状態 (8 重項状態)

$$\pi^{\pm},\ \ \pi^0,\ \ \eta,\ \ K^+,\ \ K^0,\ \ K^-,\ \ \bar{K}^0 \tag{6.5}$$

が知られている。これらのメソンはすべて不安定な粒子である。この中で π^0 と η は電磁的な相互作用で $\pi^0 \to \gamma + \gamma$ などと崩壊するため寿命は非常に短い事が知られている。ちなみに、この π^0 崩壊の実験データは 3 角形図の理論計算により非常に正確に再現されている [5]。一方、それ以外のメソンは弱い相互作用で崩壊するため、例えば $\pi^{\pm}$ はその崩壊寿命が $\sim 10^{-8}$ 秒の大きさとなっている。

- **singlet のメソン**

メソンの場合、式 (6.4) には singlet の状態が現われている。この場合、メソンには Pauli 原理は働かないので反対称である必要はない。このため singlet の状態のメソンが存在しても対称性とは矛盾していない。実際、η' という中間子が観測されていてこれが singlet のメソンに対応していると考えられている。

6.2 SU(3) の Generator とその構造

Special unitary 行列 SU(N) においてその自由度は

$$2N^2 - N^2 - 1 = N^2 - 1 \tag{6.6}$$

となる。ここで最初の $2N^2$ は複素数の変数で $N \times N$ の行列要素である事に依っている。次の N^2 は unitary の条件 $UU^\dagger = 1$ から来ている。そして最後の 1 は special すなわち $\det(U) = 1$ の条件からである。

6.2.1 SU(2) の Generator

SU(2) の場合、3 個の自由度があり、これは generator としては Pauli 行列 τ_x, τ_y, τ_z が対応している。この場合、SU(2) の群の要素 U は

$$U = e^{i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\alpha}} \tag{6.7}$$

と書かれている。ここで $\boldsymbol{\alpha}$ は任意の定数である。ここで U が special unitary であることから

$$\det(U) = e^{\frac{i}{2}\mathrm{Tr}\{\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\}} = 1 \tag{6.8}$$

よって

$$\mathrm{Tr}\{\tau_x\} = \mathrm{Tr}\{\tau_y\} = \mathrm{Tr}\{\tau_z\} = 0 \tag{6.9}$$

が条件となっている。実際、Pauli 行列

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.10}$$

はこの条件を満たしている。

6.2.2 SU(3) の Generator

SU(3) の場合、8 個の自由度がある。この generator としては Gell-Mann 行列と呼ばれる $\boldsymbol{\lambda}$ を用いる。今の場合 $\boldsymbol{\lambda}$ は

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}
\end{aligned} \tag{6.11}$$

と書かれている。これは確かに

$$\mathrm{Tr}\{\lambda_i\} = 0 \quad (i = 1,\ 8) \tag{6.12}$$

となっている。

- **λ_i の交換関係と反交換関係**

行列 λ_i には次のような交換関係と反交換関係そして Tr $\{\lambda_i\,\lambda_j\}$ の式

$$[\lambda_i,\ \lambda_j] = 2if_{ijk}\lambda_k \tag{6.13}$$

$$\{\lambda_i,\ \lambda_j\} = \frac{4}{3}\delta_{ij} + 2id_{ijk}\lambda_k \tag{6.14}$$

$$\mathrm{Tr}\{\lambda_i\,\lambda_j\} = 2\delta_{ij} \tag{6.15}$$

が成り立っている。ここで $f_{ijk},\ d_{ijk}$ は structure constant と呼ばれる定数である。

6.3 Complex Conjugate (複素共役)

SU(2) の場合、複素共役の表現は元の基本表現と同じである。しかし SU(3) では基本表現と複素共役表現では全く異なったものになっている。従って例えば、クォークの状態関数を考える場合、どうしても SU(3) が必要となる。 複素共役の表現を反クォークの状態と関連付けるからである。

6.3.1 SU(2) の複素共役の表現

SU(2) の基本的な変換性は

$$\psi' = e^{\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\psi \simeq \left(1 + \frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau}\right)\psi \tag{6.16}$$

と書かれている。ここで $\delta\boldsymbol{\alpha}$ は無限小であるとしている。この式 (6.16) の複素共役をとると

$$\psi'^{*} = \left(1 - \frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau}^{*}\right)\psi^{*} \tag{6.17}$$

となる。ここで $\boldsymbol{\tau}^{*}$ は

$$\tau_1^{*} = \tau_1, \quad \tau_2^{*} = -\tau_2, \quad \tau_3^{*} = \tau_3 \tag{6.18}$$

となっている。従って

$$(i\tau_2)\boldsymbol{\tau}^{*}(-i\tau_2) = -\boldsymbol{\tau} \tag{6.19}$$

が示される。

- **複素共役の ψ_c 状態**

  ここで複素共役の状態 ψ_c を

$$\psi_c = i\tau_2\psi^{*} \tag{6.20}$$

で定義しよう。この時、SU(2) の基本変換に対して

$$\psi'_c = i\tau_2\psi^{*\prime} = i\tau_2\left(1-\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau^*}\right)\psi^* \tag{6.21}$$

$$= \psi_c - i\tau_2\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau}^*(i\tau_2)(-i\tau_2)\psi^* = \left(1+\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\tau}\right)\psi_c \tag{6.22}$$

となり ψ_c は SU(2) の基本変換そのものとなっている。従って複素共役の状態 ψ_c は元の状態と同じ変換となっている。まとめると

$$\left\{\begin{array}{lcl}\psi & : & \mathbf{2}\\ \psi_c & : & \mathbf{2}^*=\mathbf{2}\end{array}\right. \tag{6.23}$$

となっている。従って SU(2) では $\mathbf{2}^*$ を $\mathbf{2}$ と区別する事はできない。

6.3.2 SU(3) の複素共役の表現

SU(3) の場合は SU(2) と本質的に異なっている。SU(3) の基本的な変換性は

$$\psi' = e^{\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}}\psi \simeq \left(1+\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}\right)\psi \tag{6.24}$$

$$\psi'^* = e^{-\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}^*}\psi \simeq \left(1-\frac{i}{2}\delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}^*\right)\psi^* \tag{6.25}$$

である。しかしながらこの場合

$$S\boldsymbol{\lambda}^*S^{-1} = -\boldsymbol{\lambda}^* \tag{6.26}$$

を満たすオペレータ S が存在しないことが証明されている。従って、SU(3) においては複素共役の表現は基本表現とは全く別の表現である。すなわち

$$\mathbf{3}^* \neq \mathbf{3} \tag{6.27}$$

$$\tag{6.28}$$

である。

6.4 Young Diagram

SU(3) の群の直積は通常は可約である。例えば $\mathbf{3} \otimes \mathbf{3}^*$ は 2 個の既約表現の直和

$$\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8} \tag{6.29}$$

として書かれている。この規則は Young diagram の方法で簡単に計算できる事が知られている。ここではその規則について具体的な計算手法を解説しよう。この議論は SU(3) に限って行われている。

6.4.1 Young Box とその次元

1 個の Box は **3** に対応している。$\mathbf{3}^*$ は 2 個の Box が縦に並んでいるものである。素粒子論への応用では 1 個の Box をクォークに対応させ、$\mathbf{3}^*$ は反クォークに対応している。バリオンは **3** を 3 回かける事により新しい表現を作る。メソンは $\mathbf{3}^*$ と **3** を掛ける事により作られている。これらは共に **8** 表現として実現されている。

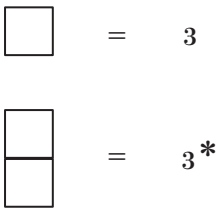

図 6.1: 基本のヤング図

ここでバリオンを作る場合の計算を書いて置こう。これは

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$$

となっていて **8** 表現が確かに表れている。注意するべき点はこの右辺に **1** 表現 (singlet) が現れている点である。この $\mathrm{SU}_f(3)$ では関係していないが、QCD の場合、重要な役割を果たしている。有限な color を持つ状態はゲージに依ってしまうので物理的に観測される状態は必ず、singlet になっている必要がある事が分かっている。これは完全反対称の状態に対応している。

6.4.2 対称群 $(\lambda,\ \mu)$ による表現

対称群においては $(\lambda,\ \mu)$ による表現が良く使われている。この $(\lambda,\ \mu)$ による表現の次元は

$$D_{(\lambda,\ \mu)} = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2) \tag{6.30}$$

となっている。以下に Young Diagram との比較を載せておこう。

- **Young Diagram と $(\lambda,\ \mu)$**

まず Young Diagram と $(\lambda,\ \mu)$ の関係をしめして置こう。右の図で N_1, N_2, N_3 は Box の数を意味している。この時

$$\lambda = N_1 - N_2$$
$$\mu = N_2 - N_3$$

である。例えば図 6.1 の Young Diagram は $(1,0)$ と $(0,1)$ となっている。このため、この次元は式 (6.30) から

$$(1,0) \ = \ \mathbf{3}$$
$$(0,1) \ = \ \mathbf{3}^*$$

となる事がわかる。

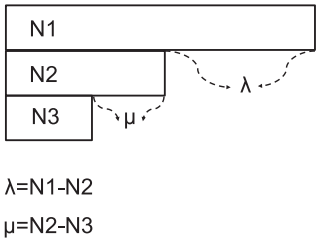

図 6.2: ヤング図と $(\lambda,\ \mu)$

● **Young Diagram と** $(\lambda,\ \mu)$ **と次元**

ここで 2 個の重要な Young Diagram と $(\lambda,\ \mu)$ の具体的な値を上げておこう。これより Young Diagram を見れはその対称性が直ちにわかることになり非常に便利でもある。Box が 3 個、縦に並んだ図が完全反対称である (0,0) に対応している。一方、その下の図は **8** 表現である。SU(2) の triplet ($\boldsymbol{\tau}$) に対して SU(3) では octet ($\boldsymbol{\lambda}$) である。

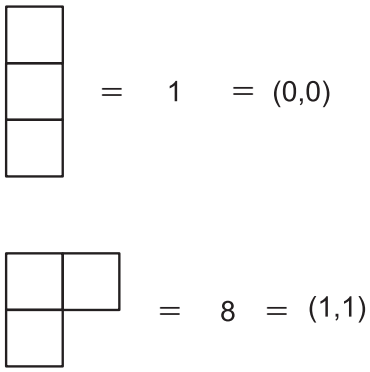

図 6.3: ヤング図と $(\lambda,\ \mu)$

6.4.3 Young Diagram の積

ここで Young Diagram の積についてその例題をあげて簡単に解説しよう。ここで a が付いた Box は左の Box の右横 (上) か下左にのみくっ付ける事ができると言う規則である。この規則を応用すると基本的な Young Diagram の積をすべて行う事ができる。これは Young Diagram の積は可約であるがそれを既約表現の和できちんと書かれていると言う事である。これは非常に有用である事が知られている。この場合、それぞれの図は

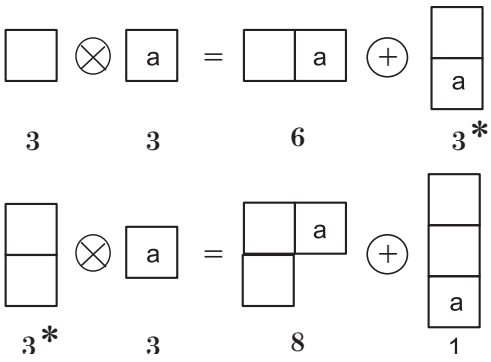

図 6.4: ヤング図の積

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{6}\oplus\mathbf{3}^*,\quad \mathbf{3}^*\otimes\mathbf{3}=\mathbf{8}\oplus\mathbf{1}$$
$$\mathbf{6}\otimes\mathbf{3}=\mathbf{8}\oplus\mathbf{10}$$

となっている。但し $\mathbf{6}\otimes\mathbf{3}$ の図は省略している。

• $\mathbf{8} \otimes \mathbf{8}$ の計算法

この $\mathbf{8} \otimes \mathbf{8}$ の積の計算法をしっかり理解するとほとんどの場合の積の計算ができるようになるものと思われる。ここで基本的な規則として、前節で解説した規則に加えて a は縦には並ぶ事は出来ない事がある。これは反対称性と関係している。次に、b は同じ列で a の左には来てはいけない規則が重要である。これらの規則の下で Young Diagram の積を行うと正しい既約表現の和が求める事ができる。

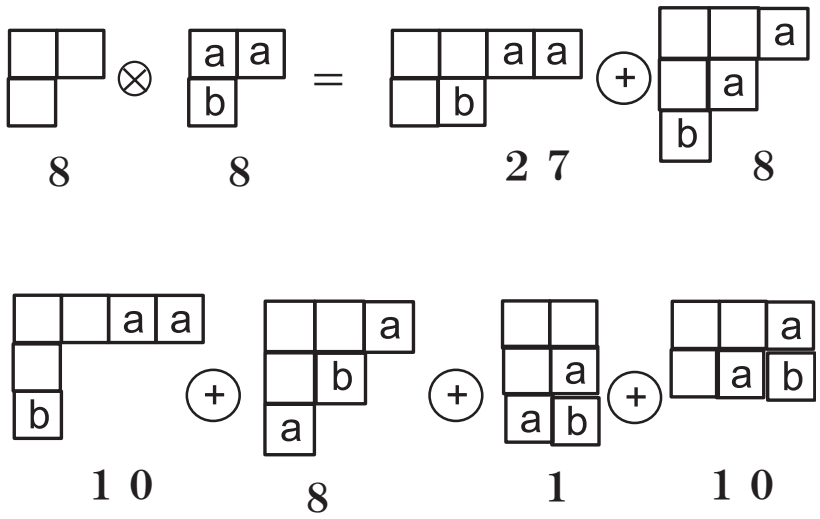

図 6.5: ヤング図の積 ($\mathbf{8} \times \mathbf{8}$)

ここでこのヤング図の積を式で書いておこう。これは

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{10}$$

となっている。

第7章　既約分解

この章での議論は回転群に限っての取り扱いとなっている。基本的には可約表現を既約テンソルの直和で書いた場合の結果をここに記している。具体的に、２個のランク１のテンソルの直積は可約であり、これを既約表現の直和と分解する方法について解説しよう。また、この章の後半では Landau-Yang の定理について解説しよう。この定理は群論と直接関係している問題である。群論における対称性をきちんと理解していると定理の証明が簡単であることがわかるのである。さらにこの定理は自然現象に直接関係していることでもあり、群論の応用としても面白い問題であると言えよう。

7.1　スピン$1\otimes 1$ は０と１と２の直和

２個のスピン１の状態の和は０と１と２の既約表現の直和になっている。この事を以下に見て行こう。

7.1.1　ベクトル

まず、通常のベクトルを考えよう。例えば空間座標 $\boldsymbol{r}$ や運動量 $\boldsymbol{p}$ は球面調和関数 $Y_{\ell m}$ の $\ell = 1$ である Y_{1m} に比例している。この事を証明するのは簡単であり、以下のように示すことができる。これは $\hat{r}_{\pm},\ \hat{r}_0$ が

$$\begin{aligned} \hat{r}_{\pm} &= \frac{1}{\sqrt{2}r}(x \pm iy) = \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi} = \mp\sqrt{\frac{4\pi}{3}}Y_{1,\pm 1}(\theta,\varphi) \\ \hat{r}_0 &= \frac{z}{r} = \cos\theta = \sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta,\varphi) \end{aligned} \tag{7.1}$$

であることから理解される。この事からベクトルはランク 1 のテンソルであることがわかる。これは通常のベクトルであればすべてこの規則に従っている。しかしながら例えばアイソスピン空間でのベクトルはこの座標系の空間に属していないのでその取扱いには注意が必要である。

7.1.2 既約分解

ここで 2 個の Y_{1m} の直積 $\{Y_1(1) \otimes Y_1(2)\}$ は

$$\{Y_1(1) \otimes Y_1(2)\} = (Y_1(1) \cdot Y_1(2)) \oplus (Y_1(1) \times Y_1(2)) \oplus [Y_1(1) \otimes Y_1(2)]^{(2)}$$

のように 3 個の既約表現の直和となっている。以下にこの式の簡単な説明をしておこう。ベクトルは 3 個の成分があるためベクトル $\boldsymbol{r}_1$ とベクトル $\boldsymbol{r}_2$ の掛け算としては 9 個の項がでてくる。この場合、それぞれの成分として、スカラーが 1 個、ベクトルが 3 個そして 2 階のテンソルが 5 個となっている。ここで、それぞれの項のランクを検証しよう。$\hat{r_{\pm}}$, $\hat{r}_0$ は

$$\begin{aligned} \hat{r_{\pm}} &= \frac{1}{\sqrt{2}r}(x \pm iy) = \frac{1}{\sqrt{2}} \sin\theta e^{i\varphi} = \mp\sqrt{\frac{4\pi}{3}} Y_{1,\pm 1}(\theta, \varphi) \\ \hat{r}_0 &= \frac{z}{r} = \cos\theta = \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \varphi) \end{aligned}$$

なので、Reduction の式は

$$\{\hat{\boldsymbol{r}}_1 \otimes \hat{\boldsymbol{r}}_2\} = (\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2) \oplus (\hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{r}}_2) \oplus [\boldsymbol{r}_1 \otimes \hat{\boldsymbol{r}}_2]^{[2]} \tag{7.2}$$

に対応している。

7.2 各項の性質

分解したそれぞれの項の性質を見て行こう。但し、第３項の２階のテンソルの項の議論はここでは省略しよう。

7.2.1 Scalar $S = \hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2$

最初の項であるスカラー $(\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2)$ を S としよう。これは

$$\begin{aligned} S &= (\hat{r_1})_+(\hat{r_2})_- + (\hat{r_1})_-(\hat{r_2})_+ + (\hat{r_1})_0(\hat{r_2})_0 \\ &= \sqrt{\frac{4\pi}{3}}\Big(-Y_{1,1}(\theta_1,\varphi_1)Y_{1,-1}(\theta_2,\varphi_2) - Y_{1,-1}(\theta_1,\varphi_1)Y_{1,1}(\theta_2,\varphi_2) \\ &\qquad +Y_{1,0}(\theta_1,\varphi_1)Y_{1,0}(\theta_2,\varphi_2)\Big) \end{aligned} \tag{7.3}$$

となる。ここで $\boldsymbol{J} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$ により全角運動量 $\boldsymbol{J}$ を定義しよう。この時、少しだけ面倒な計算を実行すると

$$\boldsymbol{J}^2 S = 0 \tag{7.4}$$

が証明される。これより S の状態は全角運動量 $\boldsymbol{J}^2$ の固有関数であり、その固有値は $J = 0$ である事がわかる。

7.2.2 Vector $\boldsymbol{V} = \hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{r}}_2$

第２項であるベクトル $(\hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{r}}_2)$ を $\boldsymbol{V}$ としよう。ここでは V_0 だけ考えよう。

$$\begin{aligned} V_0 &= i[(\hat{r_1})_+(\hat{r_2})_- - (\hat{r_1})_-(\hat{r_2})_+] \\ &= \sqrt{\frac{4\pi}{3}}\Big(-Y_{1,1}(\theta_1,\varphi_1)Y_{1,-1}(\theta_2,\varphi_2) + Y_{1,-1}(\theta_1,\varphi_1)Y_{1,1}(\theta_2,\varphi_2)\Big) \end{aligned} \tag{7.5}$$

となる。ここで、少し面倒な計算を行うと

$$\boldsymbol{J}^2 V_0 = 2V_0 = J(J+1)V_0 \tag{7.6}$$

が証明される。これより $\boldsymbol{V}$ 状態の全角運動量は $J = 1$ である事がわかる。

7.2.3 S, $\boldsymbol{V}$ の対称性

この式から 1 と 2 の入れ替えに対して S は対称、$\boldsymbol{V}$ は反対称であることがわかる。また、第 3 項目の 2 階のテンソルの項は対称であることが知られている。この対称性は球面調和関数 $Y_{\ell m}(\theta,\varphi)$ の性質と密接に関係している。実際、$Y_{\ell m}(\theta,\varphi)$ は ℓ が偶数の場合は対称であり、奇数の場合は反対称の性質と関連がある。今の場合、1 と 2 の入れ替えは空間反転と関係しているが、球面調和関数は空間反転に対して

$$Y_{\ell m}(\pi-\theta,\varphi+\pi)=(-)^{\ell}Y_{\ell m}(\theta,\varphi) \tag{7.7}$$

となっている。$\ell=0,\ 2$ は対称であり、これはランクの $0,2$ と確かに一致している。

7.3 式 (7.4), (7.6) の証明に必要な式のまとめ

$$\begin{aligned}
L_zY_{1m} &= mY_{1m} && (7.8)\\
L_+Y_{1,-1} &= \sqrt{2}Y_{1,0} && (7.9)\\
L_+Y_{1,0} &= \sqrt{2}Y_{1,1} && (7.10)\\
L_-Y_{1,1} &= \sqrt{2}Y_{1,0} && (7.11)\\
L_-Y_{1,0} &= \sqrt{2}Y_{1,-1} && (7.12)\\
\boldsymbol{L}^2 &= L_+L_- + L_z^2 - L_z && (7.13)\\
\boldsymbol{\ell}_1\cdot\boldsymbol{\ell}_2 &= \frac{1}{2}(\ell_1^+\ell_2^- + \ell_1^-\ell_2^+) + \ell_1^z\ell_2^z && (7.14)
\end{aligned}$$

7.4 Group Theory for Landau-Yang Theorem

Photon is a bose particle with a spin of 1. Thus two photon states must be symmetric under the exchange of two photon states, and this is just what we should keep in mind. Now two photon states can be reduced to the total spin states of 0, 1 and 2 where the total spin operator $\boldsymbol{J}$ is the sum of spin operator of photons $\boldsymbol{s}_1,\ \boldsymbol{s}_2$. Therefore, $\boldsymbol{J}$ is written as $\boldsymbol{J} = \boldsymbol{s}_1 + \boldsymbol{s}_2$. In what follows we prove that a massive particle with spin 1 (here we consider Z^0) cannot decay into the two photon states. This is the result of the kinematics, and we see that the two photons must be in a symmetric state due to the bose nature of photons while the total spin 1 state reduced from $1\otimes 1$ should be anti-symmetrice. This is just the essence of the Landau-Yang theorem.

7.4.1 Reduction of spin $1 \otimes 1$ states

Here we present an example of the reduction of spin $1 \otimes 1$ states in terms of $O(3)$ group theory where the spin operator $\boldsymbol{s}$ is replaced by the angular momentum operator $\boldsymbol{\ell}$, for simplicity. In this case, the equation of reduction of $1 \otimes 1$ states can be written as

$$\{\hat{\boldsymbol{r}}_1 \otimes \hat{\boldsymbol{r}}_2\} = (\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2) \oplus (\hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{r}}_2) \oplus [\boldsymbol{r}_1 \otimes \hat{\boldsymbol{r}}_2]^{[2]} \tag{7.15}$$

where $\hat{r_{\pm}},\ \hat{r}_0$ can be defined as

$$\begin{aligned} \hat{r_{\pm}} &= \frac{1}{\sqrt{2}r}(x \pm iy) = \frac{1}{\sqrt{2}} \sin\theta e^{i\varphi} = \mp\sqrt{\frac{4\pi}{3}} Y_{1,\pm 1}(\theta, \varphi) \\ \hat{r}_0 &= \frac{z}{r} = \cos\theta = \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \varphi). \end{aligned}$$

In this case, eq.(7.15) can be rewritten in terms of spherical harmonics $Y_{\ell m}$ as

$$\{Y_1(1) \otimes Y_1(2)\} = (Y_1(1) \cdot Y_1(2)) \oplus (Y_1(1) \times Y_1(2)) \oplus [Y_1(1) \otimes Y_1(2)]^{(2)}$$

• **Scalar**

Here, the first term $S = \hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2$ in the r.h.s of eq.(7.15) corresponds to a scalar state. This is obviously symmetric under the interchange of $1 \leftrightarrow 2$. Further we can prove

$$\boldsymbol{J}^2 S = 0 \tag{7.16}$$

and thus the S state has a spin zero.

• **Vector**

The second term $\boldsymbol{V} = \hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{r}}_2$ in the r.h.s of eq.(7.15) corresponds to a vector state. This is anti-symmetric under the interchange of $1 \leftrightarrow 2$. Now the spin 1 state should correspond to the vector state. Here we should prove that the vector state $\boldsymbol{V}$ has a spin 1 as the eigenvalue of total angular momentum $\boldsymbol{J}$. Namely one may prove

$$\boldsymbol{J}^2\, \boldsymbol{V} = 2\boldsymbol{V} \tag{7.17}$$

which means that the eigenvalue J is $J = 1$. In order to prove it explicitly, we may only consider V_0 term since this is, in fact, without loss of generality. Now V_0 can be written as

$$\begin{aligned} V_0 &= i[(\hat{r_1})_+(\hat{r_2})_- - (\hat{r_1})_-(\hat{r_2})_+] \\ &= \sqrt{\frac{4\pi}{3}}\,[-Y_{1,1}(\theta_1,\varphi_1)Y_{1,-1}(\theta_2,\varphi_2) + Y_{1,-1}(\theta_1,\varphi_1)Y_{1,1}(\theta_2,\varphi_2)] \end{aligned}$$

After some tedious but straightforward calculations, one can find

$$\boldsymbol{J}^2 V_0 = 2V_0 \tag{7.18}$$

which is just the same as eq.(7.17).

7.4.2 Rank k Tensor and Tensor Product

A tensor with rank k is denoted as $T^{(k,\kappa)}$, and the tensor product of two tensors $T_1^{(k_1)}$ and $T_2^{(k_2)}$ is reducible and can be written in terms of the sum of irreducible tensors as

$$T_1^{(k_1)} \otimes T_2^{(k_2)} = [T_1^{(k_1)} \otimes T_2^{(k_2)}]^{(\ell)} \oplus \cdots \oplus [T_1^{(k_1)} \otimes T_2^{(k_2)}]^{(m)} \tag{7.19}$$

where $\ell = |k_1 - k_2|$ and $m = k_1 + k_2$ are the conditions from the summation rule.

- **Rank One Tensor Product**

Here, we discuss the tensor product of rank one tensors and thus eq.(7.19) becomes

$$T_1^{(1)} \otimes T_2^{(1)} = [T_1^{(1)} \otimes T_2^{(1)}]^{(0)} \oplus [T_1^{(1)} \otimes T_2^{(1)}]^{(1)} \oplus [T_1^{(1)} \otimes T_2^{(1)}]^{(2)}. \tag{7.20}$$

Now the first term $[T_1^{(1)} \otimes T_2^{(1)}]^{(0)}$ should correspond to the scalar product and can be written as $(\boldsymbol{T}_1^{(1)} \cdot \boldsymbol{T}_2^{(1)})$. This is symmetric under the interchange of $1 \leftrightarrow 2$.

- **Vector Product**

The second term in eq.(7.20) should correspond to the vector product and can be written as $(\boldsymbol{T}_1^{(1)} \times \boldsymbol{T}_2^{(1)})$. This is anti-symmetric under the interchange of $1 \leftrightarrow 2$ since

$$(\boldsymbol{T}_1^{(1)} \times \boldsymbol{T}_2^{(1)}) = -(\boldsymbol{T}_2^{(1)} \times \boldsymbol{T}_1^{(1)}). \tag{7.21}$$

7.4.3 Landau-Yang Theorem

Photon has a spin 1 and thus it should correspond to the state of rank one tensor. Therefore, two photon states can make the states of rank 0, rank 1 and rank 2 from the group theoretical condition. However, there is a physical condition which is related to the bose nature of photon. Since photon is a bose particle, two photon states must be symmetric. Therefore, this physical requirement excludes the possibility of rank 1 tensor state (vector product) of the two photon state since it is anti-symmetric as we see above. Therefore, massive spin 1 states (such as Z^0 particle) cannot decay into two photon state. On the other hand, massive states (such as π^0 particle) can decay into two photons, which is indeed observed. This is just the Landau-Yang theorem.

第8章 Quantum Chromodynamics

Quantum chromodynamics is the theoretical frame work in which one can treat the physics of the strong interactions. This is the non-abelian gauge field theory, and it cannot be solved in the perturbation theory since the free Lagrangian densities of quarks and gluons are not gauge invariant. In the perturbation theory, we describe all the physical observables in terms of the properties of quarks and gluons, and if they are not related to physical observables, then there is no point of employing the perturbation theory.

8.1 Introduction

Physics of the strong interactions is described by quantum chromodynamics (QCD), and this is by now well established. Many experimental observations support that the number of the color must be three, and interactions between quarks should be mediated by gluons which are gauge bosons with colors. In addition to colors, quarks have six flavors of up, down, strange, charm, bottom and top.

However, it is extremely difficult to solve QCD in a non-perturbative fashion and obtain any reasonable spectrum of hadrons from QCD since quantum field theory has infinite degrees of freedom. At the present stage, one should make some kind of approximations in order to obtain physical observables. The perturbative treatment is the only possible method to calculate physical observables. However, there is a serious problem in the unperturbed QCD Hamiltonian since there are no free quark

and gluon states in physical space, and indeed the unperturbed Fock space is gauge dependent. In addition, we present an inherent problem connected to the gauge non-invariance of the unperturbed and interaction Lagrangian densities due to the non-conservation of the quark color current. Therefore there is a basic difficulty of carrying out the perturbative expansion. Here, we clarify what are the physical observables in QCD since some of known quantities are not gauge invariant and thus they cannot be observed.

8.2 Properties of QCD with $SU(N_c)$ Colors

In this section, we explain some fundamental properties of QCD which are important for the understanding of the difficulties in QCD.

8.2.1 Lagrangian Density of QCD

The Lagrangian density of QCD for quark fields ψ with $SU(N_c)$ colors is described as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m)\psi - \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} \tag{8.1}$$

where $G_{\mu\nu}$ is written as

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{8.2}$$

$$A_\mu = A^a_\mu T^a \equiv \sum_{a=1}^{N_c^2-1} A^a_\mu T^a. \tag{8.3}$$

Here A^a_μ denotes the gluon fields and T^a corresponds to the generator of $SU(N_c)$ group and satisfies the following commutation relations

$$[T^a, T^b] = iC^{abc}T^c \tag{8.4}$$

where C^{abc} denotes the structure constant of the group generators. For $SU(2)$ case, the structure constant C^{abc} becomes just the anti-symmetric symbol ϵ_{abc}. In eq.(8.1),

Tr { } means the trace of the group generators of $SU(N_c)$, and the generators T^a are normalized according to

$$\mathrm{Tr}\{T^a T^b\} = \frac{1}{2}\delta^{ab}. \tag{8.5}$$

Therefore, the last term of eq.(8.1) can be rewritten as

$$\frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} = \frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} \tag{8.6}$$

where $G^a_{\mu\nu}$ is described as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gC^{abc}A^b_\mu A^c_\nu. \tag{8.7}$$

m denotes the fermion mass, and at the massless limit, the Lagrangian density has a chiral symmetry.

8.2.2 Infinitesimal Local Gauge Transformation

QCD Lagrangian density is invariant under the following infinitesimal local gauge transformation

$$\psi' = (1 - ig\chi)\psi = (1 - igT^a\chi^a)\psi, \quad \text{with} \quad \chi = T^a\chi^a \tag{8.8}$$

$$\begin{aligned} A'_\mu &= A_\mu + ig[A_\mu, \chi] + \partial_\mu\chi \qquad \text{or} \\ A'^a_\mu &= A^a_\mu - gC^{abc}A^b_\mu\chi^c + \partial_\mu\chi^a \end{aligned} \tag{8.9}$$

where χ is infinitesimally small. By defining the covariant derivative D_μ by

$$D_\mu = \partial_\mu + igT^a A^a_\mu \tag{8.10}$$

one can see

$$D'_\mu\psi' = [\partial_\mu + igT^a(A^a_\mu - gC^{abc}A^b_\mu\chi^c + \partial_\mu\chi^a)](1 - igT^a\chi^a)\psi \tag{8.11}$$

$$= (1 - igT^a\chi^a)D_\mu\psi. \tag{8.12}$$

Therefore, one can prove that

$$\bar{\psi}' i\gamma^\mu {D'}_\mu \psi' = \bar{\psi} i\gamma^\mu D_\mu \psi \tag{8.13}$$

$${G'}_{\mu\nu} = (1 - igT^a\chi^a) G_{\mu\nu} (1 + igT^a\chi^a) \tag{8.14}$$

and one obtains

$$\mathrm{Tr}\{{G'}_{\mu\nu} G'^{\mu\nu}\} = \mathrm{Tr}\{(1 - igT^a\chi^a) G_{\mu\nu} G^{\mu\nu} (1 + igT^a\chi^a)\} = \mathrm{Tr}\{G_{\mu\nu} G^{\mu\nu}\}. \tag{8.15}$$

Therefore, one sees that the Lagrangian density of eq.(8.1) is invariant under the infinitesimal local gauge transformation.

8.2.3 Local Gauge Invariance

Now, the local gauge transformation with finite χ is defined as

$${A'}_\mu = U(\chi) A_\mu U^\dagger(\chi) - \frac{i}{g} U(\chi) \partial_\mu U^\dagger(\chi) \tag{8.16}$$

$$\psi' = U(\chi)\psi \tag{8.17}$$

where $U(\chi)$ is described in terms of χ as

$$U(\chi) = e^{-ig\chi}. \tag{8.18}$$

Here, one can easily prove the following equations

$$\bar{\psi}' i\gamma^\mu {D'}_\mu \psi' = \bar{\psi} i\gamma^\mu D_\mu \psi \tag{8.19}$$

$${G'}_{\mu\nu} = U(\chi) G_{\mu\nu} U(\chi)^{-1} \tag{8.20}$$

and by making use of the following identity

$$\sum_{a=1}^{n_a} {G'}^a_{\mu\nu} G'^{a,\mu\nu} = 2\mathrm{Tr}\{{G'}_{\mu\nu} G'^{\mu\nu}\} = 2\mathrm{Tr}\{G_{\mu\nu} G^{\mu\nu}\} = \sum_{a=1}^{n_a} G^a_{\mu\nu} G^{a,\mu\nu} \tag{8.21}$$

the gauge invariance of the Lagrangian density is easily seen.

8.3 Noether Current in QCD

The QCD Lagrangian density is invariant under the following infinitesimal global gauge transformation

$$\psi' = (1 - igT^a\theta^a)\psi \tag{8.22}$$
$$A'^a_\nu = A^a_\nu - gC^{abc}A^b_\nu\theta^c \tag{8.23}$$

where θ^a is an infinitesimally small constant. In this case, one finds

$$\delta\mathcal{L} = \mathcal{L}(\psi', \partial_\mu\psi', A'^a_\nu, \partial_\mu A'^a_\nu) - \mathcal{L}(\psi, \partial_\mu\psi, A^a_\nu, \partial_\mu A^a_\nu) = 0. \tag{8.24}$$

By making use of the equations of motion, one obtains

$$\delta\mathcal{L} = \Big[-ig(i\partial_\mu\bar{\psi}\gamma^\mu T^a\psi + i\bar{\psi}\gamma^\mu T^a\partial_\mu\psi) \tag{8.25}$$
$$-g(\partial_\mu G^{\mu\nu,c}C^{bca}A^b_\nu + G^{\mu\nu,c}C^{bca}\partial_\mu A^b_\nu)\Big]\,\theta^a = 0. \tag{8.26}$$

Therefore, one easily finds that

$$\partial_\mu\left(\bar{\psi}\gamma^\mu T^a\psi + C^{abc}G^{\mu\nu,b}A^c_\nu\right) = 0. \tag{8.27}$$

This means that the Noether current

$$I^{\mu,a} \equiv j^{\mu,a} + C^{abc}G^{\mu\nu,b}A^c_\nu \tag{8.28}$$

is indeed conserved. That is,

$$\partial_\mu I^{\mu,a} = 0 \tag{8.29}$$

where the quark color current j^a_μ is defined as

$$j^a_\mu = \bar{\psi}\gamma_\mu T^a\psi. \tag{8.30}$$

Thus, the quark color current alone cannot be conserved, and therefore there is no conservation of the quark color charge. This is consistent with the fact that the color current of quarks is not a gauge invariant quantity.

8.3.1 Conserved Charge of Color Octet State

From eqs.(8.29), one sees that the color octet vector current of one quark and one gluon state $I^{\mu,a}$ is conserved. Since $\partial_\mu I^{\mu,a}$ is a gauge invariant quantity, one can integrate it over all space

$$\int \partial_\mu I^{\mu,a} d^3r = \frac{d}{dt}\int I_0^a d^3r + \int \boldsymbol{\nabla}\cdot\boldsymbol{I}^a d^3r = \frac{dQ_I^a}{dt} + \int \boldsymbol{I}^a \cdot d\boldsymbol{S} = \frac{dQ_I^a}{dt} = 0 \quad (8.31)$$

where the color charge Q_I^a is defined as

$$Q_I^a = \int I^{0,a} d^3r. \quad (8.32)$$

Therefore, the color charge Q_I^a is indeed a conserved quantity, and there may be some chance that the color charge Q_I^a becomes a physical observable.

8.3.2 Gauge Non-invariance of Interaction Lagrangian Density

The interaction Lagrangian density of QCD that involves quark currents is written

$$\mathcal{L}_I = -g j_\mu^a A^{\mu,a}. \quad (8.33)$$

Now, the interaction Lagrangian density $\mathcal{L}_I$ is not gauge invariant, and therefore if one wishes to make any perturbation calculations involving the quark color currents, then one should check it in advance whether one can make the gauge invariant quark-quark interactions. The interaction Lagrangian density is transformed into a new shape under the infinitesimal local gauge transformation

$$\mathcal{L}_I = -g j_\mu^a (A^{\mu,a} + \partial^\mu \chi^a) \quad (8.34)$$

where the second term is a gauge dependent term. In the same way as QED case, one can rewrite the second term by making use of the conserved current as

$$-g j_\mu^a \partial^\mu \chi^a = -g\partial^\mu (j_\mu^a \chi^a) + g C^{abc} \chi^a \partial^\mu G_{\mu\nu}^b A^{\nu,c}. \quad (8.35)$$

The first term is a total derivative and thus does not contribute. However, there is no way to erase the second term which depends on χ^a. Therefore, one sees that one cannot make any simple-minded perturbative calculations of quark-quark interactions in QCD, contrary to the QED case where the electron-electron interaction is well defined and calculated. This means that there is a difficulty of defining any potential between quarks, and this is of course consistent with the picture that the color charge of quarks are gauge dependent and is not a conserved quantity.

8.4 Equation of Motion

The Lagrange equations of motion now become

$$\begin{aligned}&(i\gamma^\mu\partial_\mu - g\gamma^\mu A_\mu - m_0)\psi = 0\\ &\partial_\mu G^{\mu\nu,a} = gI^{\nu,a} = g\left(j^{\nu,a} + C^{abc}G^{\nu\rho,b}A^c_\rho\right).\end{aligned} \tag{8.36}$$

One can see that the equation of motion for the gauge fields has gauge field source terms in addition to the quark color current. Even though the equation of motion looks similar to that of QED, physics of QCD must be very different from the QED case. Now, one can introduce the color electric field $\boldsymbol{E}^a$ and the color magnetic field $\boldsymbol{B}^a$ by

$$\boldsymbol{E}^a = -\left(\frac{\partial \boldsymbol{A}}{\partial t}\right)^a - \boldsymbol{\nabla}A^a_0 - gC^{abc}\boldsymbol{A}^b A^c_0 \tag{8.37}$$

$$\boldsymbol{B}^a = \boldsymbol{\nabla}\times\boldsymbol{A}^a + \frac{1}{2}gC^{abc}\boldsymbol{A}^b\times\boldsymbol{A}^c. \tag{8.38}$$

It should be noted that the fields $\boldsymbol{E}^a$ and $\boldsymbol{B}^a$ themselves are not gauge invariant, contrary to the QED case. Now, eq.(8.36) can be rewritten in terms of $\boldsymbol{E}^a$ and $\boldsymbol{B}^a$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}^a = gj^0_a - gC^{abc}\boldsymbol{A}^b\cdot\boldsymbol{E}^c \tag{8.39}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B}^a - \frac{\partial \boldsymbol{E}^a}{\partial t} = g\boldsymbol{j}^a - gC^{abc}\left(A^b_0\boldsymbol{E}^c + \boldsymbol{A}^b\times\boldsymbol{B}^c\right). \tag{8.40}$$

From eq.(8.36), one sees that the current $I^{\nu,a}$ is a conserved quantity, $\partial_\mu I^{\mu,a} = 0$. In order to solve the dynamics of QCD, it should be inevitable to take into account the conservation of this current $I^{\nu,a}$. A question is, of course, as to in which way one should consider this effect of the current conservation in QCD dynamics, and this is still an open question.

付 録A　パリティ変換と奇関数積分

見かけ上、線形か Log 発散をもつ場合の奇関数積分について考えてみよう。この場合、被積分関数にパリティ変換を行うと一見、その発散が消えないように見える場合について議論しよう。しかし実際にはこの発散は出てこないことが証明される。

A.1　簡単な例題

今、簡単な例題として

$$I = \int_{-\Lambda}^{\Lambda} \left\{ \frac{x}{(x-a)^2+m^2} - \frac{x}{(x+a)^2+m^2} \right\} dx \tag{A.1}$$

を考えよう。これは括弧の中を先に計算すれば

$$\left\{ \frac{x}{(x-a)^2+m^2} - \frac{x}{(x+a)^2+m^2} \right\} = \frac{4x^2a}{\{(x-a)^2+m^2\}\{(x+a)^2+m^2\}} \tag{A.2}$$

となって明らかに有限である。Log 発散は存在しない。

• パリティ変換

一方、式 (A.1) の第 2 項をパリティ変換すると

$$I = \int_{-\Lambda}^{\Lambda} \frac{2x}{(x-a)^2+m^2} dx \tag{A.3}$$

となって、一見、 Log 発散のように見える。

ところがこの式は最初の積分範囲 $(-\Lambda$ から $\Lambda)$ を $(-\Lambda$ から 0 と 0 から $\Lambda)$ に書き換え、さらにパリティ変換で $-\Lambda$ から 0 を 0 から Λ に書き換えると

$$I = \int_0^{\Lambda} \left\{ \frac{2x}{(x-a)^2+m^2} - \frac{2x}{(x+a)^2+m^2} \right\} dx \tag{A.4}$$

となる。そしてこれを式 (A.2) のように書き直すとやはり有限である事がわかる。

- **奇関数積分**

これは奇関数積分の項が消えて、それ以外が残った例である。三角形図の場合は、奇関数積分で 1 次発散が消えた事も事実であるが、実際には γ 行列における計算で消えている。これが群論的な計算結果と一致している。従って三角形図では積分する前に 2 個の図を足し算して、γ 行列の計算をする事が絶対条件となっている。

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Contents

Chapter 1

Classical Field Theory of Fermions

The world of elementary particles is basically composed of fermions. Quarks, electrons and neutrinos are all fermions. On the other hand, elementary bosons are all gauge bosons, except Higgs particles though unknown at present. Therefore, if one wishes to understand field theory, then it should be the best to first study fermion field theory models.

In this chapter, we discuss the classical field theory in which "classical field" means that the field is not an operator but a c-number function. First, we treat the Schrödinger field and its equation in terms of the non-relativistic field theory model. In this case, the first quantization of $[x_i, p_j] = i\hbar\delta_{ij}$ is already done since we start from the Lagrangian density. In fact, the Lagrange equation leads to the Schrödinger equation or in other words, the Lagrangian density is constructed such that the Schrödinger equation can be derived from the Lagrange equation. The Dirac field is then discussed in terms of the Lagrangian density and the Lagrange equation. We also discuss the electromagnetic fields which interact with the Dirac field. The gauge invariance will be repeatedly discussed in this textbook, and the first introduction is given here. Finally, the field theory models with self-interacting fields are introduced and their Lagrangian density as well as Hamiltonian are described.

In this textbook, the basic parts of elementary physics can be found in Appendix, and in fact, Appendix is prepared such that it can be read in its own interests independently from the main part of the textbook.

Throughout this book, we employ the natural units

$$c = 1, \quad \hbar = 1.$$

This is, of course, due to its simplicity, and one can easily recover the right dimension of any physical quantities by making use of

$$\hbar c = 197 \ \ \mathrm{MeV} \cdot \mathrm{fm}.$$

1.1 Non-relativistic Fields

If one treats a classical field $\psi(\boldsymbol{r})$, it does not matter whether it is a relativistic field or non-relativistic one. The kinematics becomes important when one solves the equation of

motion which is relativistic or non-relativistic. If the kinematics is non-relativistic, then the equation of motion that governs the field $\psi(\boldsymbol{r})$ is the Schrödinger equation. Therefore, we should first study the Schrödinger field from the point of view of the classical field theory.

1.1.1 Schrödinger Equation

Electron in classical mechanics is treated as a point particle whose equation of motion is governed by the Newton equation. When electrons are trapped by atoms, then their motions should be described by quantum mechanics. As long as electrons move much slowly in comparison with the velocity of light c, the equation of their motion is governed by the Schrödinger equation. The Schrödinger equation for electron with its mass m in the external field $U(\boldsymbol{r})$ can be written as [102]

$$\left(i \frac{\partial}{\partial t} + \frac{1}{2m} \boldsymbol{\nabla}^2 - U(\boldsymbol{r}) \right) \psi(\boldsymbol{r}, t) = 0, \tag{1.1}$$

where $U(\boldsymbol{r})$ is taken to be a real potential. $\psi(\boldsymbol{r}, t)$ corresponds to the electron field in atoms, and $|\psi(\boldsymbol{r}, t)|^2$ can be interpreted as a probability density of finding the electron at $(\boldsymbol{r}, t)$.

Field $\psi(\boldsymbol{r}, t)$ is Complex

The Schrödinger field $\psi(\boldsymbol{r}, t)$ should be a complex function, and the complex field just corresponds to one particle state in the classical field theory. This is a well known fact, but below we will see what may happen when we assume *a priori* that the Schrödinger field $\psi(\boldsymbol{r}, t)$ should be a real function.

Real Field Condition is Unphysical

If one imposes the condition that the field $\psi(\boldsymbol{r}, t)$ should be real

$$\psi(\boldsymbol{r}, t) = \psi^\dagger(\boldsymbol{r}, t)$$

then, one sees immediately that the field $\psi(\boldsymbol{r}, t)$ becomes time-independent since eq.(1.1) and its complex conjugate equation give the following constraint for a real field $\psi(\boldsymbol{r}, t)$

$$\frac{\partial \psi(\boldsymbol{r}, t)}{\partial t} = 0.$$

Also, the field $\psi(\boldsymbol{r})$ should satisfy the following equation

$$\left(-\frac{1}{2m} \boldsymbol{\nabla}^2 + U(r) \right) \psi(\boldsymbol{r}) = 0.$$

Since the general solution of eq.(1.1) can be written as

$$\psi(\boldsymbol{r}, t) = e^{-iEt} \phi(\boldsymbol{r})$$

the field $\psi(\boldsymbol{r}, t)$ may become a real function only if the energy E of the system vanishes. That is, the energy eigenvalue of E is

$$E = 0.$$

Therefore, the real field cannot propagate and should be unphysical. This means that the real field condition of $\psi(\boldsymbol{r}, t)$ is physically too strong as a constraint.

1.1.2 Lagrangian Density for Schrödinger Fields

The Lagrangian density which can produce eq.(1.1) is easily found as

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m} \frac{\partial \psi^\dagger}{\partial x_k} \frac{\partial \psi}{\partial x_k} - \psi^\dagger U \psi, \tag{1.2}$$

where the repeated indices of k mean the summation of $k = 1, 2, 3$ and, in this text, this notation as well as the vector representation are employed depending on the situations. The repeated indices notation is mostly better for the calculation, but for memorizing the expressions or equations, the vector notation has some advantage.

The Lagrangian density of eq.(1.2) is constructed such that the Lagrange equation can reproduce the Schrödinger equation of eq.(1.1). It may also be important to note that the Lagrangian density of eq.(1.2) has a $U(1)$ symmetry, that is, it is invariant under the change of the field ψ as

$$\psi'(x) = e^{i\theta}\psi(x) \longrightarrow \mathcal{L}' = \mathcal{L},$$

where θ is a real constant. This invariance is clearly satisfied, and it is related to the conservation of vector current in terms of Noether's theorem which will be treated in the later chapters and in Appendix A.

Non-hermiticity of Lagrangian Density

At this point, we should discuss the non-hermiticity of the Lagrangian density. As one notices, the Lagrangian density of eq.(1.2) is not hermitian, and therefore some symmetry will be lost. One can build the Lagrangian density which is hermitian by replacing the first term by

$$i\psi^\dagger \frac{\partial \psi}{\partial t} \longrightarrow \left(\frac{i}{2}\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{i}{2} \frac{\partial \psi^\dagger}{\partial t} \psi \right).$$

However, it is a difficult question whether the Lagrangian density must be hermitian or not since it is not an observable. In addition, when one introduces the conjugate fields

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}}, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}$$

in accordance with the fields ψ and $\psi^\dagger$, then the symmetry between them is lost. However, the conjugate fields themselves are again not observables, and therefore there is no reason

that one should keep this symmetry. In any case, one can, of course, work with the symmetric and hermitian Lagrangian density, but physical observables are just the same as eq.(1.2). In this textbook, we employ eq.(1.2) since it is simpler.

1.1.3 Lagrange Equation for Schrödinger Fields

The Lagrange equation for field theory can be obtained by the variational principle of the action S

$$S = \int \mathcal{L}\, dt\, d^3r$$

and the Lagrange equation is derived in Appendix A. Since the field ψ is a complex field, ψ and $\psi^\dagger$ are treated as independent functional variables. The Lagrange equation for the field ψ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi}\,, \tag{1.3a}$$

where the four dimensional derivative

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is introduced for convenience. Now, the following equations can be easily evaluated

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= i\frac{\partial \psi^\dagger}{\partial t}\,, \\
\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} &= -\frac{1}{2m}\frac{\partial}{\partial x_k}\frac{\partial \psi^\dagger}{\partial x_k}\,, \\
\frac{\partial \mathcal{L}}{\partial \psi} &= -\psi^\dagger U
\end{aligned}$$

and therefore one obtains

$$\left(-i\frac{\partial}{\partial t} + \frac{1}{2m}\boldsymbol{\nabla}^2 - U(r)\right)\psi^\dagger(\boldsymbol{r}, t) = 0$$

which is just the Schrödinger equation for $\psi^\dagger$ in eq.(1.1).

It should be interesting to calculate the Lagrange equation for the field $\psi^\dagger$,

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi^\dagger}\,. \tag{1.3b}$$

In this case, one finds

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0,$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^\dagger}{\partial x_k})} = -\frac{1}{2m}\frac{\partial}{\partial x_k}\frac{\partial \psi}{\partial x_k},$$
$$\frac{\partial \mathcal{L}}{\partial \psi^\dagger} = i\frac{\partial \psi}{\partial t} - U\psi$$

and therefore one obtains

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\boldsymbol{\nabla}^2 - U(r)\right)\psi(\boldsymbol{r},t) = 0$$

which is just the same equation as eq.(1.1).

Here, we note that the Lagrangian density is not a physical observable and therefore it does not necessarily have to be determined uniquely. It is by now clear that the Lagrangian density eq.(1.2) reproduces a desired Schrödinger equation and thus can be taken as the right Lagrangian density for Schrödinger fields.

1.1.4 Hamiltonian Density for Schrödinger Fields

From the Lagrangian density, one can build the Hamiltonian density $\mathcal{H}$ which is the energy density of the field $\psi(\boldsymbol{r},t)$. The Hamiltonian density $\mathcal{H}$ is best constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\partial^\nu \psi^\dagger - \mathcal{L}g^{\mu\nu}$$

which will be derived in eq.(2.32) in Chapter 2. The energy momentum tensor $\mathcal{T}^{\mu\nu}$ satisfies the following equation of conservation law

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0$$

due to the invariance of the Lagrangian density under the translation. Therefore, the conserved charge associated with the $\mathcal{T}^{0\nu}$

$$\mathcal{Q}^\nu = \int \mathcal{T}^{0\nu}\, d^3r$$

should be a conserved quantity. Thus, it is natural that one defines the Hamiltonian in terms of the $\mathcal{Q}^0$.

Hamiltonian Density from Energy Momentum Tensor

The Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\dot{\psi}^\dagger - \mathcal{L}. \tag{1.4a}$$

Therefore, introducing the conjugate fields Π_ψ and $\Pi_{\psi^\dagger}$ by

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, \quad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi^\dagger}} = 0$$

one can write the Hamiltonian density as

$$\mathcal{H} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger} \dot{\psi^\dagger} - \mathcal{L} = \frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi. \tag{1.4b}$$

1.1.5 Hamiltonian for Schrödinger Fields

The Hamiltonian for the Schrödinger field is obtained by integrating the Hamiltonian density over all space

$$H \equiv \int \mathcal{H}\, d^3r = \int \left[\frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi \right] d^3r. \tag{1.4c}$$

By employing the Gauss theorem

$$\int_V \boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi)\, d^3r = \int_S (\psi^\dagger \boldsymbol{\nabla}_n \psi)\, dS_n$$

one can rewrite eq.(1.4c)

$$H = \int \left[-\frac{1}{2m} \psi^\dagger \boldsymbol{\nabla}^2 \psi + \psi^\dagger U \psi \right] d^3r, \tag{1.4d}$$

where the following identity is employed

$$\boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger \boldsymbol{\nabla}^2 \psi.$$

In addition, the surface integral term is neglected since it should vanish at the surface of sphere at infinity.

Now, it may be interesting to note that the Hamiltonian in eq.(1.4d) by itself does not give us much information on the dynamics. As long as we stay in the classical field theory, then the dynamics can be obtained from the equation of motion, that is, the Schrödinger equation. The static Schrödinger equation can be derived from the variational principle of the Hamiltonian with respect to ψ, and this treatment is given in Appendix A.

The Hamiltonian of eq.(1.4c) becomes important when the field ψ is quantized, that is, the field ψ is assumed to be written in terms of the annihilation operator $a_{\boldsymbol{k}}$ as discussed in Chapter 3. In this case, the Schrödinger field becomes an operator and therefore the Hamiltonian as well. This means that one has to prepare the Fock state on which the Hamiltonian can operate, and if one solves the eigenvalue equation for the Hamiltonian, then one can obtain the energy eigenvalue of the Hamiltonian corresponding to the Fock state.

However, the quantization of the Schrödinger field is not needed in the normal circumstances. The field quantization is necessary for the relativistic fields which contain negative energy solutions, and it becomes important when one wishes to treat the quantum fluctuation of the fields which corresponds to the creation and annihilation of particles.

1.1.6 Conservation of Vector Current

From the Schrödinger equation, one can derive the current conservation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0,$$

where ρ and $\boldsymbol{j}$ are defined as

$$\rho = \psi^\dagger \psi, \quad \boldsymbol{j} = \frac{i}{2m}\left[(\boldsymbol{\nabla}\psi^\dagger)\psi - \psi^\dagger \boldsymbol{\nabla}\psi\right].$$

This continuity equation of the vector current can also be derived as Noether's theorem from the Lagrangian density of eq.(1.2) which is invariant under the global gauge transformation

$$\psi' = e^{i\alpha}\psi.$$

As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\,\psi^\dagger\right], \ \text{ with } \ j^\mu = (\rho, \boldsymbol{j})$$

which just gives the above current density ρ and $\boldsymbol{j}$ when one employs the Lagrangian density of eq.(1.2).

It may be interesting to observe that the Lagrange equation, energy momentum tensor and the current conservation are all written in a relativistically covariant fashion when the properties of the Schrödinger field are derived. That is, apart from the shape of the Lagrangian density of the Schrödinger field, all the treatments are just the same as the relativistic description.

1.2 Dirac Fields

Electron in hydrogen atom moves much slowly compared with the velocity of light c. However, if one considers a hydrogen-like $^{209}_{83}\mathrm{Bi}$ atom where $Z = 83$, for example, then the motion of electron becomes relativistic since its velocity v can be given as

$$\frac{v}{c} \sim (Z\alpha)^2 \sim \left(\frac{83}{137}\right)^2 \sim 0.37$$

which is already comparable with c.

In this case, one should employ the relativistic kinematics, and therefore the Schrödinger equation should be replaced by the Dirac equation which is obtained by a natural extension of the relativistic kinematics. However, the Dirac equation contains new properties which are essentially different from the Schrödinger equation, apart from the kinematics. They have negative energy solutions and spin degrees of freedom. Both properties are very important in physics and will be repeatedly discussed in this textbook.

1.2.1 Dirac Equation for Free Fermion

The Dirac equation for free fermion with its mass m is written as [25, 26]

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta\right)\psi(\boldsymbol{r},t) = 0, \tag{1.5}$$

where ψ has four components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}.$$

$\boldsymbol{\alpha}$ and β denote the Dirac matrices and can be explicitly written in the Dirac representation as

$$\boldsymbol{\alpha} = \begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & \mathbf{0}\end{pmatrix}, \quad \beta = \begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix},$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrix.

The derivation of the Dirac equation and its application to hydrogen atom are given in Appendix D. One can learn from the procedure of deriving the Dirac equation that the number of components of the electron fields is important, and it is properly obtained in the Dirac equation. That is, among the four components of the field ψ, two degrees of freedom should correspond to the positive and negative energy solutions and another two degrees should correspond to the spin with $s = \frac{1}{2}$. It is also important to note that the factorization procedure indicates that the four component spinor is the minimum number of fields which can take into account the negative energy degree of freedom in a proper way.

Eq.(1.5) can be rewritten in terms of the wave function components by multiplying β from the left hand side

$$(i\partial_\mu\gamma^\mu - m)_{ij}\psi_j = 0 \ \text{ for } \ i = 1,2,3,4, \tag{1.6}$$

where the repeated indices of j indicate the summation of $j = 1,2,3,4$. Here, gamma matrices

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) \equiv (\beta, \beta\boldsymbol{\alpha})$$

are introduced, and the repeated indices of Greek letters μ indicate the summation of $\mu = 0,1,2,3$ as defined in Appendix A. The expression of eq.(1.6) is called $covariant$ since the Lorentz invariance of eq.(1.6) is manifest. It is indeed written in terms of the Lorentz scalars, but, of course there is no deep physical meaning in covariance.

1.2.2 Lagrangian Density for Free Dirac Fields

The Lagrangian density for free Dirac fermions can be constructed as

$$\mathcal{L} = \psi_i^\dagger\left[\gamma_0(i\partial_\mu\gamma^\mu - m)\right]_{ij}\psi_j = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi, \tag{1.7}$$

where $\bar{\psi}$ is defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0.$$

This Lagrangian density is just constructed so as to reproduce the Dirac equation of (1.6) from the Lagrange equation. It should be important to realize that the Lagrangian density of eq.(1.7) is invariant under the Lorentz transformation since it is a Lorentz scalar. This is clear since the Lagrangian density should not depend on the system one chooses.

Non-hermiticity of Lagrangian Density

This Lagrangian density is not hermitian, and it is easy to construct a hermitian Lagrangian density. However, as we discussed in the context of Schrödinger field, there is no strong reason that one should take the hermitian Lagrangian density since proper physical equations can be obtained from eq.(1.7).

1.2.3 Lagrange Equation for Free Dirac Fields

The Lagrange equation for $\psi_i^\dagger$ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \equiv \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi_i}^\dagger} + \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} \tag{1.8}$$

and one can easily calculate the following equations

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi_i}^\dagger} = 0,$$
$$\frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^\dagger}{\partial x_k})} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \psi_i^\dagger} = \left[\gamma_0(i\partial_\mu \gamma^\mu - m)\right]_{ij} \psi_j$$

and thus, this leads to the following equation

$$\left[\gamma_0(i\partial_\mu \gamma^\mu - m)\right]_{ij} \psi_j = 0$$

which is just eq.(1.6). Here, it should be noted that the ψ_i and $\psi_i^\dagger$ are independent functional variables, and the functional derivative with respect to ψ_i or $\psi_i^\dagger$ gives the same equation of motion.

1.2.4 Plane Wave Solutions of Free Dirac Equation

The free Dirac equation of eq.(1.5) can be solved exactly, and it has plane wave solutions. A simple way to solve eq.(1.5) can be shown as follows. First, one writes the wave function

ψ in the following shape

$$\psi_s(\boldsymbol{r},t) = \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \frac{1}{\sqrt{V}}\, e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.9}$$

where ζ_1 and ζ_2 are two component spinors

$$\zeta_1 = \begin{pmatrix}n_1\\ n_2\end{pmatrix}, \quad \zeta_2 = \begin{pmatrix}n_3\\ n_4\end{pmatrix}.$$

In this case, eq.(1.5) becomes

$$\begin{pmatrix}-m-E & \boldsymbol{\sigma}\cdot\boldsymbol{p}\\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m-E\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = 0 \tag{1.10}$$

which leads to

$$E^2 = m^2 + \boldsymbol{p}^2.$$

This equation has the following two solutions.

Positive Energy Solution ($E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}$)

In this case, the wave function becomes

$$\psi_s^{(+)}(\boldsymbol{r},t) = \frac{1}{\sqrt{V}}\, u_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.11a}$$

$$u_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{p}}+m}{2E_{\boldsymbol{p}}}}\begin{pmatrix}\chi_s\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_{\boldsymbol{p}}+m}\chi_s\end{pmatrix}, \quad \text{with} \quad s = \pm\frac{1}{2}, \tag{1.11b}$$

where χ_s denotes the spin wave function and is written as

$$\chi_{\frac{1}{2}} = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

Negative Energy Solution ($E_{\boldsymbol{p}} = -\sqrt{\boldsymbol{p}^2+m^2}$)

In this case, the wave function becomes

$$\psi_s^{(-)}(\boldsymbol{r},t) = \frac{1}{\sqrt{V}}\, v_{\boldsymbol{p}}^{(s)} e^{-iE_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{1.12a}$$

$$v_{\boldsymbol{p}}^{(s)} = \sqrt{\frac{|E_{\boldsymbol{p}}|+m}{2|E_{\boldsymbol{p}}|}}\begin{pmatrix}-\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E_{\boldsymbol{p}}|+m}\,\chi_s\\ \chi_s\end{pmatrix}. \tag{1.12b}$$

Some Properties of Spinor

The spinor wave function $u_{\boldsymbol{p}}^{(s)}$ and $v_{\boldsymbol{p}}^{(s)}$ are normalized according to

$$u_{\boldsymbol{p}}^{(s)\dagger} u_{\boldsymbol{p}}^{(s)} = 1,$$

$$v_{\boldsymbol{p}}^{(s)\dagger} v_{\boldsymbol{p}}^{(s)} = 1.$$

Further, they satisfy the following equations when the spin is summed over

$$\sum_{s=1}^{2} u_{\boldsymbol{p}}^{(s)} \bar{u}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}, \tag{1.13a}$$

$$\sum_{s=1}^{2} v_{\boldsymbol{p}}^{(s)} \bar{v}_{\boldsymbol{p}}^{(s)} = \frac{p_\mu \gamma^\mu + m}{2E_{\boldsymbol{p}}}. \tag{1.13b}$$

1.2.5 Quantization in Box with Periodic Boundary Conditions

In field theory, one often puts the theory into the box with its volume $V = L^3$ and requires that the wave function should satisfy the periodic boundary conditions (PBC). This is mainly because the free field solutions are taken as the basis states, and in this case, one can only calculate physical observables if one works in the box. It is clear that the free field can be defined well only if it is confined in the box.

Since the wave function $\psi_s(\boldsymbol{r}, t)$ for a free particle in the box should be proportional to

$$\psi_s(\boldsymbol{r}, t) \simeq \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\sqrt{V}}\, e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{r}}$$

the PBC equations become

$$e^{ip_x x} = e^{ip_x(x+L)}, \quad e^{ip_y y} = e^{ip_y(y+L)}, \quad e^{ip_z z} = e^{ip_z(z+L)}. \tag{1.14a}$$

Therefore, one obtains the constraints on the momentum p_k as

$$p_x = \frac{2\pi}{L}\, n_x, \quad p_y = \frac{2\pi}{L}\, n_y, \quad p_z = \frac{2\pi}{L}\, n_z, \;\; n_k = 0, \pm 1, \pm 2, \ldots . \tag{1.14b}$$

In this case, the number of states N in the large L limit becomes

$$N = \sum_{n_x, n_y, n_z} \sum_{s} = 2\, \frac{L^3}{(2\pi)^3} \int d^3 p, \tag{1.15}$$

where a factor of two comes from the spin degree of freedom.

1.2.6 Hamiltonian Density for Free Dirac Fermion

The Hamiltonian density for free fermion can be constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \sum_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \partial^\nu \psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i^\dagger)} \partial^\nu \psi_i^\dagger \right) - \mathcal{L} g^{\mu\nu}$$

which will be treated in eq.(A.12.3) of Appendix A.

Hamiltonian Density from Energy Momentum Tensor

Now, one defines the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L}. \tag{1.16}$$

Since the Lagrangian density of free fermion is given in eq.(1.7) and is rewritten as

$$\mathcal{L} = i\psi_i^\dagger \dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m\gamma_0 \right]_{ij} \psi_j$$

one can introduce the conjugate fields Π_{ψ_i} and $\Pi_{\psi_i^\dagger}$, and calculate them

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0. \tag{1.17}$$

In this case, the Hamiltonian density becomes

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) - \mathcal{L} = \bar{\psi}_i \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right]_{ij} \psi_j = \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi. \tag{1.18}$$

1.2.7 Hamiltonian for Free Dirac Fermion

The Hamiltonian for free fermion fields is obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right] \psi\, d^3r. \tag{1.19}$$

As we discussed in the Schrödinger field, the Hamiltonian itself cannot give us much information on the dynamics. One can learn some properties of the system described by the Hamiltonian, but one cannot obtain any dynamical information of the system from the Hamiltonian. In order to calculate the dynamics of the system in the classical field theory model, one has to solve the equation of motions which are obtained from the Lagrange equations for fields.

When one wishes to consider the fluctuations of the fields or, in other words, creations of particles and anti-particles, then one should quantize the fields. In this case, the Hamiltonian becomes an operator. Therefore, one has to prepare the Fock states on which the Hamiltonian can operate. Most of the difficulties of the field theory models should be to find the vacuum of the system.

1.2.8 Conservation of Vector Current

The Lagrangian density of the Dirac field has a global gauge invariance,

$$\psi' = e^{i\alpha}\psi \longrightarrow \mathcal{L}' = \mathcal{L}$$

and therefore there is a Noether current associated with the symmetry. As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\psi^\dagger\right]$$

and therefore the vector current j_μ becomes

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

Due to the global gauge invariance of the Lagrangian density, the vector current j_μ satisfies the continuity equation

$$\partial_\mu j^\mu = 0.$$

1.3 Electron and Electromagnetic Fields

The main part of the physical world is governed by the interaction between electrons and electromagnetic fields. Therefore, the Dirac equation, the Maxwell equation and their interactions are most important to understand the basic physics in many fundamental researches.

1.3.1 Lagrangian Density

When electron interacts with electromagnetic fields, the Lagrangian density becomes

$$\mathcal{L} = \bar{\psi}\big(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m\big)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{1.20}$$

where $F_{\mu\nu}$ denotes the field strength and is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

A^μ denotes the gauge field with

$$A^\mu = (A_0, \boldsymbol{A}),$$

where A_0 and $\boldsymbol{A}$ are the scalar and vector potentials, respectively. g denotes the gauge coupling constant, and in the classical electromagnetism, it corresponds to the electric charge e.

In the four dimensional field theory of QED, the coupling constant g is dimensionless, and therefore it is renormalizable in the perturbation calculation. In the two dimensional case, the coupling constant g has a mass dimension, and thus it is called *super-renormalizable*. In this case, there appear no infinities from the momentum integral in the perturbative calculations, and therefore one does not have to renormalize the coupling constant.

1.3.2 Gauge Invariance

The Lagrangian density of eq.(1.20) has an interesting feature. The free fermion Lagrangian density part

$$\bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi$$

is just the same as free Dirac Lagrangian density, and the last term in eq.(1.20)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

corresponds to the field energy term of the electromagnetic fields. The important point is that the shape of the interaction term

$$-g\bar{\psi}A_\mu\gamma^\mu\psi$$

can be determined by the requirement of the invariance under the local gauge transformation.

Local Gauge Transformation

We consider the following local gauge transformation

$$\psi' = e^{-ig\chi}\psi, \quad A'_\mu = A_\mu + \partial_\mu\chi, \tag{1.21}$$

where χ is an arbitrary real function of space and time, that is, $\chi(\boldsymbol{r}, t)$ which is therefore called *local*. It is easy to prove that the shape of the field energy term of the electromagnetic fields does not change under the local gauge transformation of eq.(1.21)

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\chi) - \partial_\nu(A_\mu + \partial_\mu\chi) = F_{\mu\nu}.$$

In addition, one can easily prove that the Lagrangian density of

$$\bar{\psi}(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi$$

does not change its shape under the local gauge transformation of eq.(1.21). That is,

$$\begin{aligned}
&\bar{\psi}'(i\partial_\mu\gamma^\mu - gA'_\mu\gamma^\mu - m)\psi' \\
&= \bar{\psi}e^{-ig\chi}e^{ig\chi}\big(i\partial_\mu\gamma^\mu + g\partial_\mu\chi\gamma^\mu - gA_\mu\gamma^\mu - g\partial_\mu\chi\gamma^\mu - m\big)\psi \\
&= \bar{\psi}\big(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m\big)\psi.
\end{aligned} \tag{1.22}$$

Therefore, a new Lagrangian density $\mathcal{L}'$ becomes equal to the original one $\mathcal{L}$

$$\mathcal{L}' = \bar{\psi}'\left(i\partial_\mu\gamma^\mu - gA'_\mu\gamma^\mu - m\right)\psi' - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} = \mathcal{L}.$$

The invariance of the Lagrangian density under the local gauge transformation determines the shape of the interaction between electron and electromagnetic fields. This is surprising, but it is, in a sense, the same as the classical mechanics as discussed in AppendixE. In this respect, it is interesting to realize that the gauge invariance that arises from the redundancy of the vector potential in solving the Maxwell equations plays an important role for determining the shape of the fundamental interactions.

1.3.3 Lagrange Equation for Dirac Field

The Dirac equation with the electromagnetic interaction can be easily obtained from the Lagrange equation for ψ

$$(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m)\psi = 0. \tag{1.23}$$

This is the Dirac equation for the hydrogen atom when the potential is static, that is

$$\boldsymbol{A} = 0$$

and

$$gA_0 = -\frac{Ze^2}{r}\,,$$

where we put $g = e$ with e the electric charge.

1.3.4 Lagrange Equation for Gauge Field

The Lagrange equation for the gauge field A_ν is written as

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\mathcal{L}}{\partial A_\nu}\,.$$

Since one can easily calculate

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial A_\nu} &= -g\bar{\psi}\gamma^\nu\psi,\\ \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{2}\,\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\times 2 = -\partial_\mu F^{\mu\nu}\end{aligned}$$

one obtains

$$\partial_\mu F^{\mu\nu} = g\bar{\psi}\gamma^\nu\psi = gj^\nu, \tag{1.24}$$

where the current density j^ν is defined as

$$j^\nu = \bar{\psi}\gamma^\nu\psi = (\bar{\psi}\gamma^0\psi, \bar{\psi}\boldsymbol{\gamma}\psi). \tag{1.25}$$

Eq.(1.24) is the Maxwell equation, and more explicitly, one can evaluate eq.(1.24)

$$[\nu = 0] \longrightarrow \frac{\partial F^{k0}}{\partial x_k} = \frac{\partial E_k}{\partial x_k} = \boldsymbol{\nabla}\cdot\boldsymbol{E} = gj_0, \tag{1.26a}$$

$$[\nu = k] \longrightarrow \frac{\partial F^{0k}}{\partial t} + \frac{\partial F^{ik}}{\partial x_i} = -\dot{E}_k + \epsilon_{kij}\frac{\partial B_j}{\partial x_i} = -\dot{E}_k + (\boldsymbol{\nabla}\times\boldsymbol{B})_k = gj_k \tag{1.26b}$$

which are just the Maxwell equations. It is of course easy to see that no magnetic monopole

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

and Faraday's law

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

are automatically satisfied in terms of the vector potential A_μ since

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \Longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0,$$
$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} A_0 \Longrightarrow \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} A_0 \right) = -\frac{\partial \boldsymbol{B}}{\partial t}\,.$$

1.3.5 Hamiltonian Density for Fermions with Electromagnetic Field

Now, one can construct the Hamiltonian density of fermion with electromagnetic field. The Hamiltonian density $\mathcal{H}$ can be defined by the energy momentum tensor $\mathcal{T}^{\mu\nu}$ [eq.(A.12.3)] as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \left(\frac{\partial \mathcal{L}}{\partial \dot{A}_k} \dot{A}_k \right) - \mathcal{L}$$

since $\mathcal{T}^{0\nu}$ is a conserved quantity. By introducing the conjugate fields Π_{ψ_i}, $\Pi_{\psi_i^\dagger}$ and Π_{A_k} as

$$\Pi_{\psi_i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}\,, \quad \Pi_{\psi_i^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i^\dagger}\,, \quad \Pi_{A_k} = \frac{\partial \mathcal{L}}{\partial \dot{A}_k}$$

one can rewrite the Hamiltonian density as

$$\mathcal{H} = \sum_i \left(\Pi_{\psi_i} \dot{\psi}_i + \Pi_{\psi_i^\dagger} \dot{\psi}_i^\dagger \right) + \sum_k \Pi_{A_k} \dot{A}_k - \mathcal{L}. \tag{1.27}$$

The conjugate fields Π_{ψ_i}, $\Pi_{\psi_i^\dagger}$ and Π_{A_k} can be calculated by employing the Lagrangian density of eq.(1.20)

$$\Pi_{\psi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger, \quad \Pi_{\psi_i^\dagger} = 0, \quad \Pi_{A_k} = \dot{A}_k + \frac{\partial A_0}{\partial x_k} = -E_k.$$

It should be noted that there is no corresponding conjugate field for A_0 in the Hamiltonian density, and thus there is no kinetic energy term present for A_0. Now, the Hamiltonian density can be calculated as

$$\mathcal{H} = \bar{\psi} \left[-i\gamma_k \frac{\partial}{\partial x_k} + m + gA_\mu \gamma^\mu \right] \psi$$
$$+ \frac{1}{2} \left[\dot{A_k}^2 - \left(\frac{\partial A_0}{\partial x_k} \right)^2 + \left(\frac{\partial A_k}{\partial x_j} \frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j} \frac{\partial A_j}{\partial x_k} \right) \right]. \tag{1.28a}$$

Eq.(1.28a) can be written in a familiar form

$$\mathcal{H} = \bar{\psi} \left(-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right) \psi - g\boldsymbol{j} \cdot \boldsymbol{A} + gj_0 A_0 + \frac{1}{2} \left[\dot{\boldsymbol{A}}^2 - (\boldsymbol{\nabla} A_0)^2 + \boldsymbol{B}^2 \right]. \tag{1.28b}$$

1.3.6 Hamiltonian for Fermions with Electromagnetic Field

The Hamiltonian can be obtained by integrating the Hamiltonian density over all space

$$H=\int\left[\bar{\psi}\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\right)\psi-g\boldsymbol{j}\cdot\boldsymbol{A}+gj_0A_0+\frac{1}{2}\left(\dot{\boldsymbol{A}}^2-(\boldsymbol{\nabla}A_0)^2+\boldsymbol{B}^2\right)\right]d^3r. \quad (1.28c)$$

Now, one makes use of the equation of motion

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=gj_0$$

in order to rewrite the A_0 in terms of the fermion current density j_0. Since there is a gauge freedom left and one should fix it to avoid the redundancy of the field variables, one may take a Coulomb gauge, for example

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0. \quad (1.29)$$

In this case, the equation of motion for the gauge field A_0 becomes

$$\boldsymbol{\nabla}^2A_0=-gj_0 \quad (1.30)$$

which is just a constraint. This is not an equation of motion any more since it does not depend on time. This constraint can be easily solved, and one obtains

$$A_0(r)=\frac{g}{4\pi}\int\frac{j_0(\boldsymbol{r}')\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}\,. \quad (1.31)$$

Now, one can make use of the following equation

$$\frac{1}{2}\int(\boldsymbol{\nabla}A_0)^2\,d^3r=-\frac{1}{2}\int(\boldsymbol{\nabla}^2A_0)A_0\,d^3r=\frac{g^2}{8\pi}\int\frac{j_0(\boldsymbol{r}')j_0(\boldsymbol{r})\;d^3r\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}\,, \quad (1.32)$$

where the surface integrals are set to zero. Also, $\boldsymbol{E}_T$ is introduced which denotes the transverse electric field

$$\boldsymbol{E}_T=-\dot{\boldsymbol{A}}$$

and it satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_T=0.$$

Therefore, the Hamiltonian of fermions with electromagnetic fields becomes

$$H=\int\left\{\bar{\psi}\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\right)\psi-g\boldsymbol{j}\cdot\boldsymbol{A}\right\}d^3r$$
$$+\frac{g^2}{8\pi}\int\frac{j_0(\boldsymbol{r}')j_0(\boldsymbol{r})\;d^3r\,d^3r'}{|\boldsymbol{r}'-\boldsymbol{r}|}+\frac{1}{2}\int\left(\boldsymbol{E}_T^2+\boldsymbol{B}^2\right)d^3r \quad (1.33)$$

which is a desired form.

1.4 Self-interacting Fermion Fields

Interactions between fermions are mediated by the gauge fields and this is the basic principle for the description of the fundamental field theory models. The reason why the gauge field theory is employed in modern physics is partly because the electromagnetic interaction is described by the gauge field theory but also because the gauge field theory is a renormalizable field theory. This is important since the renormalizable field theory has a predictive power in the perturbative calculations.

On the other hand, the field theory model with current-current interactions is not renormalizable in four dimensions since the coupling constant has the dimension of mass inverse square. Nevertheless, the model proposed by Nambu and Jona-Lasinio has been discussed frequently since it demonstrates, for the first time, the spontaneous symmetry breaking in the vacuum state in fermion field theory models. Therefore, we briefly discuss the Lagrangian density of the Nambu-Jona-Lasinio (NJL) model [93]. In addition, we treat the Thirring model which is the current current interaction model in two dimensions [109]. This model becomes important for the discussion of the spontaneous symmetry breaking which will be discussed in detail in Chapter 4.

1.4.1 Lagrangian and Hamiltonian Densities of NJL Model

The Lagrangian density of the NJL model is given as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi + \frac{1}{2}G\big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\big]. \tag{1.34}$$

In this case, the Hamiltonian density of the NJL model can be written as

$$\mathcal{H} = -i\psi^\dagger\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}\psi + m\bar{\psi}\psi - \frac{1}{2}G\big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\big]. \tag{1.35}$$

The coupling constant in this model has a dimension of inverse mass square,

$$G \sim m^{-2}. \tag{1.36}$$

Therefore, the NJL model is not renormalizable in the perturbative sense. Some of physical observables calculated in terms of the first order perturbation theory should have divergences of Λ^2. When the cut-off momentum Λ becomes very large, the physical quantity diverges very quickly, and there is no chance to renormalize this divergence into the coupling constant G.

The NJL model has been discussed often in the context of the spontaneous symmetry breaking physics [83, 84], and therefore we are bound to discuss it here since we will discuss the symmetry and its breaking in the later chapter of this book. Further, it should be fair to mention that, if one solves the field theory model exactly or non-perturbatively, then one may find that the theory has some predictive power. But this problem is too difficult to discuss further.

1.4.2 Lagrangian Density of Thirring Model

There is a popular field theory model in two dimensions with current current interactions. It is called Thirring model which has been extensively studied since it has an exact solution due to the Bethe ansatz technique. This will be treated in detail in the later chapter. Here, we should only introduce the model Lagrangian density. The Thirring model is described by the following Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m_0\bar{\psi}\psi - \frac{1}{2}\,gj^\mu j_\mu, \tag{1.37}$$

where the fermion current j_μ is given as

$$j_\mu = \bar{\psi}\gamma_\mu\psi. \tag{1.38}$$

The coupling constant g in two dimensional current current interaction model is a dimensionless constant. Therefore, it is renormalizable, and the model has a predictive power in the perturbation calculations.

1.4.3 Hamiltonian Density for Thirring Model

The Hamiltonian density of the Thirring model can be written as

$$\mathcal{H} = -i\bar{\psi}\gamma^1\partial_1\psi + m_0\bar{\psi}\psi + \frac{1}{2}\,gj^\mu j_\mu. \tag{1.39}$$

Here, the chiral representation for γ matrices in two dimensions is chosen

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 \equiv \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.40}$$

By introducing the state ψ as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{1.41}$$

the Hamiltonian density can be written

$$\mathcal{H} = -i\left(\psi_a^\dagger\frac{\partial}{\partial x}\,\psi_a - \psi_b^\dagger\frac{\partial}{\partial x}\,\psi_b\right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b. \tag{1.42}$$

Therefore, the Hamiltonian of the Thirring model can be written as

$$H = \int dx\left[-i\left(\psi_a^\dagger\frac{\partial}{\partial x}\,\psi_a - \psi_b^\dagger\frac{\partial}{\partial x}\,\psi_b\right) + m_0(\psi_a^\dagger\psi_b + \psi_b^\dagger\psi_a) + 2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b\right]. \tag{1.43}$$

In Chapter 7, we will discuss the diagonalization procedure of the Thirring model Hamiltonian in terms of the Bethe ansatz technique.

1.5 Quarks with Electromagnetic and Chromomagnetic Interactions

It should be worthwhile writing the total Lagrangian density which is composed of quarks interacting with electromagnetic fields as well as chromomagnetic fields. Normally, one considers either electromagnetic interactions or chromomagnetic interactions separately since they become important at the different physical stages. Here, we write them together since in reality there are always two different types of interactions (QED and QCD) for quarks present in nature. In addition, we include the interaction terms which violate the time reversal invariance as well as parity transformation just for academic interests.

1.5.1 Lagrangian Density

The Lagrangian density of quarks interacting with electromagnetic fields as well as chromomagnetic fields is given as

$$\mathcal{L} = \bar{\psi}_f \Big[i \left(\partial_\mu + i g_s A^a_\mu T^a + i e_f A_\mu \right) \gamma^\mu - m_0 \Big] \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$$

$$- \frac{i}{2} \tilde{d}_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 T^a \psi_f G^{\mu\nu,a} - \frac{i}{2} d_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 \psi_f F^{\mu\nu}, \tag{1.44}$$

where the summation of flavor runs $f =$ up, down, strange, charm, bottom and top quarks. T^a denotes the generator of the $SU(3)$ color group. The last two terms represent the T- and P-violating interactions. $\sigma_{\mu\nu}$ and γ_5 are defined as

$$\sigma_{\mu\nu} = \frac{i}{2} \left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \right), \quad \gamma_5 \equiv i \gamma_0 \gamma_1 \gamma_2 \gamma_3.$$

Field Strength of Electromagnetic Field

$F_{\mu\nu}$ denotes the electromagnetic field strength and is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.45}$$

where A_μ is the gauge field as given in Section 1.3.

Field Strength of Chromomagnetic Field

$G_{\mu\nu}$ denotes the chromomagnetic field strength and is given as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s C^{abc} A^b_\mu A^c_\nu, \tag{1.46}$$

where A^a_μ is the color gauge fields. C^{abc} denotes the structure constant in the $SU(3)$ group. The coupling constants g_s and e_f denote the gauge coupling constant of f-flavor quarks interacting with chromomagnetic field and electromagnetic field, respectively.

1.5.2 EDM Interactions

The last two terms in eq.(1.44) represent the interaction terms which violate the time reversal invariance as well as the space reflection at the same time. These terms are given just for references in order to understand the T-violating interactions in future in terms of EDM (Electric Dipole Moments). That is,

$$-\frac{i}{2}\,\tilde{d}_f\bar{\psi}_f\sigma_{\mu\nu}\gamma_5 T^a\psi_f G^{\mu\nu,a} : \quad \text{EDM for chromomagnetic fields,}$$
$$-\frac{i}{2}\,d_f\bar{\psi}_f\sigma_{\mu\nu}\gamma_5\psi_f F^{\mu\nu} : \quad \text{EDM for electromagnetic fields.}$$

The coupling strengths $\tilde{d}_f$ and d_f denote the strength of the time reversal and parity violating interactions of quark with the chromomagnetic fields and the electromagnetic fields, respectively. The $\tilde{d}_f$ and d_f have the dimension of the mass inverse, and, in fact, they are related to the electric dipole moment.

The existence of the EDM interactions should be determined from experiments. If there is any finite EDM interaction observed in future experiment, it should indicate an existence of a new scale which is different from the quark masses. In this respect, the observation of the EDM interaction must be physically very interesting and important indeed.

Chapter 2

Symmetry and Conservation Law

The Lagrangian density of fermions which is constructed in the previous chapter possesses various symmetries such as Lorentz invariance, time reversal symmetry and so on. These symmetries play a fundamental role for the determination of the vacuum state as well as the spectrum emerged from the model Hamiltonian. Therefore, one should be accustomed to these basic symmetries to understand the field theory.

In this chapter, we explain fundamental symmetry properties of Lorentz invariance, time reversal invariance, parity transformation, charge conjugation, translational invariance, global gauge symmetry, chiral symmetry and $SU(3)$ symmetry in field theory. In particular, the invariance of the Lagrangian density under these symmetries is discussed in detail since it is important to determine the vacuum structure.

If the Lagrangian density has a continuous symmetry, then there is a conserved current associated with this symmetry due to Noether's theorem. From the conservation of the current, one finds a conserved charge which plays an important role for the determination of physical states such as the vacuum state. All of the field theory models discussed here possess the translational invariance of the Lagrangian density, and this leads to the conserved quantity of the energy momentum tensor. From this energy momentum tensor, one can define the Hamiltonian density which is the energy density of the system. Clearly, the Hamiltonian is most important for the quantized field theory models since it can determine all of the physical states as the eigenstate of the Hamiltonian.

2.1 Introduction to Transformation Property

When one considers the transformation of the Lagrangian density or Hamiltonian density under the symmetry operator U, one first evaluates the transformation of the field ψ as

$$\psi' = U\psi.$$

Then, one calculates and sees how the Lagrangian density $\mathcal{L}$ should transform under the symmetry operator U

$$\mathcal{L}' \equiv \mathcal{L}(\psi', \partial_\mu \psi') = \mathcal{L}(U\psi, \partial_\mu (U\psi)).$$

If the Lagrangian density does not change its functional shape,

$$\mathcal{L}' = \mathcal{L}$$

then it is invariant under the transformation of U.

After the field quantization, the transformation procedure becomes somewhat different. The basic physical quantity in quantized field theory becomes the Hamiltonian $\hat{H}$, and the important point is that the Hamiltonian is now an operator and the problem becomes the eigenvalue equation for the field Hamiltonian

$$\hat{H}|\Psi\rangle = E|\Psi\rangle,$$

where $|\Psi\rangle$ is called *Fock state*. Since an operator $\mathcal{O}$ transforms under the symmetry operator U as

$$\mathcal{O}' = U\mathcal{O}U^{-1}$$

the Hamiltonian $\hat{H}$ transforms as

$$\hat{H}' = U\hat{H}U^{-1}.$$

In this case, the Fock state $|\Psi\rangle$ should transform as

$$|\Psi'\rangle = U|\Psi\rangle.$$

The transformation properties of the Lagrangian density should be kept for the quantized Hamiltonian after the field quantization. However, the vacuum state (the Fock state) may break the symmetry and indeed this can happen for the continuous global symmetry like the chiral symmetry. This physical phenomena are called *spontaneous symmetry breaking* which will be treated in Chapter 4.

2.2 Lorentz Invariance

The most important symmetry in physics must be the Lorentz invariance. The Lorentz invariance should hold in the theory of all the fundamental interactions. This is based on the observation that any physical observables should not depend on the systems one chooses if the systems $\mathcal{S}$ and $\mathcal{S}'$ are related to each other by the Lorentz transformation,

$$x'^{\mu} = \alpha^{\mu}_{\nu} x^{\nu}. \tag{2.1}$$

If the $\mathcal{S}'$ system is moving with its velocity of v along the x_1-axis, then the matrix α^{μ}_{ν} can be explicitly written as

$$\{\alpha^{\mu}_{\nu}\} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where

$$\cosh\omega = \frac{1}{\sqrt{1-v^2}}$$

is introduced. In this case, the Dirac wave function ψ should transform by the Lorentz transformation as

$$\psi'(x') = S\psi(x), \tag{2.3}$$

where S denotes a 4×4 matrix. Now, the Lagrangian density for free Dirac field is written both in $\mathcal{S}$ and $\mathcal{S}'$ systems

$$\mathcal{L} = \bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x) = \bar{\psi}'(x')(i\partial_\mu{}'\gamma^\mu - m)\psi'(x'). \tag{2.4}$$

From the equivalence between $\mathcal{S}$ and $\mathcal{S}'$ systems, one obtains

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}, \tag{2.5a}$$

$$S\gamma^\mu S^{-1}\alpha_\mu{}^\nu = \gamma^\nu. \tag{2.5b}$$

If one solves eq.(2.5b), then one can determine the shape of S explicitly when the $\mathcal{S}'$ system is moving along the x_1-axis

$$S = \exp\left(-\frac{i}{4}\,\omega\sigma_{\mu\nu}I_n^{\mu\nu}\right),$$

where $\sigma_{\mu\nu}$ and $I_n^{\mu\nu}$ are defined as

$$\sigma_{\mu\nu} = \frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right),$$
$$I_n^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.2.1 Lorentz Covariance

If physical quantities like Lagrangian density or equation of motions are written in a manifestly Lorentz invariant fashion, then they are called *Lorentz covariant*. The simplest case is that these equations are written as a Lorentz scalar. In this case, it is trivial to recognize that they are Lorentz invariant. For example, the continuity equation of the vector current reads

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

This is shown to be Lorentz invariant under the Lorentz transformation. However, it is not manifest, and therefore one defines the four vector current j^μ by

$$j^\mu = (\rho, \boldsymbol{j}).$$

In this case, one can rewrite the current conservation as

$$\partial_\mu j^\mu = 0$$

which is obviously Lorentz invariant since it is written in terms of the Lorentz scalar, and it is a Lorentz covariant expression.

However, one should not stress too much the importance of the Lorentz covariance. The Lorentz invariance is, of course, most important. However, as long as one starts from the Lorentz invariant theory, one does not have to worry about the violation of the Lorentz invariance since it can never be broken unless one makes mistakes in his calculations. In this respect, the Lorentz covariance may play an important role for avoiding careless mistakes if one carries out the perturbative calculation of the S-matrix in a covariant way.

2.3 Time Reversal Invariance

The world we live does not seem to be invariant under the time reversal transformation. Time flows always in the same direction. However, the physical law in the macroscopic world is quite different from the microscopic world, and time arrow defined by the entropy may not necessarily be related to the fundamental interactions.

Almost all of the fundamental interactions are invariant under the time reversal transformation. It is therefore important to understand the time reversal invariance (T-invariance) in field theory models.

2.3.1 T-invariance in Quantum Mechanics

Before going to field theory, we should first understand the definition of the T-invariance in quantum mechanics. When we make $t \to -t$, then the basic operators that appear in physics behave

$$t \to -t : \quad \begin{pmatrix} x_k \to x_k \\ p_k \to -p_k \\ \sigma_k \to -\sigma_k \\ E \to E \end{pmatrix}. \tag{2.6}$$

However, when the momentum p_k and the energy E are replaced by the differential operators as

$$\hat{p_k} = -i\,\frac{\partial}{\partial x_k}\,, \quad \hat{E} = i\,\frac{\partial}{\partial t} \tag{2.7}$$

then, the explicit t-dependences of the p_k and E become just opposite to eq.(2.6), and therefore one should recover them by hand. This can be realized when one makes complex conjugate of the operators $\hat{p_k}$ and $\hat{E}$

$$t \to -t \ : \ \ \hat{p_k} \to \hat{p_k}^* = i\frac{\partial}{\partial x_k} = -\hat{p_k}, \tag{2.8a}$$

$$t \to -t \ : \ \hat{E} \to \left(-i\frac{\partial}{\partial t}\right)^* = i\frac{\partial}{\partial t} = \hat{E} \tag{2.8b}$$

which can reproduce eq.(2.6). Therefore, the time reversal transformation in quantum mechanics means that the operator should be made complex conjugate as $A \to A^*$. This means that if the Hamiltonian contains an imaginary term, then this system violates the T-invariance. As one can see, the complex conjugate operation in accordance with the T-transformation should not be taken for the Pauli matrix σ_k as seen from eq.(2.6). For the Pauli matrix σ_k, one should make just the transformation of $\sigma_k \to -\sigma_k$ for $t \to -t$.

2.3.2 T-invariance in Field Theory

In field theory, momentum operators are all replaced by the differential operators, and therefore T-transformation of the field $\psi(x_k, t)$ means

$$t \to -t \ : \ \psi(x_k, t) \to \psi(x_k, -t)^* \ \text{ with } \ \sigma_k \to -\sigma_k. \tag{2.9}$$

As an example, one takes the plane wave solution of eq.(1.11) and makes the T-transformation. Then, one obtains

$$t \to -t \ : \ \psi(x_k, t) \to \left\{ \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \chi_s \\ \dfrac{\sigma_k p_k}{E_{\boldsymbol{p}} + m}\chi_s \end{pmatrix} \frac{1}{\sqrt{V}} e^{-iE_{\boldsymbol{p}}t + ip_k x_k} \right\}^*$$

$$= \psi(x_k, t) \tag{2.10}$$

which is indeed invariant under the T-transformation. The γ_μ matrices transform under the T-transformation

$$t \to -t \ : \ \begin{pmatrix} \gamma_0 \to \gamma_0 \\ \gamma_k \to -\gamma_k \end{pmatrix}. \tag{2.11}$$

Therefore, the free Dirac Lagrangian density of eq.(1.7) transforms

$$t \to -t \ : \ \mathcal{L} \to \left\{ \psi_i^\dagger \Big[\gamma_0(-i\partial_0\gamma^0 - i\partial_k\gamma^k - m) \Big]_{ij} \psi_j \right\}^* = \mathcal{L} \tag{2.12}$$

which is, of course, T-invariant as expected.

2.3.3 T-violating Interactions (Imaginary Mass Term)

At present, it is most important and fundamental to discover any interactions which violate the T-invariance. The simplest way to introduce the T-violating interaction must be an imaginary mass term,

$$\mathcal{H}_T = i\eta\bar{\psi}\psi, \tag{2.13}$$

where η denotes a real constant. In fact, the CP violating phase is originated from this type of interaction.

2.3.4 T and P-violating Interactions (EDM)

The direct examination of the T-violating interaction is based on the measurement of electric dipole moments (EDM) in isolated systems. The fundamental interaction of the T and P-violations can be written

$$\mathcal{H}_{TP} = \frac{i}{2} d_f \bar{\psi}_f \sigma_{\mu\nu} \gamma_5 \psi_f F^{\mu\nu}, \tag{2.14}$$

where d_f denotes the intrinsic EDM of the f-fermion. $F_{\mu\nu}$ and $\sigma_{\mu\nu}$ denote the electromagnetic field strength and the anti-symmetric tensor, respectively and they are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$\sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).$$

γ_5 is defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3.$$

The Hamiltonian of eq.(2.14) can be obtained by integrating the Hamiltonian density over all space, and in the nonrelativistic limit, the particle Hamiltonian becomes

$$H_{TP} \simeq -d_f \boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{2.15}$$

The measurements of the neutron EDM have been carried out extensively, and we will see in near future whether the neutron EDM is finite or not. It may be worth quoting the recent experimental measurement on the neutron EDM d_n [64]

$$d_n \simeq (1.9 \pm 5.4) \times 10^{-26}\ \mathrm{e} \cdot \mathrm{cm}.$$

2.4 Parity Transformation

The space reflection operation is called *parity* transformation $\hat{P}$, and it is defined as

$$\hat{P} x_k \hat{P}^{-1} = -x_k, \quad \hat{P} t \hat{P}^{-1} = t \tag{2.16a}$$

$$\hat{P} \gamma_k \hat{P}^{-1} = -\gamma_k, \quad \hat{P} \gamma_0 \hat{P}^{-1} = \gamma_0. \tag{2.16b}$$

In this case, ψ should also transform into ψ' as

$$\psi'(x_k, t) = \hat{P}\psi(x_k, t) = \gamma_0 \psi(x_k, t). \tag{2.16c}$$

The strong and electromagnetic interactions are invariant under the parity transformation. For example, the fermion Lagrangian density with the electromagnetic interaction of eq.(1.20)

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - g A_\mu \gamma^\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

can be seen under the parity transformation as follows.

$$\bar{\psi}\hat{P}^{-1}i\partial_0\gamma_0\hat{P}\psi = \bar{\psi}i\partial_0\gamma_0\psi, \quad \bar{\psi}\hat{P}^{-1}i\partial_k\gamma_k\hat{P}\psi = \bar{\psi}i\partial_k\gamma_k\psi,$$
$$\bar{\psi}\hat{P}^{-1}A_0\gamma_0\hat{P}\psi = \bar{\psi}A_0\gamma_0\psi, \quad \bar{\psi}\hat{P}^{-1}A_k\gamma_k\hat{P}\psi = \bar{\psi}A_k\gamma_k\psi,$$

where the following relations are employed

$$\hat{P}A_0\hat{P}^{-1} = A_0, \quad \hat{P}A_k\hat{P}^{-1} = -A_k. \tag{2.17}$$

Therefore, one sees that the fermion Lagrangian density with the electromagnetic interaction is invariant under the parity transformation.

Interaction with Parity Violation

For parity violating interactions, one takes for example

$$\mathcal{L}_I = g'\bar{\psi}\gamma_\mu\gamma_5 A^\mu\psi. \tag{2.18}$$

Under the parity transformation, one finds

$$\bar{\psi}\hat{P}^{-1}\gamma_k\gamma_5\hat{P}\psi = \bar{\psi}\gamma_k\gamma_5\psi$$

which shows that the Lagrangian density of $\mathcal{L}_I$ is odd under the parity transformation.

2.5 Charge Conjugation

The Lagrangian density for electrons interacting with the gauge field is invariant under the charge conjugation operation. The charge conjugate operation starts from the Maxwell equation which is invariant under the sign change of the vector potential.

2.5.1 Charge Conjugation in Maxwell Equation

The Maxwell equation is invariant under the sign change of the vector potential

$$\text{Charge Conjugation} \Longrightarrow A_c^\mu \equiv -A^\mu. \tag{2.19a}$$

This is clear since the Lagrangian density of the gauge field is written as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

which is obviously invariant under the operation of eq.(2.19a)

$$\mathcal{L}_c = -\frac{1}{4}(\partial_\mu A_{c\nu} - \partial_\nu A_{c\mu})(\partial^\mu {A_c}^\nu - \partial^\nu {A_c}^\mu) = \mathcal{L}.$$

When the gauge field interacts with the fermion current, then the Lagrangian density becomes

$$\mathcal{L} = -gj_\mu A^\mu - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu).$$

This Lagrangian density should be invariant under the charge conjugation operation and therefore gj_μ should change its sign

$$\text{Charge Conjugation} \Longrightarrow (gj_\mu)_c = -gj_\mu. \tag{2.19b}$$

This is a constraint on the Dirac field and it is indeed realized in the Dirac equation.

2.5.2 Charge Conjugation in Dirac Field

The invariance of the charge conjugation on the Dirac field starts from the Dirac equation with the electromagnetic interaction

$$i(\partial_\mu \gamma^\mu)_{ij}\psi_j - g(A_\mu \gamma^\mu)_{ij}\psi_j - m\psi_i = 0. \tag{2.20}$$

Now, one can make the complex conjugate of the above equation and multiply γ_0 from the left. This can be rewritten with the transposed representation of the gamma matrix γ_μ^T

$$-i(\partial^\mu \gamma_\mu^T)_{ij}\bar{\psi}_j - g(A^\mu \gamma_\mu^T)_{ij}\bar{\psi}_j - m\bar{\psi}_i = 0. \tag{2.21}$$

Now, the $\bar{\psi}$ is transformed as

$$\psi^c \equiv C\gamma^0\psi^* = C\bar{\psi}^T, \tag{2.22}$$

where C is a 4×4 matrix, and ψ^c denotes the state with charge conjugation and corresponds to an anti-particle state. Further, the operator C is assumed to satisfy the following equation

$$\gamma_\mu = -C(\gamma_\mu)^T C^{-1}. \tag{2.23}$$

In this case, one obtains

$$i(\partial_\mu \gamma^\mu)_{ij}(\psi^c)_j + g(A_\mu \gamma^\mu)_{ij}(\psi^c)_j - m(\psi^c)_i = 0. \tag{2.24}$$

This equation is just the same as eq.(2.20) if the sign of g is reversed, and indeed, the sign change of g is the requirement of the charge conjugation of eq.(2.19b). The operator C that satisfies eq.(2.23) is found to be

$$C = i\gamma_2\gamma_0 \tag{2.25}$$

which is the charge conjugation operator in the Dirac field.

2.5.3 Charge Conjugation in Quantum Chromodynamics

The Lagrangian density of QCD

$$\mathcal{L} = \bar{\psi}\Big[i(\partial^\mu + ig_s A^{\mu,a}T^a)\gamma_\mu - m_0\Big]\psi - \frac{1}{4}\,G^a_{\mu\nu}G^{\mu\nu,a}$$

with

$$G^{\mu\nu,a} = \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} - g_s C^{abc} A^{\mu,b} A^{\nu,c}$$

is invariant under the charge conjugation operation

$$A^{\mu,a}_c \equiv -A^{\mu,a}, \quad (g_s)_c \equiv -g_s.$$

This can be easily seen since

$$(G^{\mu\nu,a})_c = \partial^\mu A^{\nu,a}_c - \partial^\nu A^{\mu,a}_c - (g_s)_c C^{abc} A^{\mu,b}_c A^{\nu,c}_c = -G^{\mu\nu,a}.$$

In addition, the Dirac field part of the Lagrangian density is invariant in the same way as the QED case, and therefore one sees that the Lagrangian density of QCD is invariant under the charge conjugation operation

$$\mathcal{L}_c = \bar{\psi}_c\Big[i\big(\partial^\mu + i(g_s)_c A^{\mu,a}_c T^a\big)\gamma_\mu - m_0\Big]\psi_c - \frac{1}{4}\,(G^a_{\mu\nu})_c (G^{\mu\nu,a})_c = \mathcal{L}.$$

2.6 Translational Invariance

When one transforms the coordinate x_k into $x_k + a_k$ with a_k a constant, then the wave function $\psi(x_k)$ becomes

$$\psi(x_k) \longrightarrow \psi(x_k + a_k). \tag{2.26}$$

This translation operation $\hat{R}_{a_k}$ can be written for a very small a as

$$\hat{R}_{a_k}\psi(x_k) = \psi(x_k + a_k) = \left(1 + a_k \frac{\partial}{\partial x_k}\right)\psi(x_k). \tag{2.27}$$

For the finite a_k, one can write

$$\psi(x_k + a_k) = \lim_{n\to\infty}\left(1 + \frac{a_k}{n}\frac{\partial}{\partial x_k}\right)^n \psi(x_k) = e^{ip_k a_k}\psi(x_k). \tag{2.28}$$

Therefore, one finds the translation operation $\hat{R}_{\boldsymbol{a}}$ in three dimensions as

$$\hat{R}_{\boldsymbol{a}} = e^{ip_k a_k} = e^{i\boldsymbol{p}\cdot\boldsymbol{a}}. \tag{2.29}$$

2.6.1 Energy Momentum Tensor

If the Lagrangian density is invariant under the translation, then there is a conserved quantity associated with this symmetry, which is called *energy momentum tensor* as will be defined below. Under the infinitesimal translation of a^ν, the field ψ transforms as

$$\psi' = \psi + \delta\psi, \quad \delta\psi = (\partial_\nu\psi)a^\nu,$$
$$\partial_\mu\psi' = \partial_\mu\psi + \delta(\partial_\mu\psi), \quad \delta(\partial_\mu\psi) = (\partial_\mu\partial_\nu\psi)a^\nu + (\partial_\nu\psi)(\partial_\mu a^\nu).$$

Since the Lagrangian density is invariant under the infinitesimal translation of a^ν, one has

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi', \partial_\mu\psi') - \mathcal{L}(\psi, \partial_\mu\psi) = \frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta(\partial_\mu\psi)$$
$$= \frac{\partial\mathcal{L}}{\partial\psi}\,(\partial_\nu\psi)a^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,(\partial_\mu\partial_\nu\psi)a^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,(\partial_\nu\psi)(\partial_\mu a^\nu) = 0. \tag{2.30a}$$

By making use of the following equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\nu\psi a^\nu\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\nu\psi\right)a^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\nu\psi(\partial_\mu a^\nu)$$

and using the fact that the total divergence does not contribute to the action, one can obtain the following equation

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\psi}\,(\partial_\nu\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,(\partial_\mu\partial_\nu\psi) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\nu\psi\right)\right]a^\nu. \tag{2.30b}$$

In addition, the following identity can be employed

$$\partial_\nu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\,(\partial_\nu\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,(\partial_\mu\partial_\nu\psi)$$

and one finds

$$\delta\mathcal{L} = \partial_\mu\left[\mathcal{L}g^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial^\nu\psi\right]a_\nu = 0. \tag{2.31}$$

The same thing should hold as well for the field $\psi^\dagger$ which is an independent functional variable in the Lagrangian density, and therefore eq.(2.31) should be modified

$$\delta\mathcal{L} = \partial_\mu\left[\mathcal{L}g^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial^\nu\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\partial^\nu\psi^\dagger\right]a_\nu = 0. \tag{2.31'}$$

This means that, if one defines the energy momentum tensor $\mathcal{T}^{\mu\nu}$ as

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial^\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\partial^\nu\psi^\dagger - \mathcal{L}g^{\mu\nu} \tag{2.32}$$

then $\mathcal{T}^{\mu\nu}$ is a conserved quantity, that is

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0. \tag{2.33}$$

The reason why $\mathcal{T}^{\mu\nu}$ is called *energy momentum tensor* is because $\mathcal{T}^{0\nu}$ is related to the Hamiltonian and momentum densities.

2.6.2 Hamiltonian Density from Energy Momentum Tensor

Since the energy momentum tensor $\mathcal{T}^{0\nu}$ is a conserved quantity, one can define the Hamiltonian density $\mathcal{H}$ by $\mathcal{T}^{00}$

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\,\dot{\psi}^\dagger - \mathcal{L} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger}\,\dot{\psi}^\dagger - \mathcal{L}, \tag{2.34}$$

where the conjugate fields Π_ψ and $\Pi_{\psi^\dagger}$ are introduced as

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,, \quad \Pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\,.$$

The Hamiltonian H can be obtained by integrating the Hamiltonian density over all space

$$H = \int \mathcal{H}\, d^3r$$

and it corresponds to the total energy of the field ψ.

2.7 Global Gauge Symmetry

If one transforms the field ψ into ψ' as

$$\psi' = e^{i\alpha}\psi \quad (\alpha \text{ is a real constant}) \tag{2.35}$$

then it is called *global gauge transformation* in which α does not depend on the coordinate x. This is a simple phase transformation which is also found in quantum mechanics since physical observables do not depend on the value of α.

Now, we discuss the invariance of the global gauge symmetry in the Lagrangian density. As examples, we consider the Lagrangian density of QED and the Thirring model

$$\mathcal{L} = \bar{\psi}\big(i\partial_\mu\gamma^\mu - gA_\mu\gamma^\mu - m\big)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{2.36a}$$

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi - \frac{1}{2}\,g j^\mu j_\mu, \;\; \text{with} \;\; j_\mu = \bar{\psi}\gamma_\mu\psi. \tag{2.36b}$$

Obviously, the Lagrangian densities of eqs.(2.36) are invariant under the global gauge transformation

$$\delta\mathcal{L} \equiv \mathcal{L}(\psi', \partial_\mu\psi', \psi'^\dagger, \partial_\mu\psi'^\dagger) - \mathcal{L}(\psi, \partial_\mu\psi, \psi^\dagger, \partial_\mu\psi^\dagger) = 0.$$

In this case, the Noether current associated with the global gauge symmetry is conserved as discussed in Appendix A.11

$$\delta\mathcal{L} = i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\psi^\dagger\right] = 0 \tag{2.37a}$$

which leads to the conservation of the vector current

$$\partial_\mu j^\mu = 0 \ \text{ with } \ j^\mu = -i\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\psi^\dagger\right]. \tag{2.37b}$$

For the Lagrangian densities of eqs.(2.36), the vector current j_μ in eq.(2.37b) just becomes

$$j_\mu = \bar{\psi}\gamma_\mu\psi.$$

In any field theory models, the conservation of the vector current is known to hold at any level of quantization or regularization, and therefore the charge Q associated with the vector current is always conserved.

2.8 Chiral Symmetry

In eqs.(2.36), if the fermion is massless ($m = 0$), then there is another symmetry which is called *chiral symmetry*. If one transforms the field ψ into ψ' as

$$\psi' = e^{i\alpha\gamma_5}\psi \ \ (\alpha \ \text{ is a real constant}), \tag{2.38}$$

then one finds that the Lagrangian densities of the massless QED and the massless Thirring model are invariant under the chiral transformation. This is clear since the γ_5 anti-commutes with γ_μ

$$\{\gamma_5, \gamma_\mu\} = 0$$

and therefore one obtains for $\mu = 0, 1, 2, 3$

$$e^{-i\alpha\gamma_5}\gamma^\mu = \gamma^\mu e^{i\alpha\gamma_5}. \tag{2.39}$$

Thus, one sees that the $\bar{\psi}\gamma^\mu\psi$ transforms by the chiral symmetry as

$$\bar{\psi}'\gamma^\mu\psi' = \psi^\dagger e^{-i\alpha\gamma_5}\gamma_0\gamma^\mu e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^\mu\psi.$$

Since the Lagrangian density is invariant under the chiral transformation, the axial vector current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

is conserved, that is,

$$\partial_\mu j_5^\mu = 0.$$

2.8.1 Expression of Chiral Transformation in Two Dimensions

The chiral transformation of eq.(2.38) can be explicitly written in two dimensions for the field ψ.

Chiral Representation

In the chiral representation of the γ-matrix, the γ_5 and $e^{i\alpha\gamma_5}$ become

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^{i\alpha\gamma_5} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

Therefore, one has

$$\psi' = \begin{pmatrix} \psi'_a \\ \psi'_b \end{pmatrix} = e^{i\alpha\gamma_5} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\psi_a \\ e^{-i\alpha}\psi_b \end{pmatrix}. \tag{2.40}$$

Dirac Representation

In the Dirac representation of the γ-matrix, the γ_5 and $e^{i\alpha\gamma_5}$ become

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e^{i\alpha\gamma_5} = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}.$$

Therefore, one has

$$\psi' = \begin{pmatrix} \psi'_a \\ \psi'_b \end{pmatrix} = e^{i\alpha\gamma_5} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \psi_a\cos\alpha + i\psi_b\sin\alpha \\ \psi_b\cos\alpha + i\psi_a\sin\alpha \end{pmatrix}. \tag{2.41}$$

2.8.2 Mass Term

The mass term

$$m\bar{\psi}\psi$$

is not invariant under the chiral transformation

$$m\bar{\psi}'\psi' = m\bar{\psi}e^{2i\alpha\gamma_5}\psi \neq m\bar{\psi}\psi.$$

Therefore, if the system has a finite fermion mass, then the chiral symmetry is not preserved. In this respect, when one takes the massless limit

$$m \to 0$$

then the massless system may not necessarily be connected to the massive one if the chiral symmetry plays an important role for the determination of the vacuum. In fact, the massless limit is the singular point in the Thirring model, and the vacuum structures between the massive and massless Thirring models are completely different from each other. This is reasonable since the massless Thirring model has a vacuum which breaks the chiral symmetry while the massive Thirring model does not possesses the chiral symmetry and therefore its vacuum cannot be connected to the symmetry broken state. In addition, the massless Thirring model has no scaleful parameters, and thus physical observables should be measured in terms of the cutoff Λ, while, in the massive Thirring model, they are described by the mass m which cannot be set to zero after the system is solved.

Transformation of Mass Term in Two Dimensions

The mass term in two dimensions in the chiral representation transforms explicitly by the chiral transformation as

$$m\bar{\psi}'\psi' = m\left({\psi'}^{\dagger}_{a}{\psi'}_{b} + {\psi'}^{\dagger}_{b}{\psi'}_{a}\right) = m\left(e^{-2i\alpha}\psi^{\dagger}_{a}\psi_{b} + e^{2i\alpha}\psi^{\dagger}_{b}\psi_{a}\right) \neq m\bar{\psi}\psi$$

which shows again that the mass term is not invariant under the chiral transformation.

2.8.3 Chiral Anomaly

The conservation of the axial vector current is violated if there is a chiral anomaly. The chiral anomaly is closely related to the conflict between the local gauge invariance and the axial vector current conservation when the vacuum is regularized consistently with the local gauge invariance.

Four Dimensional QED

In four dimensional QED, the axial vector current is not conserved due to the anomaly and the conservation of the axial vector current is modified as

$$\partial_{\mu} j^{\mu}_{5} = \frac{g^2}{16\pi^2}\,\epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} F_{\mu\nu}, \tag{2.42}$$

where $\epsilon^{\rho\sigma\mu\nu}$ denotes the anti-symmetric symbol in four dimensions. $F_{\mu\nu}$ denotes the electromagnetic field strength as given in eq.(1.44).

Two Dimensional QED

The same anomaly equation is found in the two dimensional QED and is written

$$\partial_{\mu} j^{\mu}_{5} = \frac{g}{2\pi}\,\epsilon_{\mu\nu} F^{\mu\nu}, \tag{2.43}$$

where $\epsilon_{\mu\nu}$ denotes the anti-symmetric symbol in two dimensions. The explicit derivation of the anomaly equation eq.(2.43) will be treated in detail in the context of the two dimensional QED in Chapter 5.

Two Dimensional QCD

There is no chiral anomaly in the two dimensional QCD. This can be easily understood when one writes a possible anomaly equation in QCD_2

$$\partial_{\mu} j^{\mu}_{5} \Longleftrightarrow \frac{g}{2\pi}\,\epsilon_{\mu\nu} G^{\mu\nu,a}.$$

However, the right hand side has the color index while the left hand side is a color singlet object, and there is no way to construct a color singlet object in the right hand side.

Four Dimensional QCD

Contrary to the two dimensional QCD, there is an anomaly in four dimensional QCD. The axial vector current conservation is modified as

$$\partial_\mu j_5^\mu = \frac{g^2}{32\pi^2}\,\epsilon^{\rho\sigma\mu\nu} G^a_{\rho\sigma} G^a_{\mu\nu}, \tag{2.44}$$

where $G^a_{\rho\sigma}$ is the chromomagnetic field strength as given in eq.(1.46).

2.8.4 Chiral Symmetry Breaking in Massless Thirring Model

The massless Thirring model has no local gauge invariance, and therefore there is no anomaly. Thus, the axial vector current is always conserved. Therefore, the axial charge is also a conserved quantity.

In the quantum field theory of the massless Thirring model, one quantizes the fermion fields and therefore the Hamiltonian becomes an operator. Thus, the eigenvalue equation for the Hamiltonian should be solved, and the lowest state is the vacuum where all the negative energy states are occupied by the negative energy particles. The construction of the vacuum state is very difficult since one has to solve infinite many body problems in the negative energy particles. Apart from the exactness of the vacuum state, one can discuss some properties of the vacuum state. One example is the symmetry of the Lagrangian density, and the vacuum can break the symmetry possessed in the Lagrangian density. When the symmetry broken vacuum is realized because it is the lowest energy, then it is called *spontaneous symmetry breaking* phenomenon if the current associated with the symmetry is conserved. We will discuss physics of the symmetry breaking in the vacuum state in detail in chapters 4 and 7.

2.9 $SU(3)$ Symmetry

In quantum mechanics, if the particle Hamiltonian H is invariant under the unitary transformation of $SU(3)$ group,

$$UHU^{-1} = H$$

then the eigenvalues of the particle Hamiltonian H are specified by the eigenvalues of $SU(3)$ group. The same transformation can be applied to the field theory models. Suppose ψ should have 3 degenerate states, and one transforms ψ as

$$\psi' = U\psi.$$

If the Lagrangian density is invariant under the $SU(3)$ transformation, then the Hamiltonian constructed from this Lagrangian density is specified by the eigenvalues of the $SU(3)$ group. In particular, hadron masses predicted by the Hamiltonian should be specified by the

eigenvalues of the $SU(3)$ group. In the description of light baryons, one can assume that u, d and s quarks belong to the same multiplet. In this case, one can write ψ as

$$\psi(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix}. \tag{2.45}$$

By the unitary transformation of 3×3 matrix U, ψ transforms $\psi' = U\psi$ or explicitly

$$\begin{pmatrix} \psi'_u(x) \\ \psi'_d(x) \\ \psi'_s(x) \end{pmatrix} = U \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \\ \psi_s(x) \end{pmatrix}. \tag{2.46}$$

If the Hamiltonian $\hat{H}$ is invariant under the unitary transformation, then the hadron mass $\mathcal{M}$ can be described in terms of some function $G(a)$ as

$$\mathcal{M} = G([\lambda, \mu]), \tag{2.47}$$

where $[\lambda, \mu]$ denotes the quantum number of the symmetric group which specifies the representation of the $SU(3)$ group.

2.9.1 Dimension of Representation $[\lambda, \mu]$

The dimension $D_{[\lambda,\mu]}$ of the state represented by $[\lambda, \mu]$ becomes

$$D_{[\lambda,\mu]} = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2). \tag{2.48}$$

In fact, $[1, 0]$ or $[0, 1]$ are three dimensional representation which should just correspond to ψ or its anti-particle state. In this way, one sees that the $[1, 1]$ representation should have 8 states which are in fact found in nature as octet baryons

$$p, \;\; n, \;\; \Lambda, \;\; \Sigma^{\pm}, \;\; \Sigma^0, \;\; \Xi^{\pm}. \tag{2.49}$$

Indeed, their masses are found at around 1 GeV/c^2.

This success of the flavor $SU(3)$ is due to the fact that the interaction Hamiltonian is invariant under the $SU(3)$ transformation. Therefore, the flavor $SU(3)$ invariance of the Hamiltonian is broken by the mass term of the quarks. In particular, the mass of s-quark is assumed to be much larger than the masses of u- and d- quarks by one order of magnitude. However, the quark mass is still smaller than the hadron mass at least by an order of magnitude, and this is probably the main reason why the flavor $SU(3)$ works well. It is interesting to realize that, the fact that hadron masses are much larger than those of quarks indicates that the basic ingredients of generating hadron mass must come from the kinetic energy of quarks inside hadron which should give always positive energy contributions to the mass of hadron. Since the confinement of quarks must be due to the non-abelian character of the gauge fields, it should be most important to understand the properties of the non-abelian gauge field theory.

2.9.2 Useful Reduction Formula

Here, we summarize some examples of useful reduction formula of the $SU(3)$ product representations. First, we show the representation in terms of the dimension of the representation.

$$[1,0] = \mathbf{3}, \quad [0,1] = \mathbf{3}^*, \quad [0,0] = \mathbf{1}, \quad [1,1] = \mathbf{8}, \tag{2.50a}$$

$$[2,0] = \mathbf{6}, \quad [3,0] = \mathbf{10}, \quad [0,2] = \mathbf{6}^*, \quad [0,3] = \mathbf{10}^*. \tag{2.50b}$$

The following reduction formula may be useful

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus \mathbf{6}, \quad \mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}, \quad \mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10}. \tag{2.51}$$

For example, we can find the following results

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{3}^* \oplus \mathbf{6}) \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}, \tag{2.52}$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{27}. \tag{2.53}$$

Chapter 3

Quantization of Fields

In Chapter 1, we saw that the Lagrange equations for the fermion fields reproduce the Dirac equation which is the relativistic quantum mechanical equation of spin 1/2 fermions. In the Dirac equation, the field ψ would never disappear since the classical fields should always be present since they are c-number functions. The energy spectrum of the hydrogen atom can be described quite well by the Dirac equation.

However, if the hydrogen atom is in the excited state, then it naturally decays into the ground state at the final stage. During the process of transitions, the hydrogen atom emits photons. For example, when the hydrogen atom is in the $2p_{\frac{1}{2}}$-state, then it decays into the ground state of $1s_{\frac{1}{2}}$-state by emitting a photon. In this case, the photon is created during the transition. Therefore, one has to invent some scheme which takes into account the creation or annihilation of electromagnetic fields, and indeed the gauge fields should be quantized in terms of the commutation relations for the creation and annihilation operators.

The Dirac field should be always quantized because of the Pauli principle. The experimental observations in atoms show that one quantum state can be occupied only by one electron (Pauli principle). In order to accomodate the Pauli principle, one has to quantize the Dirac field in terms of anti-commutation relations for the creation and annihilation operators. The quantization of Dirac field with anti-commutation relations is also required from the presence of the negative energy states as the physical observables, and the negative energy states can be well fit into the theoretical framework in terms of the Pauli principle. The field quantization is also consistent with the observation that the pair of electron and positron can be created from virtual photons in the scattering process if some physical conditions are satisfied. In this sense, the field quantization should be made in terms of the creation and annihilation operators of fermion fields, and therefore, the field becomes an operator, and consequently the Hamiltonian becomes an operator.

$$H \Longrightarrow \hat{H} \quad (\text{operator after field quantization}).$$

In this case, one should solve an eigenvalue equation for the Hamiltonian with a corresponding eigenstate $|\Psi\rangle$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{3.1}$$

where E denotes the energy eigenvalue. The state $|\Psi\rangle$ is called *Fock state*, and in quantum field theory, the problem is now how one can solve the eigenvalue equation and determine the energy and eigenstate of the Hamiltonian. In general, it is extremely difficult to solve the eigenvalue equation in quantum field theory. The basic difficulty of the quantized field theory comes mainly from the fact that one has to construct the vacuum state $|\Omega\rangle$ which is composed of infinite numbers of negative energy particles interacting with each other. In addition, $|\Psi\rangle$ should be constructed on this vacuum state by creating particles and anti-particles, and it should satisfy eq.(3.1).

In this chapter, we first treat the quantization of free fermion fields. In most of the field theory models with interactions, the field quantization is done for free fields since one cannot directly quantize the interacting fields. Then, we discuss the quantization of the Thirring model since it can be solved exactly in terms of the Bethe ansatz method. Since the Thirring model gives a non-trivial field theory model, we can learn a lot from this field theory model. We also present the quantization of gauge fields so that we can calculate some scattering processes between electrons in the later chapter.

3.1 Quantization of Free Fermion Field

Classical fermion fields with the Dirac equation can describe the spectrum of the hydrogen atom quite well. However, the representation of the classical field has a limitation since experimental observations indicate that fermion and anti-fermion pairs can be created from the vacuum if the conditions of pair creations are satisfied. This means that the fermion fields cannot be taken as a c-number field. Instead, one should consider the fermion field as an operator. If the field becomes operator, then the value of the field should vary, depending on the state (Fock state) which should be prepared in accordance with the process one wishes to calculate in the perturbation theory.

3.1.1 Creation and Annihilation Operators

We start from the quantization of free Dirac fields. The Lagrangian density of free Dirac field is written as

$$\mathcal{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi = i\psi_i^\dagger\dot{\psi}_i + \psi_i^\dagger \left[i\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m\gamma_0\right]_{ij}\psi_j.$$

In this case, the Hamiltonian can be obtained as given in eq.(1.19)

$$H = \int \mathcal{H}\, d^3r = \int \bar{\psi}\left[-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right]\psi\, d^3r. \tag{1.19}$$

Now, we write the free Dirac field as

$$\psi(\boldsymbol{r},t) = \sum_{\boldsymbol{n},s}\frac{1}{\sqrt{L^3}}\left(a_{\boldsymbol{n}}^{(s)}u_{\boldsymbol{n}}^{(s)}e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}-iE_{\boldsymbol{n}}t} + b_{\boldsymbol{n}}^{(s)}v_{\boldsymbol{n}}^{(s)}e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}+iE_{\boldsymbol{n}}t}\right), \tag{3.2}$$

where $u_{\boldsymbol{n}}^{(s)}$ and $v_{\boldsymbol{n}}^{(s)}$ denote the spinor part of the plane wave solutions as given in Chapter 1, and can be written as

$$u_{\boldsymbol{n}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{n}}+m}{2E_{\boldsymbol{n}}}} \begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\boldsymbol{n}}}{E_{\boldsymbol{n}}+m}\chi_s \end{pmatrix}, \tag{3.3a}$$

$$v_{\boldsymbol{n}}^{(s)} = \sqrt{\frac{E_{\boldsymbol{n}}+m}{2E_{\boldsymbol{n}}}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\boldsymbol{n}}}{E_{\boldsymbol{n}}+m}\chi_s \\ \chi_s \end{pmatrix}, \tag{3.3b}$$

where

$$\boldsymbol{p}_{\boldsymbol{n}} = \frac{2\pi}{L}\boldsymbol{n}, \quad E_{\boldsymbol{n}} = \sqrt{\boldsymbol{p}_{\boldsymbol{n}}^2 + m^2}$$

and s denotes the spin index with $s = \pm\frac{1}{2}$. Inserting this field into eq.(1.19), one can express the Hamiltonian as

$$H = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} - b^{\dagger(s)}_{\boldsymbol{n}} b^{(s)}_{\boldsymbol{n}} \right) + \text{some constants.}$$

The Hamiltonian is a conserved quantity, and therefore we can quantize it. Here, the basic method to quantize the fields is to require that the annihilation and creation operators $a_{\boldsymbol{n}}^{(s)}$ and $a^{\dagger(s')}_{\boldsymbol{n}'}$ for positive energy states and $b_{\boldsymbol{n}}^{(s)}$ and $b^{\dagger(s')}_{\boldsymbol{n}'}$ for negative energy states become operators which satisfy the anti-commutation relations.

Anti-commutation Relations

The creation and annihilation operators for positive and negative energy states should satisfy the following anti-commutation relations,

$$\left\{a_{\boldsymbol{n}}^{(s)}, a^{\dagger(s')}_{\boldsymbol{n}'}\right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}, \quad \left\{b_{\boldsymbol{n}}^{(s)}, b^{\dagger(s')}_{\boldsymbol{n}'}\right\} = \delta_{s,s'}\delta_{\boldsymbol{n},\boldsymbol{n}'}. \tag{3.4}$$

All the other cases of the anti-commutations vanish, for examples,

$$\left\{a_{\boldsymbol{n}}^{(s)}, a_{\boldsymbol{n}'}^{(s')}\right\} = 0, \quad \left\{b_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')}\right\} = 0, \quad \left\{a_{\boldsymbol{n}}^{(s)}, b_{\boldsymbol{n}'}^{(s')}\right\} = 0. \tag{3.5}$$

This corresponds to the field quantization, and the quantization in terms of creation and annihilation operators should be the fundamental quantization procedure.

3.1.2 Equal Time Quantization of Field

The quantization of fields can also be written in terms of equal time anti-commutation relations for fields as,

$$\left\{\psi_i(\boldsymbol{r},t), \pi_j(\boldsymbol{r'},t)\right\} = i\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r'}), \tag{3.6}$$

where the conjugate field π_i is given by the Lagrangian density as

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} = i\psi_i^\dagger.$$

Therefore, the quantization condition of eq.(3.6) becomes

$$\left\{\psi_i(\boldsymbol{r}, t), \psi_j^\dagger(\boldsymbol{r'}, t)\right\} = \delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r'}). \tag{3.7}$$

It is important to note that the field quantization must be done always with equal time. Further, one may consider the quantization conditions of fields in terms of eqs.(3.4) and (3.5) as being more fundamental than eq.(3.6), and eq.(3.6) should be taken as the field quantization method which can be derived from eqs.(3.4) and (3.5) together with eq.(3.2).

3.1.3 Quantized Hamiltonian of Free Dirac Field

Now one finds the quantized Hamiltonian which is given as

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} - b^{\dagger(s)}_{\boldsymbol{n}} b^{(s)}_{\boldsymbol{n}} \right) + C_0, \tag{3.8}$$

where C_0 is a constant and normally it is discarded since it does not affect on physical observables. This Hamiltonian $\hat{H}$ is written in terms of the creation and annihilation operators.

It may be worthwhile noting that the Hamiltonian is obtained by integrating the Hamiltonian density over all space, and therefore it does not depend on space and also it is a conserved quantity. Instead it is not a c-number but the operator. Therefore, one should find Fock states which must be the eigenstates of the Hamiltonian. For free Dirac fields, one can easily find the eigenstates of the Hamiltonian, and indeed the vacuum state is constructed by filling out all the negative energy states by the negative energy particles as will be treated below.

Anti-particle Representation

The representation of $b^{(s)}_{\boldsymbol{n}}$ corresponds to the negative energy state. The anti-particle representation can be obtained by defining a new operator $\mathrm{b}^{(s)}_{\boldsymbol{n}}$ as

$$\mathrm{b}^{(s)}_{\boldsymbol{n}} = b^{\dagger(s)}_{-\boldsymbol{n}}.$$

In this case, the operator $\mathrm{b}^{(s)}_{\boldsymbol{n}}$ describes the annihilation of an anti-particle. In this representation, the Hamiltonian of free Dirac field becomes

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} + \mathrm{b}^{\dagger(s)}_{\boldsymbol{n}} \mathrm{b}^{(s)}_{\boldsymbol{n}} \right) + C_0{}'. \tag{3.9}$$

Perturbative Vacuum

This expression is employed in most of the field theory textbooks, and it is suitable for describing the processes of fermion creation and annihilation. However, one cannot treat the interacting vacuum state since it is assumed that the vacuum is a simple one which satisfies

$$a_{\boldsymbol{n}}^{(s)}|0\rangle = 0, \quad \mathrm{b}_{\boldsymbol{n}}^{(s)}|0\rangle = 0$$

which defines the vacuum state $|0\rangle$ in the perturbative sense. The construction of the vacuum state in interacting systems is quite difficult and mostly impossible in four dimensional field theory models.

3.1.4 Vacuum of Free Field Theory

When there is no interaction, one can easily construct the exact vacuum state. In this case, one considers the maximum number of freedom to be N, and the particles are put into the box with its length L. The momenta and energies of the negative energy particles can be written as

$$\boldsymbol{p}_n = \frac{2\pi}{L}\,\boldsymbol{n}, \quad E_{\boldsymbol{n}} = -\sqrt{m^2 + \boldsymbol{p}_{\boldsymbol{n}}^2}\,, \tag{3.10}$$

where n_k are integers and run

$$n_k = 0, \pm 1, \pm 2, \ldots, \pm N.$$

Further, one defines the cut-off momentum Λ by

$$\Lambda = \frac{2\pi}{L}\,N \tag{3.11}$$

and one lets N and L as large as required, keeping Λ finite. If the model field theory has no scaleful parameter with massless fermions, then physical observables must be measured by the Λ. But they should not depend on either N nor L.

Fock Space Vacuum

The Fock space vacuum can be written as

$$|0\rangle = \prod_{n_k} b^{\dagger(s)}_{\boldsymbol{n}}|0\rangle\rangle, \quad \text{with} \quad E_{\boldsymbol{n}} = -\sqrt{m^2 + \boldsymbol{p}_{\boldsymbol{n}}^2}\,, \tag{3.12}$$

where $|0\rangle\rangle$ denotes a null vacuum state which is defined as

$$b_{\boldsymbol{n}}^{(s)}|0\rangle\rangle = 0, \quad a_{\boldsymbol{n}}^{(s)}|0\rangle\rangle = 0. \tag{3.13}$$

This exact vacuum state of the free Dirac field is called *perturbative vacuum state* since it is often employed for the quantum field theory with interactions.

It should be noted that the construction of the exact vacuum state is most important since from this vacuum one can create any physical states by applying creation operators. However, it is, at the same time, clear that the exact vacuum state cannot be normally obtained in the field theory models in four dimensions. There are two fermion field theory models which can be solved exactly. That is, the Schwinger model and Thirring model can be solved exactly, but they are two dimensional field theory models and will be treated in the later chapter.

3.2 Quantization of Thirring Model

Now, we present the quantized Hamiltonian of the Thirring model as an example. In the chiral representation of the γ matrices, the Hamiltonian of the Thirring model becomes

$$\hat{H}=\int dx \left[-i\left(\psi_a^\dagger \frac{\partial}{\partial x}\psi_a-\psi_b^\dagger \frac{\partial}{\partial x}\psi_b\right)+m_0(\psi_a^\dagger\psi_b+\psi_b^\dagger\psi_a)+2g\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b\right]. \quad (3.14)$$

Now, the fermion fields are quantized in one space dimension with a box length of L

$$\psi(x)=\begin{pmatrix}\psi_a(x)\\ \psi_b(x)\end{pmatrix}=\frac{1}{\sqrt{L}}\sum_n \begin{pmatrix}a_n\\ b_n\end{pmatrix} e^{ip_n x}, \quad (3.15)$$

where

$$p_n=\frac{2\pi}{L}n, \quad \text{with} \quad n=0,\pm1,\dots .$$

The creation and annihilation operators satisfy the following anti-commutation relations

$$\{a_n,a_m^\dagger\}=\{b_n,b_m^\dagger\}=\delta_{nm}, \quad \{a_n,a_m\}=\{b_n,b_m\}=\{a_n,b_m\}=0. \quad (3.16)$$

Quantized Hamiltonian in Chiral Representation

In this case, the quantized Hamiltonian can be written

$$\hat{H}=\sum_n \left[p_n\left(a_n^\dagger a_n-b_n^\dagger b_n\right)+m_0\left(a_n^\dagger b_n+b_n^\dagger a_n\right)+\frac{2g}{L}\tilde{j}_a(p_n)\tilde{j}_b(p_n)\right], \quad (3.17)$$

where the currents $\tilde{j}_a(p_n)$ and $\tilde{j}_b(p_n)$ in the momentum representation are given by

$$\tilde{j}_a(p_n)=\sum_l a_l^\dagger a_{l+n}, \quad (3.18a)$$

$$\tilde{j}_b(p_n)=\sum_l b_l^\dagger b_{l+n}. \quad (3.18b)$$

3.2.1 Vacuum of Thirring Model

In general, it is very difficult to construct the vacuum state for interacting field theory models. One has to solve the eigenvalue equation of the Hamiltonian with the infinite number of the negative energy particles, and normally it is impossible.

Fortunately, however, the massless Thirring model can be solved exactly by the Bethe ansatz technique . Furthermore, the solution of the vacuum state is given analytically. Since detailed discussions will be given in Chapter 7, we give only the vacuum state which is constructed by operating creation operators.

Exact Vacuum

The exact vacuum state of the massless Thirring model $|\Omega\rangle$ can be written as

$$|\Omega\rangle = \prod_{k_i^\ell} a^\dagger_{k_i^\ell} \prod_{k_j^r} b^\dagger_{k_j^r} |0\rangle\rangle, \tag{3.19}$$

where $|0\rangle\rangle$ denotes the null vacuum state with

$$a_{k_i^\ell}|0\rangle\rangle = 0, \quad b_{k_j^r}|0\rangle\rangle = 0. \tag{3.20}$$

The momenta k_i^ℓ and k_j^r should satisfy the periodic boundary condition (PBC) equations which are solved analytically and the momenta k_i^ℓ for left mover and k_j^r for right mover are given as

$$k_1^r = \frac{2N_0}{L}\ \tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \ \ n_1 = 0, \tag{3.21a}$$

$$k_j^r = \frac{2\pi n_j}{L} + \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \ \ n_j = 1, 2, \ldots, N_0, \tag{3.21b}$$

$$k_i^\ell = \frac{2\pi n_i}{L} - \frac{2(N_0+1)}{L}\ \tan^{-1}\left(\frac{g}{2}\right) \quad \text{for} \ \ n_i = -1, -2, \ldots, -N_0. \tag{3.21c}$$

In this case, the vacuum energy becomes

$$E_v^{\text{true}} = -\Lambda\left\{N_0 + 1 + \frac{2(N_0+1)}{\pi}\tan^{-1}\left(\frac{g}{2}\right)\right\},$$

where Λ denotes the cutoff momentum.

Cut-off momentum Λ

Here, g, L and N_0 denote the coupling constant, the box length and the particle number in the negative energy state, respectively. The cut-off Λ is defined as

$$\Lambda = \frac{2\pi N_0}{L}\,.$$

Since there is no mass scale in the massless Thirring model, all the observables must be measured in terms of the Λ. The number N_0 and the box length L can be set to any large number as required, and any physical observables should not depend on neither N_0 nor L.

3.3 Quantization of Gauge Fields in QED

In this section, we present the quantization of the gauge fields A_μ in QED. This can be found in any textbooks, and therefore we discuss it briefly so that we can calculate some of the scattering S-matrix in QED processes.

Here, we employ the quantization procedure with the Coulomb gauge fixing condition. The Hamiltonian of the electromagnetic fields is written as

$$\hat{H}_{\rm em} = \frac{1}{2}\int \left[\Pi_k^2 - \left(\frac{\partial A_0}{\partial x_k}\right)^2 + \left(\frac{\partial A_k}{\partial x_j}\,\frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j}\,\frac{\partial A_j}{\partial x_k}\right)\right] d^3r, \tag{3.22}$$

where Π_k is a conjugate field to A_k and is given as

$$\Pi_k = -\dot{A}_k.$$

It should be noted that there is no term corresponding to the $\dot{A}_0$ since there is no kinetic energy term arising from the A_0 term. In this sense, the A_0 is not a dynamical variable any more.

Coulomb Gauge Fixing

Therefore, one should quantize the gauge field $\boldsymbol{A}$. However, one should be careful for the number of the degree of freedom of the gauge fields since there is a gauge fixing condition. For example, if one takes the Coulomb gauge.

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{3.23}$$

then, the gauge field $\boldsymbol{A}$ should have two degrees of freedom. In this case, the gauge field $\boldsymbol{A}$ can be expanded in terms of the free field solutions

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\,\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c^\dagger_{\boldsymbol{k},\lambda}e^{ikx}\right], \tag{3.24}$$

where

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|.$$

The polarization vector $\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)$ should satisfy the following relations

$$\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{\epsilon}(\boldsymbol{k},\lambda') = \delta_{\lambda,\lambda'} \tag{3.25}$$

since the gauge field $\boldsymbol{A}$ should satisfy eq.(3.23).

Commutation Relations

Since the gauge fields are bosons, the quantization procedure must be done in the commutation relations, instead of anti-commutation relations. Therefore, the quantization can be done by requiring that $c_{\boldsymbol{k},\lambda},\ c^{\dagger}_{\boldsymbol{k},\lambda}$ should satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda},\ c^{\dagger}_{\boldsymbol{k}',\lambda'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} \tag{3.26}$$

and all other commutation relations vanish.

In this case, the Hamiltonian $\hat{H}_{\rm em}$ of the electromagnetic fields is written in terms of the creation and annihilation operators as

$$\hat{H}_{\rm em} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right). \tag{3.27}$$

From eq.(3.27), one sees that there are two degrees of freedom for the quantized gauge fields. Since the gauge field $\boldsymbol{A}$ has always a gauge freedom, it may be the best to quantize the gauge field $\boldsymbol{A}$ in terms of the creation and annihilation operators $c_{\boldsymbol{k},\lambda},\ c^{\dagger}_{\boldsymbol{k},\lambda}$ after the gauge fixing is done.

Zero Point Energy

Eq.(3.27) contains a zero point energy. That is, the vacuum state where there is no electromagnetic field present has an infinite energy

$$E_{\rm vac} = 2\times\frac{1}{2}\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \sum|\boldsymbol{k}| \to \infty.$$

However, there is nothing serious since the vacuum state cannot be observed. Therefore, one should measure the energy of excited states from the vacuum, and thus

$$\Delta E_{\rm em} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda} + \frac{1}{2}\right) - E_{\rm vac} = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\omega_{\boldsymbol{k}}c^{\dagger}_{\boldsymbol{k},\lambda}c_{\boldsymbol{k},\lambda}$$

must be physical observables.

3.4 Quantization of Schrödinger Field

As we discussed in the first chapter, the non-relativistic fields do not have to be quantized since there are no creation and annihilation of particles in the non-relativistic kinematics.

Nevertheless, one can quantize the Schrödinger field and work out physical observables in the second quantized representation. It is, of course, the same as the classical field theory calculation, but sometime the second quantized representation is easier than the classical field version.

3.4.1 Creation and Annihilation Operators

We consider the Lagrangian density of eq.(1.2)

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi - \psi^\dagger U \psi.$$

In this case, the Schrödinger field $\psi(\boldsymbol{r}, t)$ can be expanded in terms of the free field solutions

$$\psi(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - iEt}, \tag{3.28}$$

where $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ are required to satisfy the following anti-commutation relations

$$\{a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\} = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \quad \{a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}\} = \{a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\} = 0. \tag{3.29}$$

Since one assumes that the Schrödinger field corresponds to fermions, one can carry out the field quantization in the anti-commutation relations.

3.4.2 Fermi Gas Model

From eq.(3.28), one sees that there is no anti-particle present in this model. This is clear since one starts from the vacuum which has no particle at all. On the other hand, if one starts from the Fermi surface and identifies the vacuum state in which all the states are occupied up to the Fermi energy ϵ_F, then one can discuss the particle-hole states which have some similarity with the Dirac hole state. In fact, the formulation is just the same as the Dirac vacuum, but there is of course no anti-particle. The hole state is just a hole in the Fermi sea.

Fermi Momentum

In this picture, particles corresponding to the Schrödinger fields are assumed to obey the Pauli principle. In this respect, the Schrödinger field is considered as the fermion field which should satisfy the anti-commutation relations as shown in eq.(3.29). Therefore, in the Fermi gas model, particle states are occupied up to the Fermi momentum k_F, and when the system has the number of particle N, then one finds

$$N = \sum_{n_x, n_y, n_z} = \frac{L^3}{(2\pi)^3} \int_{|\boldsymbol{k}| \le k_F} d^3k = \frac{L^3}{6\pi^2} k_F^3. \tag{3.30}$$

In this case, the Fermi energy ϵ_F is written as

$$\epsilon_F = \frac{1}{2M} k_F^2, \tag{3.31}$$

where M denotes the mass of the Schrödinger particle. Eq.(3.30) indicates that the density ρ of the system becomes

$$\rho = \frac{N}{L^3} = \frac{1}{6\pi^2} k_F^3. \tag{3.32}$$

Spin and Isospin

If the particle is a nucleon, it has spin s and isospin t. In this case, eq.(3.32) becomes

$$\rho = \frac{N}{L^3} = (2s+1)(2t+1)\,\frac{1}{6\pi^2}\,k_F^3 = \frac{2}{3\pi^2}\,k_F^3, \tag{3.33}$$

where

$$s = t = \frac{1}{2}\,. \tag{3.34}$$

3.5 Quantized Hamiltonian of QED and Eigenstates

The Hamiltonian of fermions with electromagnetic fields is given in eq.(1.33)

$$H = \int \left\{ \bar{\psi}\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right)\psi - g\boldsymbol{j}\cdot\boldsymbol{A} \right\} d^3r$$

$$+\frac{g^2}{8\pi}\int \frac{j_0(\boldsymbol{r}')j_0(\boldsymbol{r})}{|\boldsymbol{r}'-\boldsymbol{r}|}\,d^3r\,d^3r' + \frac{1}{2}\int \left(\boldsymbol{E}_T^2 + \boldsymbol{B}^2\right) d^3r.$$

Now, the fermion field ψ and the gauge field $\boldsymbol{A}$ are quantized as

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{n},s} \frac{1}{\sqrt{L^3}} \left(a_{\boldsymbol{n}}^{(s)} u_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}} + b_{\boldsymbol{n}}^{(s)} v_{\boldsymbol{n}}^{(s)} e^{i\boldsymbol{p_n}\cdot\boldsymbol{r}} \right), \tag{3.35}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\,\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\left[c_{\boldsymbol{k},\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + c_{\boldsymbol{k},\lambda}^{\dagger} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \right]. \tag{3.36}$$

3.5.1 Quantized Hamiltonian

In this case, the quantized Hamiltonian of eq.(1.33) can be written as

$$\hat{H} = \sum_{\boldsymbol{n},s} E_n \left(a^{\dagger(s)}_{\boldsymbol{n}} a^{(s)}_{\boldsymbol{n}} - b^{\dagger(s)}_{\boldsymbol{n}} b^{(s)}_{\boldsymbol{n}} \right) + \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \omega_{\boldsymbol{k}} \left(c^{\dagger}_{\boldsymbol{k},\lambda} c_{\boldsymbol{k},\lambda} + \frac{1}{2} \right)$$

$$-g\int \bar{\psi}(\boldsymbol{r})\boldsymbol{\gamma}\psi(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r})\,d^3r + \frac{g^2}{8\pi}\int \frac{(\bar{\psi}(\boldsymbol{r}')\gamma_0\psi(\boldsymbol{r}'))(\bar{\psi}(\boldsymbol{r})\gamma_0\psi(\boldsymbol{r}))}{|\boldsymbol{r}'-\boldsymbol{r}|}\,d^3r\,d^3r'. \tag{3.37}$$

The interaction terms are so complicated that we do not write them here in terms of the creation-annihilation operators in an explicit fashion. However, one notices that the interaction terms should induce particle-anti-particle creations or destructions.

3.5.2 Eigenvalue Equation

Now, one should solve eq.(3.1) with the above Hamiltonian eq.(3.37)

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{3.38}$$

where $|\Psi\rangle$ may be written even for simple bosonic excitation cases

$$|\Psi\rangle = \sum_{(\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n,t_n)} f\Big((\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n,t_n)\Big)$$

$$\times a^{\dagger(s_1)}_{\boldsymbol{p}_1}\cdots a^{\dagger(s_n)}_{\boldsymbol{p}_n} b^{(t_1)}_{\boldsymbol{q}_1}\cdots b^{(t_n)}_{\boldsymbol{q}_n}|\Omega\rangle, \tag{3.39}$$

where f is the wave function which should be determined so as to satisfy eq.(3.1). The energy eigenvalue E in eq.(3.38) may be calculated by the diagonalization procedure where the space is spanned in terms of $f\left((\boldsymbol{p}_1,s_1),\ldots,(\boldsymbol{p}_n,s_n),(\boldsymbol{q}_1,t_1),\ldots,(\boldsymbol{q}_n),t_n)\right)$. For a practical evaluation, one has to truncate the space significantly so as to carry out any numerical calculations. Numerical calculations in two dimensional QED with finite fermion mass will be discussed in Chapter 5.

3.5.3 Vacuum State $|\Omega\rangle$

$|\Omega\rangle$ denotes the vacuum state which should satisfy the eigenvalue equation of eq.(3.1)

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle,$$

where E_Ω denotes the vacuum energy, and $|\Omega\rangle$ is full of negative energy particles

$$|\Omega\rangle = \prod_{\boldsymbol{p},s} b^{\dagger(s)}_{\boldsymbol{p}}|0\rangle\rangle,$$

where $|0\rangle\rangle$ denotes the null vacuum which satisfies

$$b^{(s)}_{\boldsymbol{p}}|0\rangle\rangle = 0, \quad a^{(s)}_{\boldsymbol{p}}|0\rangle\rangle = 0.$$

One sees clearly that it is practically impossible to solve the eigenvalue equation of the Hamiltonian in eq.(3.1).

Chapter 4

Goldstone Theorem and Spontaneous Symmetry Breaking

The continuous symmetry of the Lagrangian density leads to the conservation of currents and therefore the system should have a conserved charge associated with the symmetry. The best example must be a global gauge symmetry in which the Lagrangian density is invariant under the transformation of the Dirac field ψ as

$$\psi' = e^{i\alpha}\psi \Longrightarrow \mathcal{L}' = \mathcal{L},$$

where α is a real constant. In this case, the vector current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved. That is,

$$\partial_\mu j^\mu = 0. \tag{4.1}$$

In fermion field theory models, the global gauge symmetry is not broken at any level even though the vacuum state can, in principle, break its symmetry. In particular, the gauge invariance of the local gauge field theory should hold rigorously since the violation of the local gauge invariance should lead to the breakdown of defining physical observables.

On the other hand, the chiral symmetry behaves quite differently from the global gauge symmetry. When the Lagrangian density is invariant under the chiral symmetry transformation

$$\psi' = e^{i\alpha\gamma^5}\psi \Longrightarrow \mathcal{L}' = \mathcal{L}$$

the axial vector current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

is also conserved, that is,

$$\partial_\mu j_5^\mu = 0. \tag{4.2}$$

However, the chiral symmetry is broken not only because of the chiral anomaly but also in terms of the spontaneous symmetry breaking mechanism. In the former case, the conservation of the axial vector current is violated by the anomaly term while, in the spontaneous chiral symmetry breaking, the vacuum loses its symmetry since the vacuum state prefers the lowest energy state which does not have to keep its symmetry. The discussion of the anomaly term is given in Chapter 5 in the context of the Schwinger model where the basic mechanism of the chiral anomaly and the violation of the axial vector current are described in a transparent fashion.

In this chapter, we discuss the symmetry breaking phenomena in fermion field theory models. In particular, we clarify what is physics of the spontaneous symmetry breaking and the Goldstone theorem [60, 61]. Normally, the mathematics in the theorem can be understood in a straightforward way, but its physics in connection with the theorem is always difficult since one has to examine all the possible conditions in nature when the symmetry is broken spontaneously.

First, we explain the general feature of the symmetry and its preservation in quantum many body theory, before going to the discussion of the Goldstone theorem. Then, we explain the Goldstone theorem and the problem related to the proof of existence of the Goldstone boson. Also, we present a new interpretation of the Goldstone theorem. In particular, we give a good example of the chiral symmetry breaking in the Thirring model together with the exact solution of the vacuum state.

Further, we comment on the symmetry breaking of boson field theory models together with the Higgs mechanism. However, this part is presented with reservation since there are still some problems which are not yet clarified completely in this textbook. The difficulty of the symmetry breaking physics in boson field theory is partly because the scalar boson field itself has some serious problems as will be discussed in Appendix C and partly because there is no model which can present exact solutions of the boson field theory models.

4.1 Symmetry and Its Breaking in Vacuum

Various symmetries of Hamiltonian play an important role for determining the energy eigenvalues, and in quantum mechanics, one often sees that the lowest state (ground state) preserves the symmetry of the Hamiltonian. In fact, those states which break the symmetry are, in general, higher than the symmetry preserving state for the same quantum numbers or configurations of the wave function. This can be naturally realized in quantum many body theory as our experiences tell us.

However, the physics of the symmetry breaking in quantum field theory is quite different, and phenomena which seem to be in an apparent contradiction with the picture of quantum many body theory can indeed occur. This is called *spontaneous symmetry breaking* and it has been discussed extensively in many field theory textbooks. In the spontaneous symmetry breaking, the vacuum state of the field theory models is realized with the symmetry broken state. That is, the true vacuum prefers the symmetry broken state, contrary to the naive expectation in quantum many body theory.

A question may arise as to what should be an intuitive explanation why the vacuum of the field theory models has the symmetry broken state as the most favorable state, in contradiction with the experiences of quantum many body theory. Here, we show that the symmetry broken state of the vacuum in the field theory models is naturally realized in the context of quantum many body theory. In short, the vacuum of quantum field theory is constructed by the negative energy particles, and therefore, the symmetry preserving state should have the absolute magnitude of its total energy which is smaller than the symmetry broken state. This is just consistent with the prediction of quantum many body theory. However, energies of the vacuum are all negative, and thus the lower state is, of course, the one that breaks the symmetry since the absolute magnitude of its energy is larger than that of the symmetry preserving state. This is exactly what one observes in the Thirring model as will be discussed in Chapter 7.

4.1.1 Symmetry in Quantum Many Body Theory

In quantum mechanics, the ground state energy of the particle Hamiltonian H can be written

$$E_{\rm tot} = \langle 0|H|0\rangle, \tag{4.3}$$

where we denote the ground state by $|0\rangle$ which preserves the symmetry of the Hamiltonian H. In this case, the lowest state is normally the one that keeps the symmetry. The total energy of the states which do not keep the symmetry should be found in higher energies than the symmetry preserving state. This energy of the N particle state may be written more explicitly as

$$E_{\rm tot} = \sum_{i=1}^{N} \mathcal{E}_i(\boldsymbol{k}_i), \tag{4.4}$$

where we denote the energy of the i-th particle by $\mathcal{E}_i(\boldsymbol{k}_i)$. The momentum $\boldsymbol{k}_i$ should be determined by solving the many body equations of motion

$$F_i(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N) = 0, \;\; i = 1, \ldots, N. \tag{4.5}$$

Now, suppose there is a symmetry in the Hamiltonian H. The solution of the above equations should be specified by the symmetric and the symmetry broken solutions. In quantum many body system, it is often the case that the symmetric solution $E_{\rm tot}^{sym}$ is lower than the symmetry broken solution $E_{\rm tot}^{sym.br}$.

$$E_{\rm tot}^{sym} < E_{\rm tot}^{sym.br}. \tag{4.6}$$

This, of course, depends on the interactions between particles in the system, and one can only claim that there should be some systems in which eq.(4.6) can hold.

4.1.2 Symmetry in Field Theory

In quantum field theory, the symmetry breaking phenomena occur in a completely different fashion. The lowest state which is called *vacuum* sometimes breaks the continuous symmetry since it is found to be lower than the symmetry preserving state. Since the continuous symmetry has the Noether current associated with its symmetry, the current conservation should hold true in the process of determining the lowest state of the model field theory as long as the model has no anomaly.

Why is it possible that the symmetry broken state becomes lower than the symmetry preserving state in an obvious contradiction with the picture of quantum many body theory? Here, we present a simple intuitive picture why it may occur. The basic point is that the vacuum in field theory models is constructed by particles with the negative energies which are solved in the many body Dirac equations. To be more specific, the vacuum energy $E_{\rm vac}$ can be written as

$$E_{\rm vac} = -\lim_{N\to\infty}\sum_{i=1}^{N}\mathcal{E}_i(\boldsymbol{k}_i), \tag{4.7}$$

where the energy of the i-th particle is denoted by $\mathcal{E}_i$. Since the system is infinite, one should make the number N infinity at the end of the calculation. It should be noted that one cannot make the system infinity from the beginning since in this case one cannot define the total energy of the system. This construction of the vacuum in terms of the finite N-particle system and then making the number N infinity must be well justified since the deep negative energy states in the vacuum should not have any effects on the physical properties of the vacuum state.

In order to determine the momenta of the negative energy particles, one should solve the equations of motion which may be similar to eq.(4.5)

$$G_i(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = 0, \;\; i = 1,\ldots,N. \tag{4.8}$$

Again, one can assume that there is a symmetry in the Hamiltonian H. In this case, the solution of the above equations should be specified by the symmetric solution and the symmetry broken solution. Just in the same way as the positive energy case, if one defines the total energy $E_{\rm tot}$ by

$$E_{\rm tot} = \sum_{i=1}^{N}\mathcal{E}_i(\boldsymbol{k}_i) \tag{4.9}$$

then the symmetric solution $E_{\rm tot}^{sym}$ must be lower than the symmetry broken solution $E_{\rm tot}^{sym.br}$

$$E_{\rm tot}^{sym} < E_{\rm tot}^{sym.br}. \tag{4.10}$$

However, the energy of the vacuum is all negative, and therefore one sees that the symmetry broken vacuum state must be the lowest, that is

$$E_{\rm vac}^{sym.br} < E_{\rm vac}^{sym} \tag{4.11}$$

which is just opposite to the prediction of the quantum many body theory.

In this way, the vacuum in field theory models prefers the symmetry broken state. The symmetry preserving state has the lowest energy in magnitude, but due to the negative sign in front (eq.(4.7)), the lowest energy must be the one that breaks the symmetry. This is exactly what happens in the spontaneous symmetry breaking in fermion field theory models. This appearance of the two states, one that preserves the symmetry and the other that breaks the symmetry must depend on the dynamical properties of the models one considers. Up to now, the Thirring model exhibits the two states in the vacuum, and therefore it presents a good example of the spontaneous symmetry breaking physics. In the next section, we discuss the Goldstone theorem, keeping this fact in mind.

4.2 Goldstone Theorem

The physics of the spontaneous symmetry breaking started from the Goldstone theorem. The theorem states that there should appear a massless boson when the symmetry of the vacuum state is spontaneously broken. In this process of the spontaneous symmetry breaking, the current conservation should hold. This is important since the Goldstone theorem is entirely based on the current conservation, and without the current conservation the theorem cannot be proved.

Here, without loss of generality, we can restrict our discussion to the chiral symmetry breaking of the fermion field theory models. In this case, the chiral charge Q_5 must be a conserved quantity.

4.2.1 Conservation of Chiral Charge

When the Lagrangian density has the chiral symmetry which can be represented by the unitary operator $U(\alpha)$, there is a conserved current associated with the symmetry, which is eq.(4.2). In this case, there is a conserved chiral charge Q_5

$$Q_5 = \int j_5^0(x)\, d^3r. \tag{4.12}$$

The quantized Hamiltonian $\hat{H}$ of this system is invariant under the unitary transformation $U(\alpha)$,

$$U(\alpha)\hat{H}U(\alpha)^{-1} = \hat{H}. \tag{4.13}$$

Therefore, the chiral charge operator $\hat{Q_5}$ commutes with the Hamiltonian $\hat{H}$

$$\hat{Q_5}\hat{H} = \hat{H}\hat{Q_5}. \tag{4.14}$$

4.2.2 Symmetry of Vacuum

The symmetry of the vacuum is determined in terms of its energy, and when the lowest energy state is realized, the vacuum may break the symmetry which is possessed in the

Hamiltonian. Here, the symmetry of the vacuum can be defined in the following way. The symmetric vacuum is denoted by $|0\rangle$ while the symmetry broken vacuum is denoted by $|\Omega\rangle$. They satisfy the following equations,

$$U(\alpha)|0\rangle = |0\rangle, \tag{4.15a}$$

$$U(\alpha)|\Omega\rangle \neq |\Omega\rangle. \tag{4.15b}$$

These equations can be written in terms of the chiral charge operator $\hat{Q}_5$ as

$$\hat{Q}_5|0\rangle = 0, \tag{4.16a}$$

$$\hat{Q}_5|\Omega\rangle \neq 0. \tag{4.16b}$$

4.2.3 Commutation Relation

In the Goldstone theorem, one starts from the following commutation relation which is an identity equation,

$$\left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right] = -2\int \bar{\psi}(x)\psi(x)\, d^3r. \tag{4.17}$$

Now, one takes the expectation value of the above equation with the vacuum state $|\Omega\rangle$, and obtains

$$\langle\Omega|\left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right]|\Omega\rangle = -2\langle\Omega|\int \bar{\psi}(x)\psi(x)\, d^3r|\Omega\rangle. \tag{4.18}$$

If the right hand side (fermion condensate) has a finite value, then the vacuum state $|\Omega\rangle$ must be a symmetry broken state since it should at least satisfy eq.(4.16b) because of the finite value of the left hand side.

Now, one can rewrite eq.(4.18) and assume that the right hand side is nonzero because of the finite fermion condensate

$$\sum_n (2\pi)^3\delta(\boldsymbol{p}_n)\left[\langle\Omega|j_5^0|n\rangle\langle n|\bar{\psi}\gamma_5\psi|\Omega\rangle e^{-iE_nt} - \langle\Omega|\bar{\psi}\gamma_5\psi|n\rangle\langle n|j_5^0|\Omega\rangle e^{iE_nt}\right] \neq 0, \tag{4.19}$$

where $|n\rangle$ denotes the complete set of the fermion number zero states of the field theory model one considers. Therefore, bosonic states as well as the pair of massless free fermion and anti-fermion states should be included in the intermediate states. From eq.(4.19), one sees that the right hand side is nonzero and time-independent while the left hand side is time dependent unless there is a state $|n\rangle$ that satisfies

$$E_n = 0 \ \text{ for } \ \boldsymbol{p}_n = 0. \tag{4.20}$$

Eq.(4.20) is just consistent with the dispersion relation of a massless boson.

4.2.4 Momentum Zero State

However, one easily notices that the free massless fermion and anti-fermion pair can also satisfy eq.(4.20). To be more specific, one can write the energy and momentum of the state $|n\rangle$ as

$$E_n = E_f + E_{\bar{f}}, \tag{4.21a}$$

$$\boldsymbol{p}_n = \boldsymbol{p}_f + \boldsymbol{p}_{\bar{f}}, \tag{4.21b}$$

where $\boldsymbol{p}_f$ ($\boldsymbol{p}_{\bar{f}}$) and E_f ($E_{\bar{f}}$) denote the momentum and energy of the fermion (anti-fermion), respectively. For the free massless fermion and anti-fermion pair with

$$\boldsymbol{p}_f = 0 \ \text{ and } \ \boldsymbol{p}_{\bar{f}} = 0 \tag{4.22a}$$

one obtains [41, 42, 43, 69]

$$E_f = 0 \ \text{ and } \ E_{\bar{f}} = 0 \tag{4.22b}$$

and therefore eq.(4.20) is indeed satisfied

$$\boldsymbol{p}_n = \boldsymbol{p}_f + \boldsymbol{p}_{\bar{f}} = 0 \Longrightarrow E_n = E_f + E_{\bar{f}} = 0.$$

Dispersion Relation of Massless Boson

From one information of eq.(4.20), one could derive the dispersion relation of a massless boson if the state must be covariant

$$E_n^2 - \boldsymbol{p}_n^2 = (p_n)_\mu p_n^\mu = 0.$$

The requirement of the covariance for the state $|n\rangle$ may be justified when the state is an isolated system. However, it is difficult to show the covariance from only one information on the zero momentum state which is just eq.(4.20) since the vacuum is always in the momentum zero state.

On the other hand, the dispersion relation of a massless boson

$$E = |\boldsymbol{p}|$$

contains information which should be valid for arbitrary momentum $\boldsymbol{p}$. Intuitively, it is clear that one cannot obtain the dispersion relation of a massless boson from only one information which is at $\boldsymbol{p} = 0$. Therefore, one sees that the Goldstone theorem proves the existence of a free massless fermion and anti-fermion pair for the fermion field theory models, and this is, of course, a natural statement. But there exists no massless boson.

4.2.5 Pole in S-matrix

In the spontaneous symmetry breaking, a massless pole in the S-matrix calculations is sometimes found, and they claim that the pole should be related to a massless boson (Goldstone boson). However, the S-matrix in these calculations is evaluated in the trivial (perturbative) vacuum, and it has nothing to do with a physical massless boson. Furthermore, one has to be careful that a pole in the S-matrix may not have to correspond to a bound state, and if one wishes to find a bound state pole, then one should calculate poles in exact Green's function in which the evaluation should be based on the symmetry broken vacuum state. But this is just the same as solving the system exactly.

4.3 New Interpretation of Goldstone Theorem

Here, we present a new interpretation of the Goldstone theorem and clarify what is indeed the physics of the spontaneous symmetry breaking in fermion field theory models. Since the spontaneous symmetry breaking is connected with the structure of the vacuum, we should understand the physical feature of the vacuum. However, most of the difficulties of the field theory models are concentrated in the dynamical evaluation of the vacuum state, and therefore, we should first treat the spontaneous symmetry breaking physics in terms of finite number of freedoms. After that, we should examine whether the procedure can be justified when the number of the freedom is set to infinity.

4.3.1 Eigenstate of Hamiltonian and $\hat{Q}_5$

Since the operator $\hat{Q}_5$ commutes with the Hamiltonian $\hat{H}$ as discussed in eq.(4.14),

$$\hat{Q}_5\hat{H} = \hat{H}\hat{Q}_5$$

the $\hat{Q}_5$ has the same eigenstate as the Hamiltonian. If one defines the symmetry broken vacuum state $|\Omega\rangle$ by the eigenstate of the Hamiltonian $\hat{H}$ with its energy eigenvalue E_Ω, then one can write

$$\hat{H}|\Omega\rangle = E_\Omega|\Omega\rangle. \tag{4.23}$$

In this case, one can also write the eigenvalue equation for the $\hat{Q}_5$

$$\hat{Q}_5|\Omega\rangle = q_5|\Omega\rangle \tag{4.24}$$

with its eigenvalue q_5. These equations should hold for the exact eigenstates of the Hamiltonian.

Now, if one takes the expectation value of eq.(4.17) with the symmetry broken vacuum $|\Omega\rangle$ which is the eigenstate of the Hamiltonian as well as $\hat{Q}_5$, then one obtains for the left hand side as

$$\langle\Omega|\left[\hat{Q}_5, \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right]|\Omega\rangle$$

$$= \langle\Omega|q_5 \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r - \left(\int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r\right) q_5|\Omega\rangle = 0 \tag{4.25}$$

with the help of eq.(4.24). This means that the right hand side of eq.(4.18) must vanish, that is,

$$\langle\Omega| \int \bar{\psi}(x)\psi(x)\, d^3r|\Omega\rangle = 0. \tag{4.26}$$

Therefore, the exact eigenstate of the vacuum has no fermion condensate even in the symmetry broken vacuum. The relation of eq.(4.18) has repeatedly been used, and if there is a finite fermion condensate, then the symmetry of the vacuum must be broken since the left hand side of eq.(4.18) vanishes due to eq.(4.16a) for the symmetric vacuum state. However, as seen above, the condensate must vanish even for the symmetry broken vacuum state if the vacuum is the eigenstate of the Hamiltonian, which is a natural consequence.

4.3.2 Index of Symmetry Breaking

The way out of this dilemma is simple. One should not take the expectation value of the vacuum state. Instead, the index of the symmetry breaking in connection with the condensate operator $\int \bar{\psi}(x)\psi(x)\, dx$ should be the following operator equation

$$\left(\int \bar{\psi}(x)\psi(x)\, d^3r\right)|\Omega\rangle = |\Omega'\rangle + C_1|\Omega\rangle, \tag{4.27}$$

where $|\Omega'\rangle$ denotes an operator-induced state which is orthogonal to the $|\Omega\rangle$

$$\langle\Omega|\Omega'\rangle = 0.$$

C_1 is related to the condensate value. For the exact eigenstate which breaks the chiral symmetry, one finds

$$C_1 = 0. \tag{4.28}$$

In this case, the identity equation of (4.17) can be applied to the state $|\Omega\rangle$ and one obtains

$$(\hat{Q}_5 - q_5) \int \bar{\psi}(x)\gamma_5\psi(x)\, d^3r|\Omega\rangle = -2|\Omega'\rangle \tag{4.29}$$

with the help of eq.(4.24). Indeed, eq.(4.29) holds true for the exact eigenstate. It is now clear that one should not take the expectation value of eq.(4.17) by the vacuum state. It just gives a trivial equation of "0" = "0".

4.4 Chiral Symmetry in Quantized Thirring Model

In this section, we show explicitly how the chiral symmetry in the quantized Thirring model Hamiltonian behaves in terms of the creation and annihilation operators.

4.4.1 Lagrangian Density

The Lagrangian density of the Thirring model is written as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{1}{2}\, g\bar{\psi}\gamma_\mu\psi\ \bar{\psi}\gamma^\mu\psi \tag{4.30}$$

which is invariant under the chiral transformation

$$\psi' = e^{i\alpha\gamma_5}\psi. \tag{4.31}$$

Therefore, the chiral charge

$$\hat{Q}_5 = \int \bar{\psi}(x)\gamma^0\gamma_5\psi(x)\, d^3r$$

is a conserved quantity. In fact, it commutes with the Hamiltonian $\hat{H}$

$$[\hat{H}, \hat{Q}_5] = 0.$$

4.4.2 Quantized Hamiltonian

The field ψ is quantized as

$$\psi(x) = \begin{pmatrix}\psi_a(x)\\ \psi_b(x)\end{pmatrix} = \frac{1}{\sqrt{L}}\sum_n \begin{pmatrix}a_n\\ b_n\end{pmatrix} e^{ip_n x}, \ \text{ with } \ p_n = \frac{2\pi}{L}\, n, \tag{4.32}$$

where the operators a_n and b_n should satisfy the anti-commutation relations

$$\{a_n, a_m^\dagger\} = \{b_n, b_m^\dagger\} = \delta_{nm}, \quad \{a_n, a_m\} = \{b_n, b_m\} = \{a_n, b_m\} = 0.$$

In this case, the quantized Hamiltonian of the Thirring model becomes

$$\hat{H} = \sum_n \left[p_n \left(a_n^\dagger a_n - b_n^\dagger b_n \right) + \frac{2g}{L} \left(\sum_l a_l^\dagger a_{l+n} \right) \left(\sum_m b_m^\dagger b_{m+n} \right) \right]. \tag{4.33}$$

4.4.3 Chiral Transformation for Operators

The chiral transformation of eq.(4.31) is written in terms of a_n and b_n as

$$U(\alpha)\begin{pmatrix}a_n\\ b_n\end{pmatrix} U^{-1}(\alpha) = \begin{pmatrix}e^{i\alpha}a_n\\ e^{-i\alpha}b_n\end{pmatrix}, \quad U(\alpha)\begin{pmatrix}a_n^\dagger\\ b_n^\dagger\end{pmatrix} U^{-1}(\alpha) = \begin{pmatrix}e^{-i\alpha}a_n^\dagger\\ e^{i\alpha}b_n^\dagger\end{pmatrix}. \tag{4.34}$$

In this case, one easily sees that the Hamiltonian is also invariant under the transformation of eq.(4.34)

$$\begin{aligned} U(\alpha)\hat{H}U^{-1}(\alpha) = \sum_n \Bigg[& p_n U(\alpha)\left(a_n^\dagger a_n - b_n^\dagger b_n \right) U^{-1}(\alpha) \\ & + \frac{2g}{L}\, U(\alpha)\left(\sum_l a_l^\dagger a_{l+n} \right) U^{-1}(\alpha) U(\alpha) \left(\sum_m b_m^\dagger b_{m+n} \right) U^{-1}(\alpha) \Bigg] = \hat{H}. \end{aligned} \tag{4.35}$$

4.4.4 Unitary Operator with Chiral Charge $\hat{Q}_5$

Now, the unitary operator $U(\alpha)$ can be explicitly written as

$$U(\alpha) = e^{-i\alpha\hat{Q}_5}, \tag{4.36}$$

where the chiral charge operator $\hat{Q}_5$ is expressed in terms of the creation and annihilation operators as

$$\hat{Q}_5 = \int \bar{\psi}(x)\gamma^0\gamma_5\psi(x)\, d^3r = \sum_n \left[a_n^\dagger a_n - b_n^\dagger b_n\right].$$

In this case, one can confirm the following identities

$$U(\alpha)\begin{pmatrix} a_n \\ b_n \end{pmatrix} U^{-1}(\alpha) = e^{-i\alpha\hat{Q}_5}\begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{i\alpha\hat{Q}_5} = \begin{pmatrix} e^{i\alpha}a_n \\ e^{-i\alpha}b_n \end{pmatrix}, \tag{4.37a}$$

$$U(\alpha)\begin{pmatrix} a_n^\dagger \\ b_n^\dagger \end{pmatrix} U^{-1}(\alpha) = e^{-i\alpha\hat{Q}_5}\begin{pmatrix} a_n^\dagger \\ b_n^\dagger \end{pmatrix} e^{i\alpha\hat{Q}_5} = \begin{pmatrix} e^{-i\alpha}a_n^\dagger \\ e^{i\alpha}b_n^\dagger \end{pmatrix} \tag{4.37b}$$

which are just the same as eq.(4.34).

4.4.5 Symmetric and Symmetry Broken Vacuum

In the Thirring model, the Bethe ansatz solutions show that there are symmetric vacuum $|0\rangle$ and symmetry broken vacuum $|\Omega\rangle$ states, and the energy of the symmetry broken vacuum is found to be lower than the symmetric vacuum energy. They are the eigenstate of the chiral charge $\hat{Q}_5$ and one finds

$$\hat{Q}_5|0\rangle = 0, \tag{4.38a}$$

$$\hat{Q}_5|\Omega\rangle = \pm|\Omega\rangle. \tag{4.38b}$$

Therefore, for the unitary operator $U(\alpha) = e^{-i\alpha\hat{Q}_5}$, the symmetric vacuum does not change

$$U(\alpha)|0\rangle = e^{-i\alpha\hat{Q}_5}|0\rangle = |0\rangle \tag{4.39a}$$

while the symmetry broken vacuum becomes

$$U(\alpha)|\Omega\rangle = e^{-i\alpha\hat{Q}_5}|\Omega\rangle = e^{\pm i\alpha}|\Omega\rangle \neq |\Omega\rangle \tag{4.39b}$$

which indeed satisfies the criteria of the symmetry broken vacuum state in eqs.(4.15).

4.5 Spontaneous Chiral Symmetry Breaking

There is one good example which perfectly satisfies the above requirements of the spontaneous chiral symmetry breaking and zero fermion condensate. That is the Bethe ansatz vacuum of the massless Thirring model which will be discussed in detail in Chapter 7. Here, we employ the results of the Bethe ansatz vacuum of the Thirring model and discuss the vacuum and its properties in the context of the spontaneous symmetry breaking.

4.5.1 Exact Vacuum of Thirring Model

Now, the left and right mover fermion creation operators can be denoted by $a_k^\dagger$, $b_k^\dagger$, respectively, and thus the vacuum state $|\Omega\rangle$ can be written as

$$|\Omega\rangle = \prod_{k_i^\ell} a_{k_i^\ell}^\dagger \prod_{k_j^r} b_{k_j^r}^\dagger |0\rangle\rangle, \tag{4.40}$$

where $|0\rangle\rangle$ denotes the null vacuum state with

$$a_{k_i^\ell}|0\rangle\rangle = 0, \quad b_{k_j^r}|0\rangle\rangle = 0. \tag{4.41}$$

The momenta k_j^ℓ for left mover and k_i^r for right mover should satisfy the periodic boundary condition (PBC) equations which are solved analytically, and therefore one can determine the momenta k_j^ℓ and k_i^r, as will be given in Chapter 7.

4.5.2 Condensate Operator

Now, the condensate operator $\int \bar{\psi}(x)\psi(x)\,dx$ can be written as

$$\int \bar{\psi}(x)\psi(x)\,dx = \sum_n (b_n^\dagger a_n + a_n^\dagger b_n). \tag{4.42}$$

Therefore, eq.(4.27) becomes

$$\begin{aligned}\int \bar{\psi}(x)\psi(x)\,dx|\Omega\rangle &= \sum_n (b_n^\dagger a_n + a_n^\dagger b_n)|\Omega\rangle \\ &= \sum_n \left\{ \prod_{k_i^\ell, k_i^\ell \neq n} a_{k_i^\ell}^\dagger \prod_{k_j^r} b_{k_j^r}^\dagger b_n^\dagger |0\rangle\rangle + \prod_{k_i^\ell} a_{k_i^\ell}^\dagger \prod_{k_j^r, k_j^r \neq n} b_{k_j^r}^\dagger a_n^\dagger |0\rangle\rangle \right\}. \end{aligned} \tag{4.43}$$

Clearly, the right hand side of eq.(4.43) is different from the vacuum state of the Bethe ansatz solution of eq.(4.40), and therefore denoting the right hand side of eq.(4.43) by $|\Omega'\rangle$, one obtains

$$\int \bar{\psi}(x)\psi(x)\,dx|\Omega\rangle = \sum_n (b_n^\dagger a_n + a_n^\dagger b_n)|\Omega\rangle = |\Omega'\rangle. \tag{4.44}$$

Obviously, the value of C_1 in eq.(4.27) is zero in the massless Thirring model, and indeed this confirms eq.(4.29).

It is now clear and most important to note that one cannot learn the basic dynamics of the symmetry breaking phenomena from the identity equation. If one wishes to study the symmetry breaking physics in depth, then one has to solve the dynamics of the vacuum in the field theory model properly even though it is extremely difficult to solve it exactly.

4.6 Symmetry Breaking in Two Dimensions

In two dimensional field theory models, it is well known that there should not exist any physical massless bosons because of the infra-red singularity of the propagator of the massless boson. Therefore, if one assumes the Goldstone theorem, then one finds that there should not occur any spontaneous symmetry breaking in two dimensional field theory models, which is known as Coleman's theorem [22, 31, 91].

4.6.1 Fermion Field Theory in Two Dimensions

However, as we will see in the later chapters, the vacuum states of the massless Thirring model as well as QCD in two dimensions prefer the symmetry broken vacuum states together with the current conservation. Therefore, the spontaneous symmetry breaking of the vacuum indeed takes place in two dimensional field theory models of Thirring and QCD_2. By now, this is not surprising since the Goldstone theorem does not hold in fermion field theory models. On the contrary, the spontaneous symmetry breaking in these models are consistent with the new picture of the symmetry breaking physics. As far as the spontaneous symmetry breaking physics is concerned, the two dimensional field theory is not at all special since there appears no massless boson after the symmetry is spontaneously broken in the vacuum.

4.6.2 Boson Field Theory in Two Dimensions

The spontaneous symmetry breaking should not occur in boson field theory models in two dimensions. This may be reasonable since the Goldstone theorem may hold for the boson field theory models where a massless boson should appear. However, there should not exist any physical massless boson in two dimensions, and therefore, the spontaneous symmetry breaking should be forbidden in two dimensional boson field theory models.

4.7 Symmetry Breaking in Boson Fields

In the subsequent two sections, we stray from the main stream of the spontaneous symmetry breaking physics in fermion field theory models, and come to discussions of the spontaneous symmetry breaking in boson field theory in four dimensions. In most of the field theory textbooks, the discussion of this subject can be found, and therefore, we discuss it briefly in this section.

4.7.1 Double Well Potential

Now, we discuss the spontaneous symmetry breaking in boson field theory models. This can be found in any field theory textbooks, and therefore we only sketch a simple picture why the massless boson appears in the spontaneous symmetry breaking. But it should be noted that the treatment here is approximate, and there is still some unsolved problem left when

one wishes to understand the spontaneous symmetry breaking in boson field theory models in an exact fashion. The Hamiltonian density for complex boson fields can be written as

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{\nabla}\phi^{\dagger})(\boldsymbol{\nabla}\phi) + U\left(|\phi|\right). \tag{4.45}$$

This has a $U(1)$ symmetry. However, the Hamiltonian density must be real, and therefore the $U(1)$ symmetry of the Hamiltonian density is a trivial constraint. Now, when one takes the potential as a double well type

$$U\left(|\phi|\right) = u_0\left(|\phi|^2 - \lambda^2\right)^2, \tag{4.46}$$

where u_0 and λ are constant, then the minimum of the potential $U\left(|\phi|\right)$ can be found at

$$|\phi(x)| = \lambda.$$

However, one must notice that this is a minimum of the potential, but not the minimum of the total energy.

4.7.2 Change of Field Variables

The minimum of the total energy must be found together with the kinetic energy term. Now, one rewrites the complex field as

$$\phi(x) = (\lambda + \eta(x))e^{i\frac{\xi(x)}{\lambda}}, \tag{4.47}$$

where η is assumed to be much smaller than the λ,

$$|\eta(x)| \ll \lambda.$$

In this case, one can rewrite eq.(4.45) as

$$\mathcal{H} = \frac{1}{2}\left[(\boldsymbol{\nabla}\xi)(\boldsymbol{\nabla}\xi) + (\boldsymbol{\nabla}\eta)(\boldsymbol{\nabla}\eta)\right] + U\big(|\lambda + \eta(x)|\big) + \cdots. \tag{4.48}$$

Here, one finds the massless boson ξ which is associated with the degeneracy of the vacuum energy. The important point is that this infinite degeneracy of the potential vacuum is converted into the massless boson degrees of freedom when the degeneracy of the potential vacuum is resolved by the kinetic energy term.

This is the spontaneous symmetry breaking which is indeed found by Goldstone, and he pointed out that there should appear a massless boson associated with the symmetry breaking. The degeneracy of the potential vacuum is converted into a massless boson degree of freedom. This looks plausible, and at least approximately there is nothing wrong with this treatment of the spontaneous symmetry breaking phenomena in contrast to the fermion field theory model. However, the treatment is still approximate, and one should confirm that the terms neglected in eq.(4.48) may not cause any troubles. At least, one cannot claim that

the massless boson which appears after the spontaneous symmetry breaking is an isolated particle of the system. Also, the Goldstone theorem shows that there should be a boson state as given in eq.(4.20)

$$E_n = 0 \quad \text{for} \quad \boldsymbol{p}_n = 0$$

which is consistent with a massless boson. However, to be rigorous, one may still have to prove that the state with the above constraint is an isolated system.

In this respect, it should be most important to solve the boson field theory model with the double well potential in an exact fashion, and then one may learn the essence of the symmetry breaking physics in the boson field theory in depth, and this is indeed a future problem.

4.7.3 Current Density of Fields

In addition, the boson fields ξ and η are real fields, and therefore the current density $j_0(x)$ of the boson fields ξ and η must vanish as the classical field

$$j_0(x) = i\left(\xi^\dagger(x)\,\frac{\partial \xi(x)}{\partial t} - \frac{\partial \xi^\dagger(x)}{\partial t}\,\xi(x)\right) = 0 \ \text{ since } \ \xi^\dagger(x) = \xi(x)$$

and the same equation holds for the η field as well. However, they diverge when they are quantized as will be discussed in the Appendix C. Therefore, both of the fields cannot propagate as a physical particle. In this sense, one may say that the model field theory of eq.(4.45) is not realistic.

4.8 Breaking of Local Gauge Symmetry?

At present, all of the realistic field theory models have the local gauge symmetry. Quantum electrodynamics (QED), quantum chromodynamics (QCD) and Weinberg-Salam model have the local gauge invariance. The local gauge symmetry in QED and QCD should hold rigorously, and this is just what we observe from experiments.

However, this local gauge invariance seems to be broken in the Higgs mechanism, and therefore we should discuss the essence of the spontaneous symmetry breaking of the local gauge invariance [67]. This concept is employed in the electro-weak theory by Weinberg-Salam, and the $SU(2) \otimes U(1)$ gauge field model is quite successful in describing many experimental observations.

4.8.1 Higgs Mechanism

The Lagrangian density of the complex scalar field $\phi(x)$ which interacts with the $U(1)$ gauge field can be written as

$$\mathcal{L} = \frac{1}{2}\,(D_\mu\phi)^\dagger(D^\mu\phi) - u_0\left(|\phi|^2 - \lambda^2\right)^2 - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \tag{4.49}$$

where

$$D_\mu = \partial_\mu + igA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Here, the same double well type potential as in eq.(4.46) is assumed for the complex scalar field. Now, one rewrites the complex scalar field as

$$\phi(x) = (\lambda + \eta(x))e^{i\frac{\xi(x)}{\lambda}}\,, \tag{4.50}$$

where $\eta(x)$ and $\xi(x)$ denote new fields, and therefore one can obtain a new Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - u_0\left(|\lambda+\eta|^2 - \lambda^2\right)^2$$
$$+\frac{1}{2}g^2\lambda^2\left(A_\mu + \frac{1}{g\lambda}\partial_\mu\xi\right)\left(A^\mu + \frac{1}{g\lambda}\,\partial^\mu\xi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\,. \tag{4.51}$$

This Lagrangian density is still invariant under the following gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x), \quad \xi'(x) = \xi(x) - g\lambda\chi(x),$$

where $\chi(x)$ is an arbitrary function of space and time.

4.8.2 Gauge Fixing

Now, one fixes the gauge such that

$$\xi(x) = 0$$

which is called *unitary gauge*. This means that one takes the following gauge fixing

$$\chi(x) = \frac{1}{g\lambda}\,\xi(x).$$

In this case, the Lagrangian density of eq.(4.51) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - u_0\left(|\lambda+\eta|^2 - \lambda^2\right)^2 + \frac{1}{2}g^2\lambda^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\,. \tag{4.52}$$

This new Lagrangian density shows that the gauge field becomes massive and the complex scalar field has lost one degree of freedom and becomes a real scalar field $\eta(x)$. Since the new Lagrangian density is obtained by fixing the gauge, it does not have a gauge freedom any more. The peculiarity of this gauge fixing is that the $\xi(x) = 0$ has nothing to do with the redundancy of the gauge field A_μ itself. In fact, this gauge fixing does not reduce the number of freedoms of the gauge field. If all the physical observables can be reproduced by this gauge fixing, then this gauge fixing can be justified [53].

4.8.3 What Is Physics Behind Higgs Mechanism?

In the Higgs mechanism, the gauge field acquires the mass by the spontaneous symmetry breaking of the Higgs fields. This is mainly because one takes the double well type potential for a scalar field and the potential has a minimum at

$$|\phi(x)| = \lambda.$$

Therefore, the kinetic energy part of the Higgs field which couples with the gauge field A_μ becomes a constant λ. Therefore, this part has lost a coupling between the Higgs and the gauge fields. However, this mass-term-like interactions should be still gauge invariant and it cannot be considered as a mass term of the gauge field. The mass term of the gauge field is obtained by fixing the gauge at the Lagrangian density level. Therefore, after fixing the gauge, there is no gauge freedom any more in the Lagrangian density.

Normally, one fixes the gauge at the point where one calculates the physical observables. In terms of physics, the gauge fixing becomes necessary when one wishes to determine the gauge field solutions from the equations of motion. This is clear since the gauge field has redundant variables at the level of solving the equations of motion. By fixing the gauge, one can determine the gauge field, but of course physical observables should not depend on the choice of the gauge fixing.

At the same time, one should be careful for the choice of the gauge fixing. There is no guarantee that any kind of the gauge fixing can give the same physical observables. At least, one should examine whether the gauge choice one takes can indeed reproduce the right physical observables or not.

Here, the $U(1)$ gauge field is discussed, but it is straightforward to extend it to the non-abelian case. In the non-abelian gauge field theory, gauge fields themselves are not gauge invariant, and therefore they cannot be a physical observable. However, after the spontaneous symmetry breaking of the complex scalar fields, the non-abelian gauge fields become physical observables after the gauge fixing as Weinberg-Salam model shows. The physics behind this statement is difficult to understand, and in this respect, the Higgs mechanism should be understood more in depth in future.