

2-3 例題

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(1) 直角座標での Newton 方程式

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x)$$

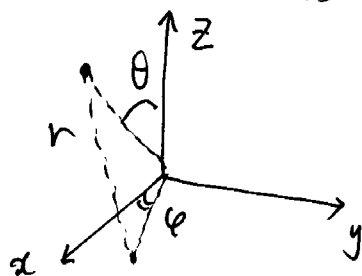
(Lagrange 方程式)

(Newton 方程式)

$$\left\{ \begin{array}{l} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = -\frac{\partial U}{\partial y} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \Rightarrow m\ddot{z} = -\frac{\partial U}{\partial z} \end{array} \right.$$

Lagrange 方程式 と Newton 方程式 は全く同じ。

(2) 極座標の場合



$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} \dot{x} = \dot{r} \sin \theta \cos \varphi + r \dot{\theta} \cos \theta \cos \varphi - r \dot{\varphi} \sin \theta \sin \varphi \\ \dot{y} = \dot{r} \sin \theta \sin \varphi + r \dot{\theta} \cos \theta \sin \varphi + r \dot{\varphi} \sin \theta \cos \varphi \\ \dot{z} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \end{cases}$$

計算より

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2$$

より

(1) 式を代入すると

よって Lagrangian は

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U(r)$$

 r, θ, ϕ に対する Lagrange の方程式は

$$r) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\therefore m \ddot{r} = m r \dot{\theta}^2 + m r \sin^2 \theta \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\theta) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\therefore \frac{d}{dt} (m r^2 \dot{\theta}) = m r^2 \sin \theta \cos \theta \dot{\phi}^2$$

 $\phi)$

$$\frac{d}{dt} (m r^2 \sin^2 \theta \dot{\phi}) = 0$$

【保存量】

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$$(a) \quad \frac{d}{dt} (m r^2 \sin^2 \theta \dot{\varphi}) = 0 \quad \text{よ}$$

$$\boxed{m r^2 \sin^2 \theta \dot{\varphi} = C_0} \quad C_0 \text{ は定数}$$

$$(b) \quad \frac{d}{dt} (r^2 \dot{\theta}) = r^2 \sin \theta \cos \theta \dot{\varphi}^2 = \frac{\cos \theta C_0^2}{m^2 r^2 \sin^3 \theta}$$

この両辺に $(r^2 \dot{\theta})$ をかけると

$$(r^2 \dot{\theta}) \frac{d}{dt} (r^2 \dot{\theta}) = \frac{\dot{\theta} \cos \theta}{m^2 \sin^3 \theta} C_0^2$$

$$\therefore \frac{1}{2} \frac{d}{dt} (r^2 \dot{\theta})^2 = - \frac{d}{dt} \left(\frac{C_0^2}{2m^2} \frac{1}{\sin^2 \theta} \right)$$

よ、

$$\boxed{(r^2 \dot{\theta})^2 = - \frac{C_0^2}{2m^2} \frac{1}{\sin^2 \theta} + C_1} \quad C_1 \text{ は定数}$$

(C_1 は定数)

よ、

$$\begin{cases} \dot{\varphi} = \frac{C_0}{m r^2 \sin^2 \theta} \\ \dot{\theta}^2 = - \frac{C_0^2}{r^4 m^2 \sin^2 \theta} + \frac{C_1}{r^4} \end{cases} \quad \text{と表す}$$

【 r に対する方程式 】

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$$m\ddot{r} = mr\dot{\theta}^2 + mrsin^2\theta \dot{\varphi}^2 - \frac{\partial U}{\partial r}$$

この式で $\dot{\varphi}$ と $\dot{\theta}^2$ を消去する

$$\begin{cases} \dot{\varphi} = \frac{C_0}{mr^2 sin^2\theta} \\ \dot{\theta}^2 = -\frac{C_0^2}{r^4 m^2 sin^2\theta} + \frac{C_1}{r^4} \end{cases} \quad \text{より}$$

$$m\ddot{r} = mr \left[-\frac{C_0^2}{r^4 m^2 sin^2\theta} + \frac{C_1}{r^4} \right] + mr sin^2\theta \left[\frac{C_0^2}{m^2 r^4 sin^4\theta} \right] - \frac{\partial U}{\partial r}$$

これを

$$m\ddot{r} = \frac{mC_1}{r^3} - \frac{\partial U}{\partial r}$$

と表す

この式に \dot{r} をかけてエネルギー積分をとる

$$\frac{d}{dt} \left(\frac{1}{2} m \dot{r}^2 \right) = - \frac{d}{dt} \left(\frac{mC_1}{2r^2} \right) - \frac{d}{dt} U(r)$$

よって

$$\frac{1}{2} m \dot{r}^2 + \frac{mC_1}{2r^2} + U(r) = E$$

と表す

E は定数

$$よって C_1 = \frac{l^2}{m^2} \quad \text{と表す}$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2} + U(r)$$

と表す

(3) 2次元極座標での Newton 方程式

$$\underline{L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)}$$

Lagrange 方程式 (2)

$$r \text{ について } \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\therefore \boxed{m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial U}{\partial r}}$$

$$\theta \text{ について } \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\therefore \boxed{\frac{d}{dt} (mr^2 \dot{\theta}) = 0}$$

よって

$$\boxed{L \equiv mr^2 \dot{\theta}}$$

角運動量
(保存量)

2次元で

$$\boxed{m\ddot{r} = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r}}$$

と表す

【注】 角運動量 L (2次元)

$$\boxed{L = \mathbf{r} \times m \mathbf{\dot{r}}}$$

2次元で

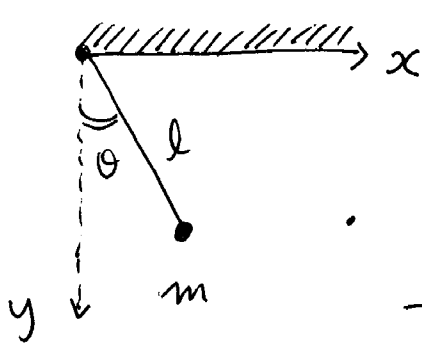
$$\text{よって } L_z = (\mathbf{r} \times m \mathbf{\dot{r}})_z = m(x\dot{y} - y\dot{x})$$

$$= m \left\{ r\dot{r} \cos\theta \sin\theta + r^2 \dot{\theta} \cos^2\theta - (r\dot{r} \sin\theta \cos\theta - r^2 \dot{\theta} \sin^2\theta) \right\}$$

$$= mr^2 \dot{\theta} = L //$$

(4) 単振り子

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$$\begin{cases} x = l \sin \theta \\ y = l \cos \theta \end{cases} \quad \begin{cases} \dot{x} = l \dot{\theta} \cos \theta \\ \dot{y} = -l \dot{\theta} \sin \theta \end{cases}$$

• 質点 m の運動エネルギー T は

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\theta}^2$$

• 質点に及ぶ重力は

$$U = -mgy = -mgl \cos \theta$$

∴ Lagrangian L は $L = T - U$

$$\therefore \boxed{L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta}$$

Lagrange の方程式は θ に対して

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \quad \therefore m l^2 \ddot{\theta} = -mgl \sin \theta$$

$$\therefore \boxed{\ddot{\theta} + \frac{g}{l} \sin \theta = 0}$$

微小振動の時 \longleftrightarrow

$$\theta \ll 1 \quad \sin \theta \approx \theta$$

$$\therefore \boxed{\ddot{\theta} + \omega^2 \theta = 0} \quad \omega = \sqrt{\frac{g}{l}}$$

一般解は $\theta = A \sin \omega t + B \cos \omega t$