Chapter 1

Classical Field Theory of Fermions

The world of elementary particles is basically composed of fermions. Quarks, electrons and neutrinos are all fermions. On the other hand, elementary bosons are all gauge bosons, except Higgs particles though unknown at present. Therefore, if one wishes to understand field theory, then it should be the best to first study fermion field theory models.

In this chapter, we discuss the classical field theory in which "classical field" means that the field is not an operator but a c-number function. First, we treat the Schrödinger field and its equation in terms of the non-relativistic field theory model. In this case, the first quantization of $[x_i, p_j] = i\hbar\delta_{ij}$ is already done since we start from the Lagrangian density. In fact, the Lagrange equation leads to the Schrödinger equation or in other words, the Lagrangian density is constructed such that the Schrödinger equation can be derived from the Lagrange equation. The Dirac field is then discussed in terms of the Lagrangian density and the Lagrange equation. We also discuss the electromagnetic fields which interact with the Dirac field. The gauge invariance will be repeatedly discussed in this textbook, and the first introduction is given here. Finally, the field theory models with self-interacting fields are introduced and their Lagrangian density as well as Hamiltonian are described.

In this textbook, the basic parts of elementary physics can be found in Appendix, and in fact, Appendix is prepared such that it can be read in its own interests independently from the main part of the textbook.

Throughout this book, we employ the natural units

$$c = 1, \qquad \hbar = 1.$$

This is, of course, due to its simplicity, and one can easily recover the right dimension of any physical quantities by making use of

$$\hbar c = 197 \ \ \mathrm{MeV} \cdot \mathrm{fm}.$$

1.1 Non-relativistic Fields

If one treats a classical field $\psi(\boldsymbol{r})$, it does not matter whether it is a relativistic field or non-relativistic one. The kinematics becomes important when one solves the equation of motion which is relativistic or non-relativistic. If the kinematics is non-relativistic, then the equation of motion that governs the field $\psi(\boldsymbol{r})$ is the Schrödinger equation. Therefore, we should first study the Schrödinger field from the point of view of the classical field theory.

1.1.1 Schrödinger Equation

Electron in classical mechanics is treated as a point particle whose equation of motion is governed by the Newton equation. When electrons are trapped by atoms, then their motions should be described by quantum mechanics. As long as electrons move much slowly in comparison with the velocity of light c, the equation of their motion is governed by the Schrödinger equation. The Schrödinger equation for electron with its mass m in the external field $U(\boldsymbol{r})$ can be written as

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2m}\boldsymbol{\nabla}^2-U(\boldsymbol{r})\right)\psi(\boldsymbol{r},t)=0 \tag{1.1}$$

where $U(\boldsymbol{r})$ is taken to be a real potential. $\psi(\boldsymbol{r},t)$ corresponds to the electron field in atoms, and $|\psi(\boldsymbol{r},t)|^2$ can be interpreted as a probability density of finding the electron at $(\boldsymbol{r},t)$.

Field $\psi(\boldsymbol{r},t)$ is Complex

The Schrödinger field $\psi(\boldsymbol{r},t)$ should be a complex function, and the complex field just corresponds to one particle state in the classical field theory. This is a well known fact, but below we will see what may happen when we assume *a priori* that the Schrödinger field $\psi(\boldsymbol{r},t)$ should be a real function.

Real Field Condition is Unphysical

If one imposes the condition that the field $\psi(\boldsymbol{r},t)$ should be real

$$\psi(\boldsymbol{r},t)=\psi^\dagger(\boldsymbol{r},t)$$

then, one sees immediately that the field $\psi(\boldsymbol{r},t)$ becomes time-independent since eq.(1.1) and its complex conjugate equation give the following constraint for a real field $\psi(\boldsymbol{r},t)$

$$\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=0.$$

Also, the field $\psi(\boldsymbol{r})$ should satisfy the following equation

$$\left(-\frac{1}{2m}\boldsymbol{\nabla}^2+U(r)\right)\psi(\boldsymbol{r})=0.$$

Since the general solution of eq.(1.1) can be written as

$$\psi(\boldsymbol{r},t) = e^{-iEt}\phi(\boldsymbol{r})$$

the field $\psi(\boldsymbol{r},t)$ may become a real function only if the energy E of the system vanishes. That is, the energy eigenvalue of E is

$$E = 0.$$

Therefore, the real field cannot propagate and should be unphysical. This means that the real field condition of $\psi(\boldsymbol{r},t)$ is physically too strong as a constraint.

1.1.2 Lagrangian Density for Schrödinger Fields

The Lagrangian density which can produce eq.(1.1) is easily found as

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m}\frac{\partial \psi^\dagger}{\partial x_k}\frac{\partial \psi}{\partial x_k} - \psi^\dagger U \psi \tag{1.2}$$

where the repeated indices of k mean the summation of $k = 1, 2, 3$ and, in this text, this notation as well as the vector representation are employed depending on the situations. The repeated indices notation is mostly better for the calculation, but for memorizing the expressions or equations, the vector notation has some advantage.

The Lagrangian density of eq.(1.2) is constructed such that the Lagrange equation can reproduce the Schrödinger equation of eq.(1.1). It may also be important to note that the Lagrangian density of eq.(1.2) has a $U(1)$ symmetry, that is, it is invariant under the change of the field ψ as

$$\psi'(x) = e^{i\theta}\psi(x) \quad \rightarrow \quad \mathcal{L}' = \mathcal{L}$$

where θ is a real constant. This invariance is clearly satisfied, and it is related to the conservation of vector current in terms of Noether's theorem which will be treated in the later chapters and in Appendix A.

Non-hermiticity of Lagrangian Density

At this point, we should discuss the non-hermiticity of the Lagrangian density. As one notices, the Lagrangian density of eq.(1.2) is not hermitian, and therefore some symmetry will be lost. One can build the Lagrangian density which is hermitian by replacing the first term by

$$i\psi^\dagger \frac{\partial \psi}{\partial t} \rightarrow \left(\frac{i}{2}\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{i}{2}\frac{\partial \psi^\dagger}{\partial t}\psi \right).$$

However, it is a difficult question whether the Lagrangian density must be hermitian or not since it is not an observable. In addition, when one introduces the conjugate fields

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}}, \qquad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}$$

in accordance with the fields ψ and $\psi^\dagger$, then the symmetry between them is lost. However, the conjugate fields themselves are again not observables, and therefore there is no reason that one should keep this symmetry. In any case, one can, of course, work with the symmetric and hermitian Lagrangian density, but physical observables are just the same as eq.(1.2). In this textbook, we employ eq.(1.2) since it is simpler.

1.1.3 Lagrange Equation for Schrödinger Fields

The Lagrange equation for field theory can be obtained by the variational principle of the action S

$$S = \int \mathcal{L} dt d^3 r$$

and the Lagrange equation is derived in Appendix A. Since the field ψ is a complex field, ψ and $\psi^\dagger$ are treated as independent functional variables. The Lagrange equation for the field ψ is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi} \tag{1.3a}$$

where the four dimensional derivative

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is introduced for convenience. Now, the following equations can be easily evaluated

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\frac{\partial \psi^\dagger}{\partial t}$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_k})} = -\frac{1}{2m}\frac{\partial}{\partial x_k}\frac{\partial \psi^\dagger}{\partial x_k}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\psi^\dagger U$$

and therefore one obtains

$$\left(-i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 - U(r)\right)\psi^\dagger(r,t) = 0$$

which is just the Schrödinger equation for $\psi^\dagger$ in eq.(1.1).

It should be interesting to calculate the Lagrange equation for the field $\psi^\dagger$,

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial (\frac{\partial \psi^\dagger}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial \psi^\dagger}. \tag{1.3b}$$

In this case, one finds

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial (\frac{\partial \psi^\dagger}{\partial x_k})} = -\frac{1}{2m}\frac{\partial}{\partial x_k}\frac{\partial \psi}{\partial x_k}$$

$$\frac{\partial \mathcal{L}}{\partial \psi^\dagger} = i\frac{\partial \psi}{\partial t} - U\psi$$

and therefore one obtains

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\boldsymbol{\nabla}^2 - U(\boldsymbol{r})\right)\psi(\boldsymbol{r},t) = 0$$

which is just the same equation as eq.(1.1).

Here, we note that the Lagrangian density is not a physical observable and therefore it does not necessarily have to be determined uniquely. It is by now clear that the Lagrangian density eq.(1.2) reproduces a desired Schrödinger equation and thus can be taken as the right Lagrangian density for Schrödinger fields.

1.1.4 Hamiltonian Density for Schrödinger Fields

From the Lagrangian density, one can build the Hamiltonian density $\mathcal{H}$ which is the energy density of the field $\psi(\boldsymbol{r},t)$. The Hamiltonian density $\mathcal{H}$ is best constructed from the energy momentum tensor $\mathcal{T}^{\mu\nu}$

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\partial^\nu \psi^\dagger - \mathcal{L}g^{\mu\nu}$$

which will be derived in eq.(2.32) in chapter 2. The energy momentum tensor $\mathcal{T}^{\mu\nu}$ satisfies the following equation of conservation law

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0$$

due to the invariance of the Lagrangian density under the translation. Therefore, the conserved charge associated with the $\mathcal{T}^{0\nu}$

$$\mathcal{Q}^\nu = \int \mathcal{T}^{0\nu} d^3r$$

should be a conserved quantity. Thus, it is natural that one defines the Hamiltonian in terms of the $\mathcal{Q}^0$.

Hamiltonian Density from Energy Momentum Tensor

The Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H} \equiv \mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger}\dot{\psi}^\dagger - \mathcal{L}. \tag{1.4a}$$

Therefore, introducing the conjugate fields Π_ψ and $\Pi_{\psi^\dagger}$ by

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, \qquad \Pi_{\psi^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0$$

one can write the Hamiltonian density as

$$\mathcal{H} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger}\dot{\psi}^\dagger - \mathcal{L} = \frac{1}{2m}\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi. \tag{1.4b}$$

1.1.5 Hamiltonian for Schrödinger Fields

The Hamiltonian for the Schrödinger field is obtained by integrating the Hamiltonian density over all space

$$H \equiv \int \mathcal{H} d^3r = \int \left[\frac{1}{2m}\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger U \psi\right] d^3r. \tag{1.4c}$$

By employing the Gauss theorem

$$\int_V \boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi) d^3r = \int_S (\psi^\dagger \boldsymbol{\nabla}_n \psi) dS_n$$

one can rewrite eq.(1.4c)

$$H = \int \left[-\frac{1}{2m}\psi^\dagger \boldsymbol{\nabla}^2 \psi + \psi^\dagger U \psi\right] d^3r \tag{1.4d}$$

where the following identity is employed

$$\boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \psi^\dagger \boldsymbol{\nabla}^2 \psi.$$

In addition, the surface integral term is neglected since it should vanish at the surface of sphere at infinity.

Now, it may be interesting to note that the Hamiltonian in eq.(1.4d) by itself does not give us much information on the dynamics. As long as we stay in the classical field theory, then the dynamics can be obtained from the equation of motion, that is, the Schrödinger equation. The static Schrödinger equation can be derived from the variational principle of the Hamiltonian with respect to ψ, and this treatment is given in Appendix A.

The Hamiltonian of eq.(1.4c) becomes important when the field ψ is quantized, that is, the fluctuation of the field ψ is taken into account. In this case, the Schrödinger field becomes an operator and therefore the Hamiltonian as well. This means that one has to prepare the Fock state on which the Hamiltonian can operate, and if one solves the eigenvalue equation for the Hamiltonian, then one can obtain the energy eigenvalue of the Hamiltonian corresponding to the Fock state.

However, the quantization of the Schrödinger field is not needed in the normal circumstances. The field quantization is necessary for the relativistic fields which contain negative energy solutions, and it becomes important when one wishes to treat the quantum fluctuation of the fields which corresponds to the creation and annihilation of particles.

1.1.6 Conservation of Vector Current

From the Schrödinger equation, one can derive the current conservation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

where ρ and $\boldsymbol{j}$ are defined as

$$\rho = \psi^\dagger \psi, \qquad \boldsymbol{j} = \frac{i}{2m}\left[(\boldsymbol{\nabla}\psi^\dagger)\psi - \psi^\dagger \boldsymbol{\nabla}\psi\right].$$

This continuity equation of the vector current can also be derived as Noether's theorem from the Lagrangian density of eq.(1.2) which is invariant under the global gauge transformation

$$\psi' = e^{i\alpha}\psi.$$

As treated in Appendix A, the Noether current is written as

$$j^\mu \equiv -i\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\psi^\dagger\right], \qquad \text{with} \quad j^\mu = (\rho, \boldsymbol{j})$$

which just gives the above current density ρ and $\boldsymbol{j}$ when one employs the Lagrangian density of eq.(1.2).

It may be interesting to observe that the Lagrange equation, energy momentum tensor and the current conservation are all written in a relativistically covariant fashion when the properties of the Schrödinger field are derived. That is, apart from the shape of the Lagrangian density of the Schrödinger field, all the treatments are just the same as the relativistic description.