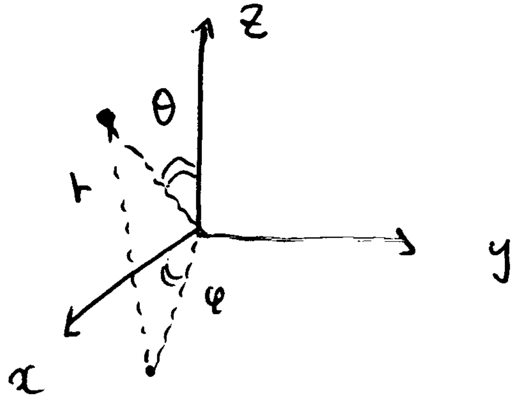
2-4 座標系

$$
\begin{cases}
\text{◎ デカルト座標} & (x, y, z) \\
\text{◎ 極座標} & (r, \theta, \varphi)
\end{cases}
$$



$$
\begin{cases}
x = r\sin\theta\cos\varphi \\
y = r\sin\theta\sin\varphi \\
z = r\cos\theta
\end{cases}
$$

[θ, φ の範囲]

$$
\boxed{0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi}
$$

(θは 2πまではとれない!!)

$$
\begin{cases}
dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi \\
dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi \\
dz = \cos\theta\, dr - r\sin\theta\, d\theta
\end{cases}
$$

$$
\begin{aligned}
(ds)^2 &\equiv (dx)^2 + (dy)^2 + (dz)^2 \\
&= (dr)^2 + r^2 (d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2
\end{aligned}
$$

[付録]

$$
\begin{cases}
\dfrac{dx}{dt} = \sin\theta\cos\varphi\dfrac{dr}{dt} + r\cos\theta\cos\varphi\dfrac{d\theta}{dt} - r\sin\theta\sin\varphi\dfrac{d\varphi}{dt} \\[2ex]
\dfrac{dy}{dt} = \sin\theta\sin\varphi\dfrac{dr}{dt} + r\cos\theta\sin\varphi\dfrac{d\theta}{dt} + r\sin\theta\cos\varphi\dfrac{d\varphi}{dt} \\[2ex]
\dfrac{dz}{dt} = \cos\theta\dfrac{dr}{dt} - r\sin\theta\dfrac{d\theta}{dt}
\end{cases}
$$

ここで

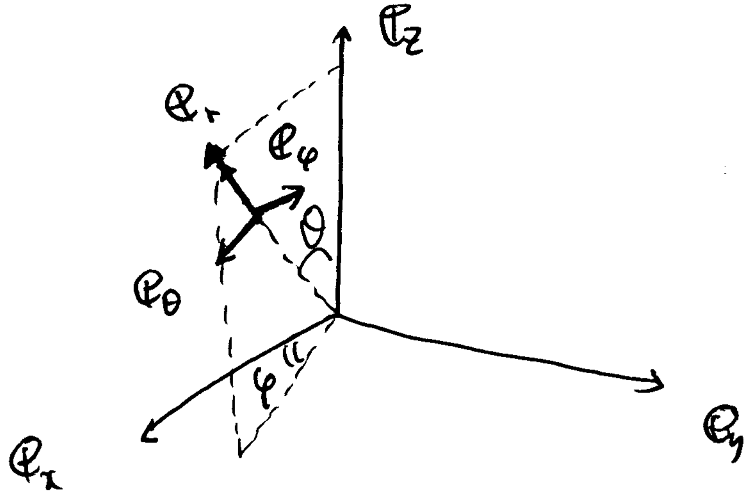
$$
\begin{cases}
\mathbf{e}_x \equiv \dfrac{dx}{dt} \Big/ \left|\dfrac{dx}{dt}\right| \\[2ex]
\mathbf{e}_y \equiv \dfrac{dy}{dt} \Big/ \left|\dfrac{dy}{dt}\right| \\[2ex]
\mathbf{e}_z \equiv \dfrac{dz}{dt} \Big/ \left|\dfrac{dz}{dt}\right|
\end{cases}
$$

(絶対値)

$$
\begin{cases}
\mathbf{e}_r \equiv \dfrac{dr}{dt} \Big/ \left|\dfrac{dr}{dt}\right| \\[2ex]
\mathbf{e}_\theta \equiv \dfrac{r\,d\theta}{dt} \Big/ \left|\dfrac{r\,d\theta}{dt}\right| \\[2ex]
\mathbf{e}_\varphi \equiv \dfrac{r\sin\theta\,d\varphi}{dt} \Big/ \left|r\sin\theta\dfrac{d\varphi}{dt}\right|
\end{cases}
$$

[単位ベクトル]

$$
\begin{cases}
[\,\boldsymbol{e}_x,\ \boldsymbol{e}_y,\ \boldsymbol{e}_z\,] & \text{(デカルト座標)} \\
[\,\boldsymbol{e}_r,\ \boldsymbol{e}_\theta,\ \boldsymbol{e}_\varphi\,] & \text{(極座標)}
\end{cases}
$$



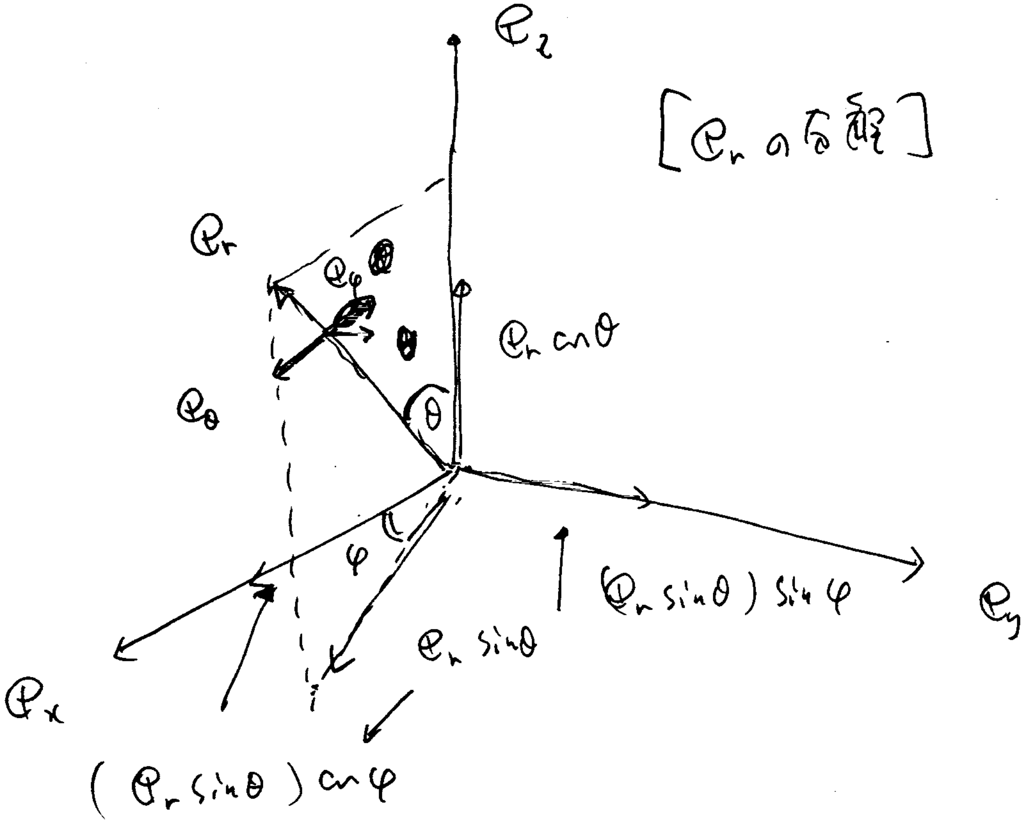
$$
\begin{cases}
\boldsymbol{e}_x = \sin\theta\cos\varphi\ \boldsymbol{e}_r + \cos\theta\cos\varphi\ \boldsymbol{e}_\theta - \sin\varphi\ \boldsymbol{e}_\varphi \\
\boldsymbol{e}_y = \sin\theta\sin\varphi\ \boldsymbol{e}_r + \cos\theta\sin\varphi\ \boldsymbol{e}_\theta + \cos\varphi\ \boldsymbol{e}_\varphi \\
\boldsymbol{e}_z = \cos\theta\ \boldsymbol{e}_r - \sin\theta\ \boldsymbol{e}_\theta
\end{cases}
$$

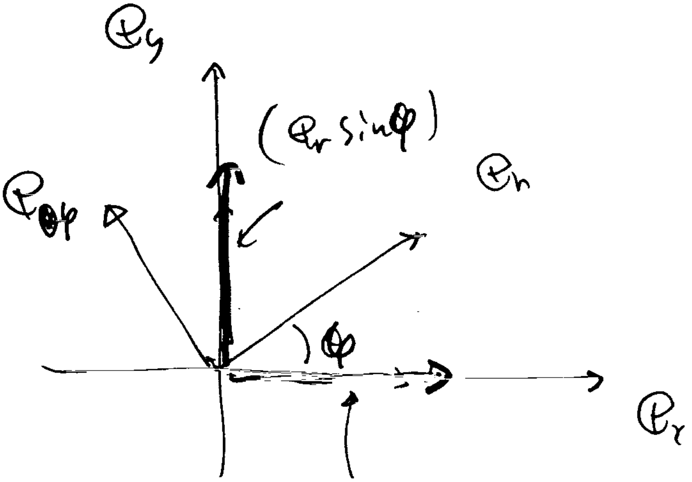
$\Downarrow$ 逆にといて

$$
\begin{cases}
\boldsymbol{e}_r = \sin\theta\cos\varphi\ \boldsymbol{e}_x + \sin\theta\sin\varphi\ \boldsymbol{e}_y + \cos\theta\ \boldsymbol{e}_z \\
\boldsymbol{e}_\theta = \cos\theta\cos\varphi\ \boldsymbol{e}_x + \cos\theta\sin\varphi\ \boldsymbol{e}_y - \sin\theta\ \boldsymbol{e}_z \\
\boldsymbol{e}_\varphi = -\sin\varphi\ \boldsymbol{e}_x + \cos\varphi\ \boldsymbol{e}_y
\end{cases}
$$

(付録)

[e_r の分解]





$$\left(\begin{aligned} e_x &= e_r \cos\varphi - e_\varphi \sin\varphi \\ e_y &= e_r \sin\varphi + e_\varphi \cos\varphi \end{aligned}\right)$$

◎ $\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\varphi$ の直交性

$$\begin{cases} \mathbb{e}_r \cdot \mathbb{e}_\theta = 0 \\ \mathbb{e}_r \cdot \mathbb{e}_\varphi = 0 \\ \mathbb{e}_\theta \cdot \mathbb{e}_\varphi = 0 \end{cases} \qquad \begin{cases} \mathbb{e}_r \times \mathbb{e}_\theta = \mathbb{e}_\varphi \\ \mathbb{e}_\theta \times \mathbb{e}_\varphi = \mathbb{e}_r \\ \mathbb{e}_\varphi \times \mathbb{e}_r = \mathbb{e}_\theta \end{cases}$$

◎ 極座標での速度

$$\underline{\mathbb{r} = r\,\mathbb{e}_r}$$

$$\underline{\dot{\mathbb{r}} = \dot{r}\,\mathbb{e}_r + r\,\dot{\mathbb{e}}_r}$$

ところが

$$\begin{cases} \dot{\mathbb{e}}_r = \dot{\theta}\,\mathbb{e}_\theta + \dot{\varphi}\sin\theta\,\mathbb{e}_\varphi \\ \dot{\mathbb{e}}_\theta = -\dot{\theta}\,\mathbb{e}_r + \dot{\varphi}\cos\theta\,\mathbb{e}_\varphi \\ \dot{\mathbb{e}}_\varphi = -\dot{\varphi}\sin\theta\,\mathbb{e}_r - \dot{\varphi}\cos\theta\,\mathbb{e}_\theta \end{cases}$$

(ここで $\dot{\mathbb{e}}_x = \dot{\mathbb{e}}_y = \dot{\mathbb{e}}_z = 0$)

これより

$$\boxed{\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + r\dot{\varphi}\sin\theta\,\mathbf{e}_\varphi}$$

◎ 極座標での加速度

$$\ddot{\mathbf{r}} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + r\ddot{\theta}\,\mathbf{e}_\theta + r\dot{\theta}\,\dot{\mathbf{e}}_\theta$$
$$+ \dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta\,\mathbf{e}_\varphi + r\dot{\varphi}\dot{\theta}\cos\theta\,\mathbf{e}_\varphi$$
$$+ r\dot{\varphi}\sin\theta\,\dot{\mathbf{e}}_\varphi$$

∴

$$\boxed{\begin{aligned}\ddot{\mathbf{r}} &= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\mathbf{e}_r \\ &+ \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\mathbf{e}_\theta \\ &+ \left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right)\mathbf{e}_\varphi\end{aligned}}$$

◎ 円筒座標：(r, φ, z)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$

$$\underline{(ds)^2 = (dr)^2 + r^2(d\varphi)^2 + (dz)^2}$$

$$\underline{\mathbf{r} = r\,\mathbf{e}_r + z\,\mathbf{e}_z}$$

$$\boxed{\begin{aligned} \dot{\mathbf{e}}_r &= \dot{\varphi}\,\mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi &= -\dot{\varphi}\,\mathbf{e}_r \end{aligned}}$$

を使うと

$$\boxed{\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\,\mathbf{e}_r + r\dot{\varphi}\,\mathbf{e}_\varphi + \dot{z}\,\mathbf{e}_z \\ \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\varphi}^2)\,\mathbf{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\,\mathbf{e}_\varphi + \ddot{z}\,\mathbf{e}_z \end{aligned}}$$

『一般座標』

物理の記述はどの座標系でも

同等なハズである

例えば (x, y, z) (r, θ, φ), (r, φ, z) …

$\boxed{\text{1ケの質点}}$ ⟺ 3ケの自由度があるという

$(x, y, z) \Rightarrow \boxed{(q_1, q_2, q_3)}$

一般座標という

$q_k \quad (k=1, 2, 3)$ と書く

(x, y, z) は (q_1, q_2, q_3) の関数

$$\begin{cases} x = x(q_1\, q_2\, q_3) \\ y = y(q_1\, q_2\, q_3) \\ z = z(q_1\, q_2\, q_3) \end{cases}$$

$$x = x(q_1, q_2, q_3)$$

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\cdot\frac{dq_1}{dt} + \frac{\partial x}{\partial q_2}\cdot\frac{dq_2}{dt} + \frac{\partial x}{\partial q_3}\cdot\frac{dq_3}{dt}$$

$$\boxed{\dot{x} = \sum_{k=1}^{3} \frac{\partial x}{\partial q_k}\cdot \dot{q}_k}$$ となる

これより

$$\boxed{\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k}} \quad (k=1,2,3)$$

同様にして

$$\boxed{\begin{aligned} \frac{\partial \dot{y}}{\partial \dot{q}_k} &= \frac{\partial y}{\partial q_k} \\ \frac{\partial \dot{z}}{\partial \dot{q}_k} &= \frac{\partial z}{\partial q_k} \end{aligned}}$$

● Newton方程式を一般座標で書き直す

[x, y, z 座標]

$$\begin{cases} m\ddot{x} = -\dfrac{\partial U}{\partial x} \\ m\ddot{y} = -\dfrac{\partial U}{\partial y} \\ m\ddot{z} = -\dfrac{\partial U}{\partial z} \end{cases}$$

(Newton方程式)

● 次の物理量を定義する

$$I_k = m\left(\ddot{x}\frac{\partial x}{\partial q_k} + \ddot{y}\frac{\partial y}{\partial q_k} + \ddot{z}\frac{\partial z}{\partial q_k}\right) \quad (k=1,2,3)$$

Newton方程式を使ってかき直すと

$$I_k = -\left(\frac{\partial U}{\partial x}\frac{\partial x}{\partial q_k} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_k} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_k}\right)$$

$$= -\frac{\partial U}{\partial q_k}$$

$$\therefore \boxed{I_k = -\frac{\partial U}{\partial q_k}} \quad (k=1,2,3)$$

$L = T - U$ として

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \quad (k=1,2,3)$$

を示す。

← Newton 方程式

(q_1, q_2, q_3) が極座標のとき (r, θ, φ)

$$\begin{cases} T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta \cdot \dot{\varphi}^2\right) \\ U = U(r) \end{cases}$$

ところが

$$\ddot{x}\,\frac{\partial x}{\partial q_k} = \frac{d}{dt}\left(\dot{x}\underbrace{\frac{\partial x}{\partial q_k}}_{\Downarrow\ \frac{\partial \dot{x}}{\partial q_k}}\right) - \dot{x}\underbrace{\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right)}_{\text{微分の順序を入れかえると}\ \frac{\partial \dot{x}}{\partial q_k}}$$

$$\therefore\ \ddot{x}\,\frac{\partial x}{\partial q_k} = \frac{d}{dt}\left(\dot{x}\,\frac{\partial \dot{x}}{\partial \dot{q}_k}\right) - \dot{x}\,\frac{\partial \dot{x}}{\partial q_k}$$

$$= \frac{1}{2}\,\frac{d}{dt}\left(\frac{\partial \dot{x}^2}{\partial \dot{q}_k}\right) - \frac{1}{2}\left(\frac{\partial \dot{x}^2}{\partial q_k}\right)$$

よって

$$I_k = m\left(\ddot{x}\,\frac{\partial x}{\partial q_k} + \ddot{y}\,\frac{\partial y}{\partial q_k} + \ddot{z}\,\frac{\partial z}{\partial q_k}\right)$$

$$= \frac{d}{dt}\,\frac{\partial}{\partial \dot{q}_k}\left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right]$$

$$- \frac{\partial}{\partial q_k}\left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right]$$

運動エネルギーを T とすると

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

従って

$$I_k = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial \dot{q}_k}$$

ところが $I_k = -\frac{\partial U}{\partial q}$ であったから

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k}$$

$$\therefore \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial}{\partial q_k} (T - U)$$

ここで $U(x, y, z)$ は座標のみによって $\dot{q}_k$ にはよらないと仮定する

$$\boxed{\frac{\partial U}{\partial \dot{q}_k} = 0} \leftarrow 仮定$$

よって

$$\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_k} = \frac{\partial}{\partial q_k}(T-U)$$

ここで $L \equiv T-U$ と定義すると

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}} \quad (k=1,2,3)$$

↓

Lagrange方程式という

これは

$$\boxed{m\ddot{\mathbf{r}} = -\nabla U}$$

を

一般座標 q_k で書き直した。

【注】

Lagrange 方程式は 3つある

$$
\begin{cases}
\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_1} = \dfrac{\partial L}{\partial q_1} \\[2ex]
\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_2} = \dfrac{\partial L}{\partial q_2} \\[2ex]
\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_3} = \dfrac{\partial L}{\partial q_3}
\end{cases}
$$

この場合、(q_1, q_2, q_3) は

どの座標系でもよい

$q_1 = x$, $q_2 = y$, $q_3 = z$ ととると

通常の Newton 方程式になる

[Examples]

① 1次元調和振動子

$$T = \frac{1}{2} m \dot{x}^2 \ , \ U = \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Lagrange方程式 $\quad \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{x}} = \dfrac{\partial L}{\partial x}$

$$\Downarrow$$

$$\underline{m \ddot{x} = -kx}$$

② 3次元自由粒子 (極座標)

$$\begin{cases} T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \cdot \dot{\varphi}^2 \right) \\ U = 0 \end{cases} \quad \therefore \ L = T - U = T$$

$$\therefore \ L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \cdot \dot{\varphi}^2 \right)$$

(a) r」 $\quad \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{r}} = \dfrac{\partial L}{\partial r}$

$$\underline{m \ddot{r} = m r \dot{\theta}^2 + m r \sin^2\theta \, \dot{\varphi}^2}$$

(b) θ 」 $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\therefore \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\dot{\varphi}^2\sin\theta\cos\theta$$

$$\underline{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\dot{\varphi}^2\sin\theta\cos\theta = 0}$$

(c) φ 」 $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

$$\therefore \frac{d}{dt}\left(mr^2\sin^2\theta\cdot\dot{\varphi}\right) = 0$$

$$\Downarrow$$

$$mr^2\sin^2\theta\cdot\dot{\varphi} = \text{定数}$$

上式3つより加速度を求めると

$$\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2 \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta \\ a_\varphi = r\sin\theta\,\ddot{\varphi} + 2\sin\theta\,\dot{r}\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} \end{cases}$$

[$\ddot{r}$ で求めたものと一致する]