Appendix E

Maxwell Equation and Gauge Transformation

Fundamental equations for electromagnetic fields are the Maxwell equation, and they are written for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$\nabla \cdot \boldsymbol{E} = \rho \tag{E.0.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{E.0.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{E.0.3}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{E.0.4}$$

where ρ and $\boldsymbol{j}$ denote the charge and current densities, respectively. These are already equations for the fields and therefore they are quantum mechanical equations. In this respect, it is important to realize that the first quantization procedure ($[x_j, p_i] = i\hbar\delta_{ij}$) is already done in the Maxwell equation.

E.1 Gauge Invariance

The Maxwell equation is written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. Now, if one introduces the vector potential $\boldsymbol{A}$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{E.1.1}$$

then eq.(E.0.2) can be always satisfied since

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \nabla \times \boldsymbol{A} = \nabla \times \nabla \cdot \boldsymbol{A} = 0.$$

Therefore, one often employs the vector potential in order to solve the Maxwell equation. However, one notices in this case that the number of the degrees of freedom is still 3, that is, A_x, A_y, A_z in spite of the fact that we made use of one equation [eq.(E.0.2)]. This means that there must be a redundancy in the vector potential. This is the gauge freedom, that is, if one transforms

$$\boldsymbol{A} = \boldsymbol{A}' + \boldsymbol{\nabla}\chi \qquad (E.1.2a)$$

then the magnetic field $\boldsymbol{B}$ does not depend on χ

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A}' + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\chi = \boldsymbol{\nabla} \times \boldsymbol{A}'$$

where χ is an arbitrary function that depends on $(\boldsymbol{r}, t)$. Now, Faraday's law [eq.(E.0.3)] can be rewritten by using the vector potential,

$$\boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0. \qquad (E.1.3)$$

This means that one can write the electric field $\boldsymbol{E}$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \qquad (E.1.4)$$

where A_0 is an arbitrary function of $(\boldsymbol{r}, t)$ and is called electrostatic potential. Since $\boldsymbol{E}$ in eq.(E.1.4) must be invariant under the gauge transformation of eq.(E.1.2a), it suggests that A_0 should be transformed under the gauge transformation as

$$A_0 = A_0' - \frac{\partial \chi}{\partial t}. \qquad (E.1.2b)$$

In this case, the electric field is invariant under the gauge transformation of eqs.(E.1.2)

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{\nabla}\left(A_0' - \frac{\partial \chi}{\partial t}\right) - \frac{\partial}{\partial t}\left(\boldsymbol{A}' + \boldsymbol{\nabla}\chi\right) = -\boldsymbol{\nabla} A_0' - \frac{\partial \boldsymbol{A}'}{\partial t}$$

and eq.(E.1.4) can automatically reproduce Faraday's law since

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} A_0 - \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

E.2 Derivation of Lorenz Force in Classical Mechanics

The interaction of electrons with the electromagnetic forces in nonrelativistic kinematics can be determined from the gauge invariance. This is remarkable and therefore we explain the derivation below since it is indeed interesting to learn the basic mechanism of the interaction. First, one starts from a free electron Lagrangian in classical mechanics

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2. \tag{E.2.1}$$

When one wishes to add any interaction of electron with $\boldsymbol{A}$ and A_0 to the above Lagrangian, one sees that the Lagrangian must be linear functions of $\boldsymbol{A}$ and A_0. This is clear since the Lagrangian must be gauge invariant under eqs.(E.1.2). From the parity and time reversal invariance, one can write down the new Lagrangian

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + g(\dot{\boldsymbol{r}} \cdot \boldsymbol{A} - A_0) \tag{E.2.2}$$

where g is a constant which cannot be determined from the gauge condition. When one makes the gauge transformation

$$\boldsymbol{A} = \boldsymbol{A}' + \boldsymbol{\nabla}\chi, \quad A_0 = A_0' - \frac{\partial \chi}{\partial t}$$

one obtains

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + g(\dot{\boldsymbol{r}} \cdot \boldsymbol{A}' - A_0') + g\frac{d\chi}{dt}. \tag{E.2.3}$$

Since the total derivative in the Lagrangian does not have any effects on the equation of motion, eq.(E.2.2) is invariant under the gauge transformation. It is amazing that the shape of the Lagrangian for electrons interacting with the electromagnetic fields is determined from the gauge invariance.

It is now easy to calculate the equation of motion for electron,

$$m\ddot{\boldsymbol{r}} = g\dot{\boldsymbol{r}} \times \boldsymbol{B} + g\boldsymbol{E} \tag{E.2.4}$$

where the first term in the right hand side corresponds to the Lorenz force.

E.3 Number of Independent Functional Variables

The Maxwell equations are described in terms of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. Once the charge density ρ and the current density $\boldsymbol{j}$ are given, then one can determine the fields $\boldsymbol{E}$, $\boldsymbol{B}$. It should be important to count the number of the unknown functional variables and the number of equations.

E.3.1 Electric and Magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$

In terms of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$, it is easy to count the number of the functional variables. The number is six since one has

$$E_x, \quad E_y, \quad E_z, \quad B_x, \quad B_y, \quad B_z. \tag{E.3.1}$$

On the other hand, the number of equations looks eight since the Gauss law [eq.(E.0.1)] and no magnetic monopole [eq.(E.0.2)] give two equations, and Faraday's law [eq.(E.0.3)] and Ampere's law [eq.(E.0.4)] seem to have six equations. However, Faraday's law gives only two equations since there is one constraint because

$$\boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial \boldsymbol{B}}{\partial t}=0 \quad\rightarrow\quad \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial \boldsymbol{B}}{\partial t}\right)=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{E}+\frac{\partial(\boldsymbol{\nabla}\cdot\boldsymbol{B})}{\partial t}=0 \tag{E.3.2}$$

In addition, Ampere's law has two equations since there is one constraint due to the continuity equation because

$$\boldsymbol{\nabla}\times\boldsymbol{B}-\boldsymbol{j}-\frac{\partial \boldsymbol{E}}{\partial t}=0 \quad\rightarrow\quad \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{B}-\boldsymbol{j}-\frac{\partial \boldsymbol{E}}{\partial t}\right)=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{B}-\boldsymbol{\nabla}\cdot\boldsymbol{j}-\frac{\partial \rho}{\partial t}=0. \tag{E.3.3}$$

Therefore, the number of the Maxwell equations is six which agrees with the number of the independent functional variables as expected.

Integrated Gauss's Law

In the electro-static exercise problems, one often employs the integrated Gauss law

$$\int_S \boldsymbol{E}\cdot d\boldsymbol{S}=\int_V \boldsymbol{\nabla}\cdot\boldsymbol{E}d^3r=\int_V \rho d^3r=Q. \tag{E.3.4}$$

For the spherical charge distribution of ρ, for example, one can determine the electric field E_r in spite of the fact that one has employed only one equation of the Gauss law. This is of course clear because the symmetry makes it possible to adjust the number of the independent functional variable E_r which is one and the number of equation which is also one.

E.3.2 Vector Field A_μ and Gauge Freedom

When one introduces the vector field A_μ as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \tag{E.3.5}$$

then the number of the independent fields is four since

$$A_0, \quad A_x, \quad A_y, \quad A_z. \tag{E.3.6}$$

On the other hand, the number of equations is three since the Gauss law [eq.(E.0.1)] gives one equation

$$-\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} A_0 + \frac{\partial \boldsymbol{A}}{\partial t} \right) = \rho \tag{E.3.7}$$

and Ampere's law gives two equations as discussed above due to the continuity equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{j} - \frac{\partial}{\partial t} \left(\boldsymbol{\nabla} A_0 + \frac{\partial \boldsymbol{A}}{\partial t} \right). \tag{E.3.8}$$

It is of course easy to see that no magnetic monopole

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{E.3.9}$$

and Faraday's law

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} A_0 \right) = -\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{E.3.10}$$

are automatically satisfied in terms of the vector potential A_μ.

Gauge Freedom

Therefore, the number of the unknown functional variables is four, but the number of equations is three, and they are not the same. This redundancy of the vector field is just related to the gauge freedom, and if one wishes to solve the Maxwell equations in terms of the vector potential A_μ, then one should reduce the number of the functional variables of the vector potential by fixing the gauge freedom.

Electromagnetic Wave

As an example, if there is no source term present [$\rho = 0$ and $\boldsymbol{j} = 0$], then the solution of the Maxwell equations with the Coulomb gauge fixing gives the electromagnetic wave which is composed of the transverse field only

$$A_0 = 0, \qquad A_z = 0, \qquad (A_x, A_y) \neq 0 \tag{E.3.11}$$

where the direction of $\boldsymbol{k}$ is chosen to be $z-$direction.

E.4 Lagrangian Density of Electromagnetic Fields

For the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the total energy of the system becomes

$$E = \frac{1}{2}\int (E_k E_k + B_k B_k) d^3r = \frac{1}{2}\int \left[\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right)^2 + \left(\frac{\partial A_k}{\partial x_j}\frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}\right)\right] d^3r \tag{E.4.1}$$

Now, one introduces the field strength $F_{\mu\nu}$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{E.4.2}$$

which is gauge invariant. In this case, one sees that $F_{\mu\nu}$ just corresponds to the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$F_{0k} = F^{k0} = -F_{k0} = -F^{0k} = E_k, \qquad F_{ij} = F^{ij} = -F_{ji} = -F^{ji} = -\epsilon_{ijk} B_k.$$

The Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}(E_k E_k - B_k B_k) = \frac{1}{2}\left[\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right)^2 - \left(\frac{\partial A_k}{\partial x_j}\frac{\partial A_k}{\partial x_j} - \frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}\right)\right] \tag{E.4.3}$$

which leads to the following Lagrangian density

$$\mathcal{L} = \frac{1}{4}\left(-F_{0k}F^{0k} - F_{k0}F^{k0} - F_{jk}F^{jk}\right) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{E.4.4}$$

The Lagrange equation for A_ν is given as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \equiv \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{A}_0} + \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\frac{\partial A_\nu}{\partial x_k})} = \frac{\partial \mathcal{L}}{\partial A_\nu} \tag{E.4.5}$$

which becomes

$$[\nu = 0] \quad \rightarrow \qquad \frac{\partial}{\partial x_k}\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right) = 0 \quad \rightarrow \quad \nabla \cdot \boldsymbol{E} = 0$$

$$[\nu = k] \quad \rightarrow \qquad \frac{\partial}{\partial t}\left(\dot{A}_k + \frac{\partial A_0}{\partial x_k}\right) + \frac{\partial}{\partial x_j}(\epsilon_{jki} B_i) = 0 \quad \rightarrow \quad \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = 0.$$

They are just the Maxwell equations [eqs.(E.0.1) and (E.0.4)] without any source terms. Since no magnetic monopole and Faraday's law [eqs.(E.0.2) and (E.0.43] can be automatically satisfied in terms of the vector potential A_μ, the Lagrangian density of eq.(E.4.4) is the right one that reproduces the Maxwell equations.

E.5 Boundary Condition for Photon

When there is no source term present ($\rho = 0,\ \boldsymbol{j} = 0$), then eq.(E.3.8) becomes

$$\left(\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = 0 \qquad (E.5.1)$$

where the Coulomb gauge fixing condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$$

is employed. In this case, one sees that eq.(E.5.1) is a quantum mechanical equation for photon. Yet, one does not discuss the bound state of photon. This is clear since photon cannot be confined. There is no bound state of photon in quantum mechanics and eq.(E.5.1) has always the plane wave solution

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\boldsymbol{\epsilon}_\lambda\left[c_{\boldsymbol{k},\lambda}e^{-ikx} + c_{\boldsymbol{k},\lambda}^\dagger e^{ikx}\right] \qquad (E.5.2)$$

where the polarization vectors $\boldsymbol{\epsilon}_\lambda$ has two components

$$\boldsymbol{\epsilon}_1 = (1,0,0), \qquad \boldsymbol{\epsilon}_2 = (0,1,0) \qquad (E.5.3)$$

when the direction of $\boldsymbol{k}$ is chosen to be $z-$direction.

This is basically due to the fact that photon is massless and therefore one cannot specify the system one measures. It always propagates with the speed of light ! But still the equation derived from the Maxwell equation is a quantum mechanical equation of motion, though relativistic.