力学に必要な数学公式集

1. 微分：

(a) 定義：$y = f(x)$ の時、Δx を微少量として $\Delta y \equiv f(x+\Delta x) - f(x)$ とする.

$$f'(x) \equiv \frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(b) 絶対覚えるべき式：

$$(x^\alpha)' = \alpha x^{\alpha-1} \qquad (\alpha：\text{任意の実数})$$

(c) 合成微分 *** (知らないと困る式) *** : $y = f\left(x(t)\right)$ の時

$$\frac{df\left(x(t)\right)}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \equiv \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t} = \frac{dy}{dx}\frac{dx}{dt}$$

但し、$\Delta y \equiv f(x(t+\Delta t)) - f(x(t)) = f(x+\Delta x) - f(x), \quad \Delta x \equiv x(t+\Delta t) - x(t)$

(d) 絶対覚えるべき式： $f(x) = (ax+b)^\alpha$ の時、

$$f'(x) = a\alpha(ax+b)^{\alpha-1}$$

(e) Taylor 展開： $n! \equiv 1 \times 2 \times \cdots \times n$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + ... + \frac{1}{n!}f^{(n)}(x)h^n + ..$$

$x = 0$ の時：

$$f(h) = f(0) + f'(0)h + \frac{1}{2}f''(0)h^2 + ... + \frac{1}{n!}f^{(n)}(0)h^n + ..$$

2. 偏微分 ：

(a) 偏微分の定義：関数 $f(x,y)$ に対して

$$\frac{\partial f(x,y)}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \qquad (y\text{をとめて}x\text{で微分})$$

$$\frac{\partial f(x,y)}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \qquad (x\text{をとめて}y\text{で微分})$$

(b) 全微分：関数 $f\left(x(t), y(t), t\right)$

但し $\Delta x \equiv x(t+\Delta t) - x(t), \quad \Delta y \equiv y(t+\Delta t) - y(t)$

この時、$\Delta t \to 0$ で $\Delta x \to 0, \quad \Delta y \to 0$

$$\begin{aligned}
\frac{df\left(x(t), y(t), t\right)}{dt} &\equiv \lim_{\Delta t \to 0} \frac{f\left(x(t+\Delta t), y(t+\Delta t), t+\Delta t\right) - f\left(x(t), y(t), t\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\{ \frac{f\left(x(t+\Delta t), y(t+\Delta t), t+\Delta t\right) - f\left(x(t), y(t+\Delta t), t+\Delta t\right)}{\Delta t} \right. \\
&\qquad + \frac{f\left(x(t), y(t+\Delta t), t+\Delta t\right) - f\left(x(t), y(t), t+\Delta t\right)}{\Delta t} \\
&\qquad \left. + \frac{f\left(x(t), y(t), t+\Delta t\right) - f\left(x(t), y(t), t\right)}{\Delta t} \right\} \\
&= \lim_{\Delta t \to 0} \left\{ \frac{f\left(x(t+\Delta t), y(t+\Delta t), t+\Delta t\right) - f\left(x(t), y(t+\Delta t), t+\Delta t\right)}{\Delta x} \frac{\Delta x}{\Delta t} \right. \\
&\qquad + \frac{f\left(x(t), y(t+\Delta t), t+\Delta t\right) - f\left(x(t), y(t), t+\Delta t\right)}{\Delta y} \frac{\Delta y}{\Delta t} \\
&\qquad \left. + \frac{f\left(x(t), y(t), t+\Delta t\right) - f\left(x(t), y(t), t\right)}{\Delta t} \right\} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}
\end{aligned}$$

3. ベクトル：

$$\mathbf{a} = (a_x, a_y, a_z), \qquad \mathbf{b} = (b_x, b_y, b_z)$$

・絶対値：$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

・内積：

$$\mathbf{a} \cdot \mathbf{b} \equiv a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}||\mathbf{b}| \cos\theta$$

・直交性：

$\mathbf{a} \cdot \mathbf{b} = 0$ の時、$\mathbf{a}$ と $\mathbf{b}$ は直交するという

・外積：

$$\mathbf{a} \times \mathbf{b} \equiv (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{b} \times \mathbf{a}$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta, \qquad \mathbf{a} \times \mathbf{a} = 0$$

・ベクトルの公式：

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\,\boldsymbol{a}\,||\,\boldsymbol{b}\,| \cos\theta$$

$$|\,\boldsymbol{a} \times \boldsymbol{b}\,| = |\,\boldsymbol{a}\,||\,\boldsymbol{b}\,| \sin\theta$$

$$(\boldsymbol{a} \times \boldsymbol{b})^2 = |\,\boldsymbol{a}\,|^2 |\,\boldsymbol{b}\,|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

$$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{|\,\boldsymbol{a}\,|^2 + |\,\boldsymbol{b}\,|^2 + 2\,|\,\boldsymbol{a}\,||\,\boldsymbol{b}\,| \cos\theta}$$

・単位ベクトル： $\mathbf{n} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

(1) 直交座標： $\mathbf{e}_x$， $\mathbf{e}_y$， $\mathbf{e}_z$

$$\mathbf{e}_x = (1,0,0), \qquad \mathbf{e}_y = (0,1,0) \qquad \mathbf{e}_z = (0,0,1)$$

$$\begin{cases} |\mathbf{e}_x| = 1, \quad |\mathbf{e}_y| = 1, \quad |\mathbf{e}_z| = 1 \\ \mathbf{e}_x \cdot \mathbf{e}_y = 0, \quad \mathbf{e}_y \cdot \mathbf{e}_z = 0, \quad \mathbf{e}_z \cdot \mathbf{e}_x = 0 \\ \mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z, \quad \mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x, \quad \mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y \end{cases}$$

(2) 円筒座標： $\mathbf{e}_r$， $\mathbf{e}_\varphi$， $\mathbf{e}_z$

$$\begin{cases} |\mathbf{e}_r| = 1, \quad |\mathbf{e}_\varphi| = 1, \quad |\mathbf{e}_z| = 1 \\ \mathbf{e}_r = \cos\varphi\ \mathbf{e}_x + \sin\varphi\ \mathbf{e}_y \\ \mathbf{e}_\varphi = -\sin\varphi\ \mathbf{e}_x + \cos\varphi\ \mathbf{e}_y \end{cases}$$

$$\begin{cases} \mathbf{e}_r \cdot \mathbf{e}_\varphi = 0, \quad \mathbf{e}_\varphi \cdot \mathbf{e}_z = 0, \quad \mathbf{e}_z \cdot \mathbf{e}_r = 0 \\ \mathbf{e}_r \times \mathbf{e}_\varphi = \mathbf{e}_z, \quad \mathbf{e}_\varphi \times \mathbf{e}_z = \mathbf{e}_r, \quad \mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\varphi \end{cases}$$

(3) 極座標： $\mathbf{e}_r$， $\mathbf{e}_\theta$， $\mathbf{e}_\varphi$

$$\mathbf{e}_r = \frac{\mathbf{r}}{r}$$

$$\begin{cases} |\mathbf{e}_r| = 1, \quad |\mathbf{e}_\theta| = 1, \quad |\mathbf{e}_\varphi| = 1 \\ \mathbf{e}_r = \sin\theta\cos\varphi\ \mathbf{e}_x + \sin\theta\sin\varphi\ \mathbf{e}_y + \cos\theta\ \mathbf{e}_z \\ \mathbf{e}_\theta = \cos\theta\cos\varphi\ \mathbf{e}_x + \cos\theta\sin\varphi\ \mathbf{e}_y - \sin\theta\ \mathbf{e}_z \\ \mathbf{e}_\varphi = -\sin\varphi\ \mathbf{e}_x + \cos\varphi\ \mathbf{e}_y \end{cases}$$

$$\begin{cases} \mathbf{e}_x = \sin\theta\cos\varphi\ \mathbf{e}_r + \cos\theta\cos\varphi\ \mathbf{e}_\theta - \sin\varphi\ \mathbf{e}_\varphi \\ \mathbf{e}_y = \sin\theta\sin\varphi\ \mathbf{e}_r + \cos\theta\sin\varphi\ \mathbf{e}_\theta + \cos\varphi\ \mathbf{e}_\varphi \\ \mathbf{e}_z = \cos\theta\ \mathbf{e}_r - \sin\theta\ \mathbf{e}_\theta \end{cases}$$

$$\begin{cases} \mathbf{e}_r \cdot \mathbf{e}_\theta = 0, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\varphi = 0, \quad \mathbf{e}_\varphi \cdot \mathbf{e}_r = 0 \\ \mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\varphi, \quad \mathbf{e}_\theta \times \mathbf{e}_\varphi = \mathbf{e}_r, \quad \mathbf{e}_\varphi \times \mathbf{e}_r = \mathbf{e}_\theta \end{cases}$$

・ベクトル $\mathbf{A}$ はどの座標系でも分解できる：

$$\begin{cases} \mathbf{A} = A_x\ \mathbf{e}_x + A_y\ \mathbf{e}_y + A_z\ \mathbf{e}_z \\ \mathbf{A} = A_r\ \mathbf{e}_r + A_\theta\ \mathbf{e}_\theta + A_\varphi\ \mathbf{e}_\varphi \end{cases}$$

この時、$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ を $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ でかくと

$$\begin{cases} A_r = A_x \sin\theta\cos\varphi + A_y \sin\theta\sin\varphi + A_z\cos\theta \\ A_\theta = A_x \cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\theta \\ A_\varphi = -A_x\sin\varphi + A_y\cos\varphi \end{cases}$$

・単位ベクトルの時間微分：

1. 直交座標：

$$\dot{\mathbf{e}}_x = \dot{\mathbf{e}}_y = \dot{\mathbf{e}}_z = 0$$

2. 円筒座標：

$$\begin{cases} \dot{\mathbf{e}}_r = \dot{\varphi}\ \mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi = -\dot{\varphi}\ \mathbf{e}_r \end{cases}$$

3. 極座標：

$$\begin{cases} \dot{\mathbf{e}}_r = \dot{\theta}\ \mathbf{e}_\theta + \dot{\varphi}\sin\theta\ \mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\theta = -\dot{\theta}\ \mathbf{e}_r + \dot{\varphi}\cos\theta\ \mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi = -\dot{\varphi}\sin\theta\ \mathbf{e}_r - \dot{\varphi}\cos\theta\ \mathbf{e}_\theta \end{cases}$$

・$\mathbf{r} = r\mathbf{e}_r$ の時間微分

$$\dot{\mathbf{r}} = \dot{r}\ \mathbf{e}_r + r\dot{\theta}\ \mathbf{e}_\theta + r\dot{\varphi}\sin\theta\ \mathbf{e}_\varphi$$

$$\begin{cases} \ddot{\mathbf{r}} = a_r\ \mathbf{e}_r + a_\theta\ \mathbf{e}_\theta + a_\varphi\ \mathbf{e}_\varphi \\ a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta \\ a_\varphi = r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta \\ \quad = \dfrac{1}{r\sin\theta}\dfrac{d}{dt}(r^2\dot{\varphi}\sin^2\theta) \end{cases}$$

4. 複素数：

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Euler の公式

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\begin{cases} \cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{cases}$$

5. 三角関数：

$$\begin{cases} \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) = \cos x \cos y \mp \sin x \cos y \\ \tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \end{cases}$$

$$\begin{aligned} a\sin\theta + b\cos\theta &= \sqrt{a^2+b^2}\sin(\theta+\alpha) \\ &= \sqrt{a^2+b^2}(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \end{aligned}$$

$$\left(\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \right)$$

$$\begin{cases} \sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2} \\ \sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2} \\ \cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2} \\ \cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2} \end{cases}$$

$$\begin{cases} \sin A \sin B = -\dfrac{1}{2}\left[\cos(A+B) - \cos(A-B)\right] \\ \sin A \cos B = \dfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \\ \cos A \sin B = \dfrac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \\ \cos A \cos B = \dfrac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \end{cases}$$

$$\begin{cases} \sin 2\theta = 2\sin\theta\cos\theta \\ \sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta) \end{cases}$$

Taylor 展開

$$\begin{cases} \sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + \ldots \\ \cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + \ldots \\ \tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 + \ldots \end{cases}$$

微分

$$\begin{cases} \dfrac{d\sin x}{dx} = \cos x \\ \dfrac{d\cos x}{dx} = -\sin x \\ \dfrac{d\tan x}{dx} = \dfrac{1}{\cos^2 x} \end{cases}$$

6. 指数関数と対数関数：

$$e^x \cdot e^y = e^{(x+y)}, \qquad (e^x)^y = e^{xy}, \qquad e = 2.7182818$$

$$\log xy = \log x + \log y, \qquad \log x^y = y \log x, \qquad \ln x \equiv \log_e x$$

Taylor 展開

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\left(\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots \right)$$

微分

$$\begin{cases} \dfrac{de^x}{dx} = e^x \\ \dfrac{d\ln x}{dx} = \dfrac{1}{x} \end{cases}$$