

**Symmetry and Its Breaking
in
Quantum Field Theory**

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PREFACE

Physics is always difficult, though it is extremely interesting. Many times I thought I understood it sufficiently profoundly, but after some time, it turned out that my understanding of physics was far from satisfactory. In particular, field theory has special complexities which may not be common to other fields of research. The symmetry and its breaking are most exotic and sometimes almost mysterious to even those who can normally understand the basic physics in a clear manner.

In this textbook, I focused on presenting a simple and clear picture of the symmetry and its breaking in quantum field theory. For this purpose, I explained physics of elementary field theory of fermions interacting by gauge fields as well as by four body fermion fields. In this respect, the interpretation of the basic field theory is repeatedly done such that physicists including graduate students may understand the essential points of the symmetry breaking in this textbook.

Also, this book is intended for researchers who look for the basic problems in their investigations. In many fields of research, field theory is used as a computational tool. In this regard, I present some elaborate technical tools which are quite useful and sometimes incentive for new ideas in fundamental researches.

In physics, deeper understanding is more important than quicker understanding. In particular, graduate students should realize that, if someone else can understand the basic physics very quickly, then he is most likely a good interpreter of the textbook knowledge. Slow but deep understanding of physics is most important since it should definitely take much time to understand physics in depth. The shortest path of understanding physics is only one of many paths, and interesting physics may well be found in the paths which are far from the shortest one.

Physics must be simple once we understand it all. For example, I believe that QCD can surely describe the strong interaction physics. However, it may well be difficult to justify the perturbative calculation of the interactions between quarks, unless the gauge independence of the quark-quark interactions is guaranteed. In other words, when the unperturbed as well as interaction Hamiltonians are gauge dependent, we should make it sure that any physical quantities evaluated perturbatively are indeed gauge invariant, which seems to be very difficult.

In this textbook, there are quite a few issues which are still debating. I believe that the present understanding of the basic field theory in this textbook must be reasonably good, and as far as physics of the symmetry and its breaking is concerned, it should be the best of all. The spontaneous symmetry breaking of the global symmetry is by now understood in this textbook in terms of a simple physics terminology, and there is nothing mysterious from the standard way of understanding physics. However, it is still not yet settled whether the local gauge symmetry can be broken in terms of Higgs mechanism or not. At least, the gauge fixing for the non-gauge field is physically not at all easy to understand. For this problem, we

need a lot to think over in future what should be physical observables in the Higgs mechanism.

This textbook contains a brief description of the lattice field theory even though it is not directly connected to the symmetry breaking physics. Still it may be interesting for readers to understand the basic point of the lattice field theory. For example, the continuum field theory must be richer than the lattice version, and it is most likely true that the lattice field theory can give only limited information on the continuum field theory, particularly when the latter keeps some symmetry while the former does not.

In Appendix, I explain some elementary physics so that readers may grasp the essence of the symmetry breaking phenomena in fermion field theory with little advanced knowledge. In some sense, Appendix can be read in its own interests since it includes non-relativistic quantum mechanics, Dirac equation and Maxwell equation, in addition to the notations which are often used in field theory. At the same time, Appendix contains some new physics interpretation for bosons, Dirac fields and quantization procedure. In particular, I believe that the first quantization of $[x, p_x] = i\hbar$, etc. may well be the result of the Dirac equation in that the Dirac Lagrangian density can be derived from the gauge principle as well as the Maxwell equations without involving the first quantization procedure. In the final chapter of Appendix, I briefly explain the renormalization in QED which is the most successful theory in quantum field theory. The perturbation theory is not the main issue of this textbook, but nevertheless readers may learn the essence of the renormalization scheme and renormalization group in quantum field theory.

The motive force of writing this textbook is initiated by Frank Columbus who understands the importance of the new picture of spontaneous symmetry breaking physics prior to experts and has encouraged me to write it into a textbook form. Indeed, I started to write this book from intensive discussions and hard works with my collaborators on this subject to achieve deeper but simpler understanding of the symmetry and its breaking in quantum field theory.

I should be grateful to all of my collaborators, in particular, Tomoko Asaga, Makoto Hiramoto, Takashi Homma, Seiji Kanemaki, Sachiko Oshima and Hidenori Takahashi for their great contributions to this book. Quite a few physicists and students also helped me a great deal for their critical reading of this manuscript. However, it is trivial to note that any mistakes in this book are entirely due to my carelessness.

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Contents

1	Classical Field Theory of Fermions	1
1.1	Non-relativistic Fields	2
1.1.1	Schrödinger Equation	2
1.1.2	Lagrangian Density for Schrödinger Fields	3
1.1.3	Lagrange Equation for Schrödinger Fields	4
1.1.4	Hamiltonian Density for Schrödinger Fields	5
1.1.5	Hamiltonian for Schrödinger Fields	6
1.1.6	Conservation of Vector Current	7
1.2	Dirac Fields	8
1.2.1	Dirac Equation for Free Fermion	8
1.2.2	Lagrangian Density for Free Dirac Fields	9
1.2.3	Lagrange Equation for Free Dirac Fields	10
1.2.4	Plane Wave Solutions of Free Dirac Equation	10
1.2.5	Quantization in Box with Periodic Boundary Conditions	11
1.2.6	Hamiltonian Density for Free Dirac Fermion	12
1.2.7	Hamiltonian for Free Dirac Fermion	13
1.2.8	Conservation of Vector Current	13
1.3	Electron and Electromagnetic Fields	14
1.3.1	Lagrangian Density	14
1.3.2	Gauge Invariance	14
1.3.3	Lagrange Equation for Dirac Field	15
1.3.4	Lagrange Equation for Gauge Field	16
1.3.5	Hamiltonian Density for Fermions with Electromagnetic Field	17
1.3.6	Hamiltonian for Fermions with Electromagnetic Field	18
1.4	Self-interacting Fermion Fields	19
1.4.1	Lagrangian and Hamiltonian Densities of NJL Model	19
1.4.2	Lagrangian Density of Thirring Model	20
1.4.3	Hamiltonian Density for Thirring Model	20
1.5	Quarks with Electromagnetic and Chromomagnetic Interactions	21
1.5.1	Lagrangian Density	21
1.5.2	EDM Interactions	22

2	Symmetry and Conservation Law	23
2.1	Introduction to Transformation Property	24
2.2	Lorentz Invariance	25
2.2.1	Lorentz Covariance	26
2.3	Time Reversal Invariance	27
2.3.1	T-invariance in Quantum Mechanics	27
2.3.2	T-invariance in Field Theory	28
2.3.3	T-violating Interactions (Imaginary Mass Term)	28
2.3.4	T and P-violating Interactions (EDM)	29
2.4	Parity Transformation	30
2.5	Charge Conjugation	31
2.5.1	Charge Conjugation in Maxwell Equation	31
2.5.2	Charge Conjugation in Dirac Field	31
2.5.3	Charge Conjugation in Quantum Chromodynamics	32
2.6	Translational Invariance	33
2.6.1	Energy Momentum Tensor	33
2.6.2	Hamiltonian Density from Energy Momentum Tensor	34
2.7	Global Gauge Symmetry	35
2.8	Chiral Symmetry	36
2.8.1	Expression of Chiral Transformation in Two Dimensions	36
2.8.2	Mass Term	37
2.8.3	Chiral Anomaly	38
2.8.4	Chiral Symmetry Breaking in Massless Thirring Model	39
2.9	$SU(3)$ Symmetry	40
2.9.1	Dimension of Representation $[\lambda, \mu]$	40
2.9.2	Useful Reduction Formula	41
3	Quantization of Fields	43
3.1	Quantization of Free Fermion Field	44
3.1.1	Equal Time Quantization of Field	44
3.1.2	Creation and Annihilation Operators	45
3.1.3	Quantized Hamiltonian of Free Dirac Field	46
3.1.4	Vacuum of Free Field Theory	47
3.2	Quantization of Thirring Model	48
3.2.1	Vacuum of Thirring Model	48
3.3	Quantization of Gauge Fields in QED	50
3.4	Quantization of Schrödinger Field	52
3.4.1	Field Quantization	52
3.4.2	Creation and Annihilation Operators	52
3.4.3	Fermi Gas Model	53
3.5	Quantized Hamiltonian of QED and Eigenstates	54
3.5.1	Quantized Hamiltonian	54

3.5.2	Eigenvalue Equation	54
3.5.3	Vacuum State $ \Omega\rangle$	55
4	Goldstone Theorem and Spontaneous Symmetry Breaking	57
4.1	Symmetry and Its Breaking in Vacuum	59
4.1.1	Symmetry in Quantum Many Body Theory	59
4.1.2	Symmetry in Field Theory	60
4.2	Goldstone Theorem	62
4.2.1	Conservation of Chiral Charge	62
4.2.2	Symmetry of Vacuum	62
4.2.3	Commutation Relation	63
4.2.4	Momentum Zero State	63
4.2.5	Pole in S-matrix	64
4.3	New Interpretation of Goldstone Theorem	65
4.3.1	Eigenstate of Hamiltonian and \hat{Q}_5	65
4.3.2	Index of Symmetry Breaking	66
4.4	Chiral Symmetry in Quantized Thirring Model	67
4.4.1	Lagrangian Density	67
4.4.2	Quantized Hamiltonian	67
4.4.3	Chiral Transformation for Operators	67
4.4.4	Unitary Operator with Chiral Charge \hat{Q}_5	68
4.4.5	Symmetric and Symmetry Broken Vacuum	68
4.5	Spontaneous Chiral Symmetry Breaking	69
4.5.1	Exact Vacuum of Thirring Model	69
4.5.2	Condensate Operator	69
4.6	Symmetry Breaking in Two Dimensions	71
4.6.1	Fermion Field Theory in Two Dimensions	71
4.6.2	Boson Field Theory in Two Dimensions	71
4.7	Symmetry Breaking in Boson Fields	72
4.7.1	Double Well Potential	72
4.7.2	Change of Field Variables	72
4.7.3	Current Density of Fields	73
4.8	Breaking of Local Gauge Symmetry ?	74
4.8.1	Higgs Mechanism	74
4.8.2	Gauge Fixing	75
4.8.3	What Is Physics Behind Higgs Mechanism ?	75
5	Quantum Electrodynamics	77
5.1	General Properties of QED	78
5.1.1	QED Lagrangian Density	78
5.1.2	Local Gauge Invariance	79
5.1.3	Equation of Motion	79
5.1.4	Noether Current and Conservation Law	79

5.1.5	Gauge Invariance of Interaction Lagrangian	80
5.1.6	Gauge Fixing	81
5.1.7	Gauge Choices	81
5.1.8	Gauge Dependence without $\partial_\mu j^\mu = 0$	83
5.2	S-matrix in QED	85
5.2.1	Definition of S-matrix	85
5.2.2	Fock Space of Free Fields	87
5.2.3	Electron-Electron Interactions	87
5.2.4	Feynman Rules for QED	90
5.3	Schwinger Model (Massless QED ₂)	91
5.3.1	QED with Massless Fermions in Two Dimensions	91
5.3.2	Gauge Fixing	92
5.3.3	Quantized Hamiltonian of Schwinger Model	93
5.3.4	Bosonization of Schwinger Model	94
5.3.5	Chiral Anomaly	95
5.3.6	Regularization of Vacuum Energy	96
5.3.7	Bosonized Hamiltonian of Schwinger Model	97
5.4	Quantized QED ₂ Hamiltonian in Trivial Vacuum	99
5.4.1	Hamiltonian and Gauge Fixing	99
5.4.2	Field Quantization in Anti-particle Representation	99
5.4.3	Dirac Representation of γ -matrices	100
5.4.4	Quantized Hamiltonian of QED ₂	100
5.4.5	Boson Fock States	102
5.4.6	Boson Wave Function	102
5.4.7	Boson Mass	103
5.5	Bogoliubov Transformation in QED ₂	105
5.5.1	Bogoliubov Transformation	105
5.5.2	Boson Mass in Bogoliubov Vacuum	108
5.5.3	Chiral Condensate	109
5.6	QED ₂ in Light Cone	110
5.6.1	Light Cone Quantization	110
6	Quantum Chromodynamics	113
6.1	Properties of QCD with $SU(N_c)$ Colors	114
6.1.1	Lagrangian Density of QCD	114
6.1.2	Infinitesimal Local Gauge Transformation	115
6.1.3	Local Gauge Invariance	115
6.1.4	Noether Current in QCD	116
6.1.5	Conserved Charge of Color Octet State	116
6.1.6	Gauge Non-invariance of Interaction Lagrangian	117
6.1.7	Equations of Motion	118
6.1.8	Hamiltonian Density of QCD	118

6.1.9	Hamiltonian of QCD	119
6.2	Hamiltonian of QCD in Two Dimensions	120
6.2.1	Gauge Fixing	120
6.2.2	Quantization of Fields	121
6.2.3	Quantized Hamiltonian of QCD ₂ with $SU(N_c)$	122
6.2.4	Bogoliubov Transformed Hamiltonian	122
6.2.5	Determination of Bogoliubov Angle	123
6.2.6	Fermion Condensate	123
6.2.7	Boson Mass	124
6.2.8	Condensate and Boson Mass in $SU(N_c)$	125
6.3	't Hooft Model	127
6.3.1	$1/N_c$ Expansion	127
6.3.2	Examination of 't Hooft Model	128
6.4	Spontaneous Symmetry Breaking in QCD ₂	129
6.5	Explicit Expression of H'	131
7	Thirring Model	133
7.1	Bethe Ansatz Method for Massive Thirring Model	134
7.1.1	Free Fermion System	135
7.1.2	Bethe Ansatz State in Two Particle System	136
7.1.3	Bethe Ansatz State in N Particle System	137
7.2	Bethe Ansatz Method for Field Theory	139
7.2.1	Vacuum State of Massive Thirring Model	139
7.2.2	Excited States	140
7.2.3	Lowest Excited State (Boson)	140
7.2.4	Higher Excited States	140
7.2.5	Continuum States	141
7.3	Bethe Ansatz Method for Massless Thirring Model	142
7.3.1	Vacuum State of Massless Thirring Model	143
7.3.2	Symmetric Vacuum State	143
7.3.3	True Vacuum (Symmetry Broken) State	144
7.3.4	$1p - 1h$ State	145
7.3.5	Momentum Distribution of Negative Energy States	146
7.4	Bosonization of Thirring Model	148
7.4.1	Massless Thirring Model	148
7.4.2	Massive Thirring Model	150
7.4.3	Physics of Zero Mode	150
7.5	Massive Thirring vs Sine-Gordon Models	152
7.5.1	Sine-Gordon Field Theory Model	152
7.5.2	Correlation Functions	153
7.5.3	Correspondence	154
7.6	Bogoliubov Method for Thirring Model	155

7.6.1	Massless Thirring Model	155
7.6.2	Bogoliubov Transformation	155
7.6.3	Bogoliubov Transformed Hamiltonian	156
7.6.4	Eigenvalue Equation for Boson	157
7.6.5	Solution of Separable Interactions	157
7.6.6	Boson Spectrum	158
7.6.7	Axial Vector Current Conservation	159
7.6.8	Fermion Condensate	159
7.6.9	Massive Thirring Model	160
7.6.10	NJL Model	161
8	Lattice Field Theory	163
8.1	General Remark on Discretization of Space	164
8.1.1	Equal Spacing	164
8.1.2	Continuum Limit	164
8.1.3	Continuum Limit and Renormalization Group	166
8.1.4	Continuum Limit in Non-perturbative Treatment	166
8.2	Bethe Ansatz Method in Heisenberg Model	167
8.2.1	Exchange Operator $P_{i,j}$	167
8.2.2	Heisenberg XXZ for One Magnon State	168
8.2.3	Heisenberg XXZ for Two Magnon States	169
8.2.4	Heisenberg XXZ for m Magnon States	171
8.3	Equivalence between Heisenberg XYZ and Massive Thirring Models	172
8.3.1	Jordon-Wigner Transformation	172
8.3.2	Continuum Limit	173
8.3.3	Heisenberg XXZ and Massless Thirring Models	175
8.4	Gauge Fields on Lattice	176
8.4.1	Discretization of Space	176
8.4.2	Wilson's Action	176
8.4.3	Wilson Loop	178
8.4.4	Critical Review on Wilson's Results	180
8.4.5	Problems in Wilson's Action	181
8.4.6	Confinement of Quarks	182
A	Introduction to Field Theory	183
A.1	Natural Units	185
A.2	Hermite Conjugate and Complex Conjugate	186
A.3	Scalar and Vector Products (Three Dimensions) :	187
A.4	Scalar Product (Four Dimensions)	188
A.4.1	Metric Tensor	188
A.5	Four Dimensional Derivatives ∂_μ	189
A.5.1	\hat{p}^μ and Differential Operator	189
A.5.2	Laplacian and d'Alembertian Operators	189

A.6	γ -Matrices	190
A.6.1	Pauli Matrices	190
A.6.2	Representation of γ -matrices	191
A.6.3	Useful Relations of γ -Matrices	191
A.7	Transformation of State and Operator	192
A.8	Fermion Current	193
A.9	Trace in Physics	194
A.9.1	Definition	194
A.9.2	Trace in Quantum Mechanics	194
A.9.3	Trace in $SU(N)$	194
A.9.4	Trace of γ -Matrices and \not{p}	195
A.10	Lagrange Equation	196
A.10.1	Lagrange Equation in Classical Mechanics	196
A.10.2	Hamiltonian in Classical Mechanics	196
A.10.3	Lagrange Equation for Fields	197
A.11	Noether Current	198
A.11.1	Global Gauge Symmetry	198
A.11.2	Chiral Symmetry	199
A.12	Hamiltonian Density	200
A.12.1	Hamiltonian Density from Energy Momentum Tensor	200
A.12.2	Hamiltonian Density from Conjugate Fields	201
A.12.3	Hamiltonian Density for Free Dirac Fields	201
A.12.4	Hamiltonian for Free Dirac Fields	202
A.12.5	Role of Hamiltonian	202
A.13	Variational Principle in Hamiltonian	204
A.13.1	Schrödinger Field	204
A.13.2	Dirac Field	205
B	Non-relativistic Quantum Mechanics	207
B.1	Procedure of First Quantization	207
B.2	Mystery of Quantization or Hermiticity Problem ?	209
B.2.1	Free Particle in Box	209
B.2.2	Hermiticity Problem	209
B.3	Schrödinger Fields	211
B.3.1	Currents of Bound State	211
B.3.2	Free Fields (Static)	211
B.3.3	Degree of Freedom of Schrödinger Field	212
B.4	Hydrogen-like Atoms	213
B.5	Harmonic Oscillator Potential	214
B.5.1	Creation and Annihilation Operators	215

C	Relativistic Quantum Mechanics of Bosons	217
C.1	Klein-Gordon Equation	217
C.2	Negative Energy State	218
C.2.1	Classical Field	218
C.2.2	Quantized Field	220
C.2.3	Complex Scalar Field	221
C.3	Degree of Freedom of Boson Fields	222
C.4	Photon and Gauge Bosons	223
D	Relativistic Quantum Mechanics of Fermions	225
D.1	Derivation of Dirac Equation	225
D.2	Negative Energy States	226
D.3	Hydrogen Atom	227
D.3.1	Conserved Quantities	227
D.3.2	Energy Spectrum	228
D.3.3	Ground State Wave Function ($1s_{\frac{1}{2}}$ - state)	228
D.4	Lamb Shifts	230
D.4.1	Quantized Vector Field	230
D.4.2	Non-relativistic Hamiltonian	230
D.4.3	Second Order Perturbation Energy	231
D.4.4	Mass Renormalization and New Hamiltonian	231
D.4.5	Lamb Shift Energy	232
E	Maxwell Equation and Gauge Transformation	233
E.1	Gauge Invariance	233
E.2	Derivation of Lorenz Force in Classical Mechanics	235
E.3	Number of Independent Functional Variables	236
E.3.1	Electric and Magnetic fields \mathbf{E} and \mathbf{B}	236
E.3.2	Vector Field A_μ and Gauge Freedom	237
E.4	Lagrangian Density of Electromagnetic Fields	238
E.5	Boundary Condition for Photon	239
F	Regularizations and Renormalizations	241
F.1	Euler's Regularization	241
F.1.1	Abelian Summation	241
F.1.2	Regularized Abelian Summation	241
F.2	Chiral Anomaly	242
F.2.1	Charge and Chiral Charge of Vacuum	242
F.2.2	Large Gauge Transformation	243
F.2.3	Regularized Charge	243
F.2.4	Anomaly Equation	244
F.3	Index of Renormalizability	245
F.3.1	Renormalizable	245

F.3.2	Unrenormalizable	245
F.3.3	Summary of Renormalizability	246
F.4	Infinity in Physics	247
G	Path Integral Formulation	249
G.1	Path Integral for Quantum Mechanics	250
G.1.1	General Property	250
G.1.2	Harmonic Oscillator Case	251
G.2	Path Integral in Field Theory	252
G.3	Coulomb Force in QED in Path Integral	253
G.4	Averaging Procedure	255
H	New Concept of Quantization	257
H.1	Derivation of Lagrangian Density of Dirac Field from Gauge Invariance and Maxwell Equation	257
H.1.1	Lagrangian Density for Maxwell Equation	257
H.1.2	Four Component Spinor	258
H.2	Shape of Lagrangian Density	259
H.2.1	Mass Term	259
H.2.2	First Quantization	260
H.3	Two Component Spinor	260
H.4	Klein-Gordon Equation	261
H.5	Incorrect Quantization in Polar Coordinates	262
H.6	Interaction with Gravity	262
I	Renormalization in QED	263
I.1	Hilbert Space of Unperturbed Hamiltonian	263
I.2	Self-energy of Fermion	264
I.2.1	Dimensional Regularization	264
I.2.2	Mass Renormalization	265
I.3	Vacuum Polarization	266
I.4	Vertex Corrections	267
I.5	Renormalization Group Equation	268
I.5.1	Dimensional Regularization	268
I.5.2	Cutoff Momentum Regularization	269
I.5.3	Difference in Renormalization Group Equations	269
I.6	Physics of Renormalization Group	270
I.6.1	Perturbative Treatment	270
I.6.2	Non-perturbative Treatment	271
I.7	Renormalization Group Equation in QCD	272
I.7.1	Fock Space of Free Fields	272
I.7.2	Virtual Fock Space	272
I.7.3	Asymptotic Freedom	273