# Appendix G

# Path Integral Formulation

The path integral in quantum mechanics can be obtained by rewriting the quantum mechanical amplitude $K(x, x' : t)$ into many dimensional integrations over discretized coordinates $x_n$. We discuss some examples which can be related to physical observables. However, the path integral expression cannot be connected to the dynamics of classical mechanics, even though, superficially, there is some similarity between them. Then, we show Feynman's formulation of the path integral in quantum electrodynamics (QED), which is based on many dimensional integrations over the parameters $q_{\boldsymbol{k}, \lambda}$ appearing in the vector potential. This should be indeed connected to the second quantization in field theory models. However, the field theory path integral formulation in most of the textbooks is normally defined in terms of many dimensional integrations over fields. In this case, the path integral does not correspond to the field quantization. Here, we clarify what should be the problems of the path integral formulation over the field variables and why the integrations over fields do not correspond to the field quantization.

## 1  Path Integral in Quantum Mechanics

The path integral formulation in one dimensional quantum mechanics starts from the amplitude $K(x, x' : t)$ which is defined by

$$K(x, x' : t) = \langle x' | e^{-iHt} | x \rangle \qquad (G.1.1)$$

where the system is specified by the Hamiltonian $H$. In the field theory textbooks, one often finds the expression of the amplitude $K(x, x' : t)$ in terms of the transition between the state $|x, t\rangle$ and $|x', t'\rangle$ as

$$\langle x', t' | x, t \rangle \rightarrow \langle x' | e^{-iH(t'-t)} | x \rangle. \qquad (G.1.2)$$

However, the state $|x, t\rangle$ is not an eigenstate of the Hamiltonian and therefore, one cannot prove the rightarrow of eq.(G.1.2). Instead, we should rewrite the amplitude $K(x, x' : t)$ so as to understand its physical meaning

$$K(x, x' : t) = \langle x' | e^{-iHt} | x \rangle = \sum_n \psi_n(x') \psi_n^\dagger(x) e^{-iE_n t} \qquad (G.1.3)$$

where $\psi_n(x)$ and $E_n$ should be the eigenstate and the eigenvalue of the Hamiltonian $H$. We note that eq.(G.1.3) is not yet directly related to physical observables.

## 1.1 Path Integral Expression

We start from the amplitude $K(x', x : t)$

$$K(x', x : t) = \langle x'|e^{-iHt}|x\rangle$$

where the Hamiltonian in one dimension is given as

$$H = \frac{\hat{p}^2}{2m} + U(x) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + U(x). \qquad (G.1.4)$$

Here, a particle with its mass $m$ is bound in the potential $U(x)$. Now, one can make $n$ partitions of $t$ and $x' - x$, and therefore we label the discretized coordinate $x$ as

$$x = x_0, \ x_1, \ x_2, \ \cdots, \ x' = x_n. \qquad (G.1.5)$$

In this case, we assume that each $x_i$ and $p$ should satisfy the following completeness relations

$$\int_{-\infty}^{\infty} dx_i |x_i\rangle\langle x_i| = 1, \ \langle x_i|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx_i}, \ \int_{-\infty}^{\infty} dp|p\rangle\langle p| = 1, \ (i = 1, \cdots, n). \qquad (G.1.6)$$

Therefore, $K(x', x : t)$ becomes

$$K(x', x : t) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \times$$

$$\langle x'|e^{-iH\Delta t}|x_{n-1}\rangle\langle x_{n-1}|e^{-iH\Delta t}|x_{n-2}\rangle \cdots \langle x_1|e^{-iH\Delta t}|x\rangle \qquad (G.1.7)$$

where $\Delta t$ is defined as $\Delta t = \frac{t}{n}$. Further, one can calculate the matrix elements, for example, as

$$\langle x_1|e^{-iH\Delta t}|x\rangle = \langle x_1|\exp\left[-i\left(-\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2} + U(x)\Delta t\right)\right]|x\rangle$$

$$\simeq \exp(-iU(x)\Delta t)\langle x_1|\exp\left(i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}\right)|x\rangle + O((\Delta t)^2). \qquad (G.1.8)$$

In addition, $\langle x_1|e^{i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}}|x\rangle$ can be evaluated by inserting a complete set of momentum states

$$\langle x_1|\exp\left(i\frac{\Delta t}{2m}\frac{\partial^2}{\partial x^2}\right)|x\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi}e^{-i\frac{p^2}{2m}\Delta t}e^{-ip(x-x_1)} = \sqrt{\frac{m}{2i\pi\Delta t}}e^{-i\frac{m(x-x_1)^2}{2\Delta t}}. \qquad (G.1.9)$$

Therefore, one finds now the path integral expression for $K(x', x : t)$

$$K(x', x : t) = \lim_{n\to\infty}\left(\frac{m}{2i\pi\Delta t}\right)^{\frac{n}{2}} \times$$

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \exp\left\{ i \sum_{k=1}^{n} \left( \frac{m(x_k - x_{k-1})^2}{2\Delta t} - U(x_k)\Delta t \right) \right\} \qquad (G.1.10a)$$

where $x_0 = x$ and $x_n = x'$, respectively. Since the classical action $S$ is given as

$$S = \int_0^t dt \left( \frac{1}{2} m \dot{x}^2 - U(x) \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta t \left\{ \frac{m}{2} \left( \frac{x_k - x_{k-1}}{\Delta t} \right)^2 - U(x_k) \right\} \qquad (G.1.11)$$

the amplitude can be symbolically written as

$$K(x', x : t) = \mathcal{N} \int [\mathcal{D}x] \exp\left\{ i \int_0^t \left( \frac{1}{2} m \dot{x}^2 - U(x) \right) dt \right\} \qquad (G.1.10b)$$

where $\mathcal{N} \int [\mathcal{D}x]$ is defined as

$$\mathcal{N} \int [\mathcal{D}x] \equiv \lim_{n \to \infty} \left( \frac{m}{2i\pi\Delta t} \right)^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1}.$$

This is indeed amazing in that the quantum mechanical amplitude is connected to the Lagrangian of the classical mechanics for a particle with its mass $m$ in the same potential $U(x)$. Since the procedure of obtaining eq.(G.1.10a) is just to rewrite the amplitude by inserting the complete set of the $|x_n\rangle$ states, there is no mathematical problem involved in evaluating eq.(G.1.10a).

## 1.2   Physical Meaning of Path Integral

However, the physical meaning of the procedure is not at all easy to understand. It is clear that eq.(G.1.10a) is well defined and there is no problem since it simply involves mathematics. However, there is a big jump from eq.(G.1.10a) to eq.(G.1.10b), even though it looks straightforward. Eq.(G.1.10b) indicates that the first term of eq.(G.1.10b) in the curly bracket is the kinetic energy of the particle in classical mechanics. In this case, however, $x_k$ and $x_{k-1}$ cannot be varied independently as one sees it from classical mechanics since it is related to the time derivative. On the other hand, they must be varied independently in the original version of eq.(G.1.10a) since it has nothing to do with the time derivative in the process of the evaluation. This is clear since, in quantum mechanics, time and coordinate are independent from each other. Therefore, it is difficult to interpret the first term of eq.(G.1.10a) as the kinetic energy term in classical mechanics. Secondly, the procedure of rewriting the amplitude is closely connected to the fact that the kinetic energy of the Hamiltonian is quadratic in $p$, that is, it is described as $\frac{p^2}{2m}$. In this respect, the fundamental ingredients of the path integral formulation must lie in eq.(G.1.9) which relates the momentum operator $p^2$ to the time derivative of the coordinate $x$ under the condition that $x_k$ and $x_{k-1}$ are sufficiently close to each other. In this sense, if the kinetic energy operator were linear in $p$ like the Dirac equation, then there is no chance to rewrite the amplitude since the Gaussian integral is crucial in evaluating the integral.

Therefore, it should be difficult to claim that eq.(G.1.10a) can correspond to the dynamics of classical mechanics, even though, superficially, there is some similarity between them. In other words, it is hard to prove that the quantum mechanical expression of $K(x', x : t)$ is related to any dynamics of classical mechanics. One may say that $K(x', x : t)$ happens to have a similar shape to classical Lagrangian, mathematically, but, physically it has nothing to do with the dynamics of classical mechanics.

### 1.2.1　No Summation of Classical Path

In some of the path integral textbooks, one finds the interpretation that the quantum mechanical dynamics can be obtained by summing up all possible paths in the classical mechanical trajectories. However, if one starts from eq.(G.1.10b) and tries to sum up all the possible paths, then one has to find the functional dependence of the coordinates on time and should integrate over all the possible coordinate configurations as the function of time. This should be quite different from the expression of eq.(G.1.10a). If one wishes to sum up all the possible paths in eq.(G.1.10b), then one may have to first consider the following expression of the coordinate $x$ as the function of time

$$
x = x_{cl}(t') + \sum_{n=1}^{\infty} y_n \sin\left(\frac{2\pi n}{t}\right) t'
$$

where $x_{cl}(t')$ denotes the classical coordinate that satisfies the Newton equation of motion with the initial conditions of $x_{cl}(0) = x$ and $x_{cl}(t) = x'$. The amplitude $y_n$ is the expansion coefficient. Therefore, the integrations over all the paths should mean that one should integrate over

$$
\prod_{n=1}^{\infty} \int_{-\infty}^{\infty} dy_n
$$

and, in this case, one can easily check that the calculated result of the integration over all the paths cannot reproduce the proper quantum mechanical result for the harmonic oscillator case. This simply means that the integration over $dx_1 \cdots dx_{n-1}$ in eq.(G.1.10a) and the integration over classical paths are completely different from each other, which is a natural result. This fact is certainly known to some careful physicists, but most of the path integral textbooks are reluctant to putting emphasis on the fact that the classical trajectories should not be summed up in the path integral formulation. Rather, they say that the summation of all the classical paths should correspond to the quantization by the path integral, which is a wrong and misleading statement.

## 1.3　Advantage of Path Integral

What should be any merits of the path integral formulation ? Eqs.(G.1.10) indicate that one can carry out the first quantization of the classical system once the Lagrangian is given where the kinetic energy term should have a quadratic shape, that is, $c\dot{x}^2$ with $c$

some constant. In this case, one can obtain the quantized expression just by tracing back from eq.(G.1.10b) to eq.(G.1.10a).

There is an advantage of the path integral formulation. That is, one does not have to solve the differential equation. Instead, one should carry out many dimensional integrations. It is, of course, not at all clear whether the many dimensional integrations may have some advantage over solving the differential equation or not. However, one can, at least, claim that the procedure of the many dimensional integrations is indeed an alternative method to solving the Schrödinger equation.

## 1.4 Harmonic Oscillator Case

When the potential $U(x)$ is a harmonic oscillator

$$U(x) = \frac{1}{2} m \omega^2 x^2$$

then one can evaluate the amplitude analytically after some lengthy calculations

$$K(x', x : t) = \sqrt{\frac{m\omega}{2i\pi \sin \omega t}} \exp \left\{ i \frac{m\omega}{2} \left[ (x'^2 + x^2) \cot \omega t - \frac{2x'x}{\sin \omega t} \right] \right\}. \qquad (G.1.12)$$

On the other hand, one finds

$$K(x, x : t) = \langle x | e^{-iHt} | x \rangle = \sum_n e^{-iE_n t} |\psi_n(x)|^2. \qquad (G.1.13)$$

Therefore, if one integrates $K(x, x : t)$ over all space, then one obtains

$$\int_{-\infty}^{\infty} dx K(x, x : t) = \sum_{n=0}^{\infty} e^{-iE_n t} = \int_{-\infty}^{\infty} dx \sqrt{\frac{m\omega}{2i\pi \sin \omega t}} e^{-im\omega x^2 \tan \frac{\omega t}{2}} = \frac{1}{2i \sin \frac{\omega t}{2}}. \quad (G.1.14)$$

Since the last term can be expanded as

$$\frac{1}{2i \sin \frac{\omega t}{2}} = \frac{e^{-\frac{i}{2}\omega t}}{1 - e^{-i\omega t}} = e^{-\frac{i}{2}\omega t} \sum_{n=0}^{\infty} (e^{-i\omega t})^n = \sum_{n=0}^{\infty} e^{-i\omega t(n+\frac{1}{2})} \qquad (G.1.15)$$

one obtains by comparing two equations

$$E_n = \omega \left( n + \frac{1}{2} \right) \qquad (G.1.16)$$

which is just the right energy eigenvalue of the harmonic oscillator potential in quantum mechanics.

It should be important to note that the evaluation of eq.(G.1.12) is entirely based on the expression of eq.(G.1.10a) which is just the quantum mechanical equation. Therefore, this example of the harmonic oscillator case shows that the rewriting of the amplitude $K(x', x : t)$ is properly done in obtaining eq.(G.1.10a). This does not prove any connection of the $K(x', x : t)$ to the classical mechanics.

# 2 Path Integral in Field Theory

The basic notion of the path integral was introduced by Feynman [1, 2, 3], and the formulation of the path integral in quantum mechanics is given in terms of many dimensional integrations of the discretized coordinates $x_n$. As one sees from eq.(G.1.10), the amplitude is expressed in terms of many dimensional integrations with the weight factor of $e^{iS}$ where $S$ is the action of the classical mechanics. This was, of course, surprising and interesting. However, as Feynman noted in his original papers, the path integral expression is not more than the ordinary quantum mechanics.

When the classical particle interacts with the electromagnetic field $\boldsymbol{A}$, the amplitude of the particle can be expressed in terms of the many dimensional integrations of the action of the classical particle. However, the electromagnetic field $\boldsymbol{A}$ is already a quantum mechanical object, and therefore, there is no need for the first quantization in the Maxwell equation. However, when one wishes to treat physical processes which involve the absorption or emission of photon, then one has to quantize the electromagnetic field $\boldsymbol{A}$ which is called field quantization or second quantization.

## 2.1 Field Quantization

The field quantization of the electromagnetic field $\boldsymbol{A}$ can be done by expanding the field $\boldsymbol{A}$ in terms of the plane wave solutions

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \boldsymbol{\epsilon}(\boldsymbol{k}, \lambda) \left[ c_{\boldsymbol{k},\lambda} e^{-ikx} + c_{\boldsymbol{k},\lambda}^{\dagger} e^{ikx} \right]. \qquad (G.2.1)$$

The field quantization requires that $c_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}^{\dagger}$ should be operators which satisfy the following commutation relations

$$[c_{\boldsymbol{k},\lambda}, \ c_{\boldsymbol{k}',\lambda'}^{\dagger}] = \delta_{\boldsymbol{k},\boldsymbol{k}'} \delta_{\lambda,\lambda'} \qquad (G.2.2)$$

and all other commutation relations should vanish. This is the standard way of the second quantization procedure even though it is not understood well from the fundamental principle. However, it is obviously required from experiments since electron emits photon when it decays from the $2p_{\frac{1}{2}}$ state to the $1s_{\frac{1}{2}}$ state in hydrogen atom.

## 2.2 Field Quantization in Path Integral (Feynman's Ansatz)

In his original paper, Feynman proposed a new method to quantize the electromagnetic field $\boldsymbol{A}$ in terms of the path integral formulation [1, 2, 3]. Here, we should first describe his formulation of the path integral. For the fermion part, he employed the particle expression, and therefore the path integral is defined in terms of quantum mechanics.

For the gauge field, Feynman started from the Hamiltonian formulation of the electromagnetic field. The Hamiltonian of the electromagnetic field can be expressed in terms

of the sum of the harmonic oscillators

$$H_{el} = \frac{1}{2} \sum_{\boldsymbol{k},\lambda} \left( p_{\boldsymbol{k},\lambda}^2 + k^2 q_{\boldsymbol{k},\lambda}^2 \right) \qquad (G.2.3)$$

where $p_{\boldsymbol{k},\lambda}$ is a conjugate momentum to $q_{\boldsymbol{k},\lambda}$. Here, it should be noted that the $q_{\boldsymbol{k},\lambda}$ corresponds to the amplitude of the vector potential $\boldsymbol{A}(x)$. The classical $c_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}^\dagger$ can be expressed in terms of $p_{\boldsymbol{k},\lambda}$ and $q_{\boldsymbol{k},\lambda}$ as

$$c_{\boldsymbol{k},\lambda} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left( p_{\boldsymbol{k},\lambda} - i\omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda} \right) \qquad (G.2.4a)$$

$$c_{\boldsymbol{k},\lambda}^\dagger = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left( p_{\boldsymbol{k},\lambda} + i\omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda} \right). \qquad (G.2.4b)$$

In this case, the Hamiltonian can be written in terms of $c_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}^\dagger$ as

$$H_{el} = \frac{1}{2} \sum_{\boldsymbol{k},\lambda} \omega_{\boldsymbol{k}} \left( c_{\boldsymbol{k},\lambda}^\dagger c_{\boldsymbol{k},\lambda} + c_{\boldsymbol{k},\lambda} c_{\boldsymbol{k},\lambda}^\dagger \right). \qquad (G.2.5)$$

It should be important to note that the Hamiltonian of eq.(G.2.3) is originated from the Hamiltonian of field theory and it has nothing to do with the classical Hamiltonian of Newton dynamics. For the electromagnetic field, there is no corresponding Hamiltonian of the Newton dynamics.

### 2.2.1 Feynman's Ansatz

Feynman proposed a unique way of carrying out the field quantization [1, 2, 3]. Since the Hamiltonian of the electromagnetic field can be written as the sum of the harmonic oscillators, he presented the path integral formulation for the electromagnetic field

$$K(q_{\boldsymbol{k},\lambda}, q_{\boldsymbol{k},\lambda}', t) \equiv \mathcal{N} \int [\mathcal{D}q_{\boldsymbol{k},\lambda}] \exp \left\{ \frac{i}{2} \int_0^t \sum_{\boldsymbol{k},\lambda} \left( \dot{q}_{\boldsymbol{k},\lambda}^2 - k^2 q_{\boldsymbol{k},\lambda}^2 \right) dt \right\} \qquad (G.2.6)$$

which should correspond to the quantization of the variables $q_{\boldsymbol{k},\lambda}$, and this corresponds to the quantization of the $c_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}^\dagger$. Thus, it is the second quantization of the electromagnetic field.

In this expression, there is an important assumption for the coordinates $q_{\boldsymbol{k},\lambda}$ which are the parameters appearing in the vector potential. That is, the states $|q_{\boldsymbol{k},\lambda}\rangle$ should make a complete set. Only under this assumption, one can derive the quantization of the harmonic oscillators. Since this is the parameter space, it may not be easy to prove the completeness of the states $|q_{\boldsymbol{k},\lambda}\rangle$. Nevertheless, Feynman made use of the path integral expression to obtain the Feynman rules in the perturbation theory for QED. It may also be important to note that the path integral in Feynman's method has nothing to do with the integration of the configuration space. It is clear that one should not integrate out over the configuration space in the path integral since the field quantization should be done for the parameters $c_{\boldsymbol{k},\lambda}$ and $c_{\boldsymbol{k},\lambda}^\dagger$.

## 2.3 Electrons Interacting through Gauge Fields

When one treats the system in which electrons are interacting through electromagnetic fields, one can write the whole system in terms of the path integral formulation. In this case, however, we treat electrons in the non-relativistic quantum mechanics. The electromagnetic fields are treated just in the same way as the previous section.

$$K(q_{\boldsymbol{k},\lambda}, q'_{\boldsymbol{k},\lambda}, \boldsymbol{r}, \boldsymbol{r}', t) \equiv \mathcal{N} \int [\mathcal{D}\boldsymbol{r}][\mathcal{D}q_{\boldsymbol{k},\lambda}] \times$$

$$\exp\left\{ i \int_0^t \left( \frac{1}{2}m\dot{\boldsymbol{r}}^2 - g\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}) + \frac{1}{2}\sum_{\boldsymbol{k},\lambda} \left( \dot{q}_{\boldsymbol{k},\lambda}^2 - k^2 q_{\boldsymbol{k},\lambda}^2 \right) \right) dt \right\} \qquad (G.2.7)$$

where the vector potential $\boldsymbol{A}$ is given in eq.(G.2.1)

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \frac{\boldsymbol{\epsilon}(\boldsymbol{k}, \lambda)}{\sqrt{V}\omega_{\boldsymbol{k}}} \left[ \dot{q}_{\boldsymbol{k},\lambda} \cos(\boldsymbol{k} \cdot \boldsymbol{r}) + \omega_{\boldsymbol{k}} q_{\boldsymbol{k},\lambda} \sin(\boldsymbol{k} \cdot \boldsymbol{r}) \right] \qquad (G.2.8)$$

which is now rewritten in terms of the variables $q_{\boldsymbol{k},\lambda}$.

It should be noted that the path integral formulation works only for the electromagnetic field since its field Hamiltonian can be described by the sum of the harmonic oscillators. This is a very special case, and there is only little chance that one can extend his path integral formulation to other field theory models. In particular, it should be hopeless to extend the path integral formulation to the field quantization of quantum chromodynamics (QCD) since QCD includes fourth powers of $q_{\boldsymbol{k},\lambda}$. In this case, one cannot carry out the Gaussian integral in the parameter space of $q_{\boldsymbol{k},\lambda}$.

# 3 Problems in Field Theory Path Integral

In this section, we discuss the problems in the standard treatment of the path integral formulation in field theory models. Normally, one starts from writing the path integral formulation in terms of the many dimensional integrations over field variables.

## 3.1 Real Scalar Field as Example

For simplicity, we take a real scalar field in 1+1 dimensions. In most of the field theory textbooks, the amplitude $Z$ is written as

$$Z = \mathcal{N} \int [\mathcal{D}\phi(t,x)] \exp\left[ i \int \mathcal{L}(\phi, \partial_\mu \phi) dt dx \right] \qquad (G.3.1)$$

where the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ is given as

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2}\left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2}m^2\phi^2. \qquad (G.3.2)$$

If we rewrite the path integral definition explicitly in terms of the field variable integrations like eq.(G.1.10), we find

$$Z = \mathcal{N} \lim_{n \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k,\ell=1}^{n} d\phi_{k,\ell} \times$$

$$\exp\left[ i \sum_{k,\ell=1}^{n} \Delta t \Delta x \left( \frac{(\phi_{k,\ell} - \phi_{k-1,\ell})^2}{2(\Delta t)^2} - \frac{(\phi_{k,\ell} - \phi_{k,\ell-1})^2}{2(\Delta x)^2} - \frac{1}{2} m^2 \phi_{k,\ell}^2 \right) \right] \qquad (G.3.3)$$

where $\phi_{k,\ell}$ is defined as

$$\phi_{k,\ell} = \phi(t_k, x_\ell), \quad \text{with} \quad t_1 = t, \cdots, t_n = t' \quad \text{and} \quad x_1 = x, \cdots, x_n = x'. \qquad (G.3.4)$$

Also, $\Delta t$ and $\Delta x$ are defined as

$$\Delta t = \frac{(t - t')}{n}, \quad \Delta x = \frac{(x - x')}{n}. \qquad (G.3.5)$$

Now, we should examine the physical meaning of the expression of the amplitude $Z$ in eq.(G.3.3), and clarify as to what are the problems in eq.(G.3.3) in connection to the field quantization. The first problem is that eq.(G.3.3) does not contain any quantity which is connected to the initial and final states. This is clear since, in eq.(G.3.3), one should integrate over fields as defined and therefore no information of $\phi(t, x)$ and $\phi(t', x')$ is left while, in quantum mechanics version, the amplitude is described by the quantities $\psi_n(x)$ and $\psi_n^\dagger(x')$ which are the eigenstates of the Hamiltonian. The second problem is that the calculated result of the amplitude $Z$ must be only a function of $m, \Delta t, \Delta x$, that is

$$Z = f(m, \Delta t, \Delta x). \qquad (G.3.6)$$

This shows that the formulation which is started from many dimensional integrations over the field $\phi(t, x)$ has nothing to do with the second quantization. In addition, $Z$ depends on the artificial parameters $\Delta t$ and $\Delta x$, and this clearly shows that it cannot be related to any physical observables.

This is in contrast with the formulation of eq.(G.2.7) where the amplitude $K$ is specified by the quantum number of the parameter space $q_{\boldsymbol{k},\lambda}$ which is connected to the state with a proper number of photons, and it must be a function of $m, g, q_{\boldsymbol{k},\lambda}$

$$K = f(m, g, q_{\boldsymbol{k},\lambda}, q'_{\boldsymbol{k},\lambda}). \qquad (G.3.7)$$

In addition, the $K$ does not depend on the parameters $\Delta t$ and $\Delta x$, which is a natural result as one can see it from the quantum mechanical path integral formulation.

Finally, we note that the treatment of Feynman is based on the total QED Hamiltonian which is a conserved quantity. On the other hand, eq.(G.3.3) is based on the action which is obtained by integrating the Lagrangian density over space and time. As one knows, the Lagrangian density is not directly related to physical observables. Therefore, unless one

can confirm that the path integral is reduced to the quantum mechanical amplitude like eq.(G.1.10), one cannot make use of the field theory path integral formulation. In fact, one cannot rewrite the expression of eq.(G.3.3) in terms of the field theory Hamiltonian density $\mathcal{H}$, contrary to the path integral formulation in quantum mechanics. This shows that the amplitude $Z$ has nothing to do with the amplitude $K$ in eq.(G.2.7). In this respect, the amplitude $Z$ has no physical meaning, and therefore one can not calculate any physical quantities from the path integral formulation of $Z$ in field theory.

## 3.2 Lattice Field Theory

Most of the numerical calculations in the lattice field theory are based on the path integral formulation of eq.(G.3.3). Unfortunately, the path integral formulation of eq.(G.3.3) has lost its physical meaning, and therefore there is little chance that one can obtain any physics out of numerical simulations of the lattice field theory. In this respect, it is, in a sense, not surprising that the calculation of Wilson's area law [4] in QED is incorrect [5] as we saw in chapter 8.

Since Wilson's calculation is presented analytically, it may be worth writing again the result of the Wilson loop calculation in QED as given in chapter 8

$$W \equiv \mathcal{N} \prod_m \prod_\mu \int_{-\infty}^{\infty} dB_{m\mu} \exp\left( i \sum_P B_{n\mu} + \frac{1}{2g^2} \sum_{n\mu\nu} e^{if_{n\mu\nu}} \right). \qquad (8.49)$$

In the strong coupling limit, one can evaluate eq.(8.49) analytically as

$$W \simeq (g^2)^{-\Delta S/a^2} \qquad (G.3.8)$$

where $\Delta S$ should be an area encircled by the loop. This has the same behavior as that of eq.(G.3.6) since the lattice constant $a$ is equal to $a = \Delta x = \Delta t$, that is,

$$W = f(g, \Delta S, \Delta x, \Delta t). \qquad (G.3.9)$$

This amplitude $W$ has the dependence of the artificial parameters $\Delta t$ and $\Delta x$. This is completely different from eq.(G.3.7), and therefore one sees that the calculation of eq.(8.49) has no physical meaning, contrary to Feynman's treatment which has a right behavior as the function of the field parameters $q_{\boldsymbol{k},\lambda}$.

## 3.3 Physics of Field Quantization

The quantization of fields is required from experiment. Yet, it is theoretically quite difficult to understand the basic physics of the field quantization. The fundamental step of the quantization is that the Hamiltonian one considers becomes an operator after the quantization. The reason why one considers the Hamiltonian is because it is a conserved quantity. In this respect, one cannot quantize the Hamiltonian density since it is not a conserved quantity yet. This is an important reason why one must quantize the field in

terms of the creation and annihilation operator $c_{k,\lambda}$ and $c_{k,\lambda}^\dagger$ in QED. In this respect, it is clear that the field quantization must be done in terms of $c_{k,\lambda}$ and $c_{k,\lambda}^\dagger$ with which the Hamiltonian in classical QED can be described.

## 3.4 No Connection between Fields and Classical Mechanics

Here, we should make a comment on the discretized coordinates and fields. The discretized space is, of course, artificial, and there is no physics in the discretized fields and equations. In some textbooks [6], the field equation is derived from the picture that the field is constructed by the sum of springs in which the discretized coordinates of neighboring sites are connected by the spring. This picture can reproduce the field equation for a massless scalar field by adjusting some parameters, even though one started from a non-relativistic classical mechanics. However, this is obviously a wrong picture for a scalar field theory since the field equation has nothing to do with classical mechanics. It is somewhat unfortunate that it may have had some impact on the picture concerning lattice gauge field calculations as an excuse to make use of the discretized classical fields. As we saw in the previous section, the path integral formulation has nothing to do with the dynamics of classical mechanics, and it is, of course, clear that the field theory path integral is never connected to any dynamics of the classical field theory.

# 4 Partition Function $Z$ in Field Theory

As we saw in section G.2, Feynman's path integral formulation in terms of many dimensional integrations in parameter space is indeed a plausible method to quantize the fields in QED. However, people commonly use the expression of the amplitude $Z$ as defined in eq.(G.3.1).

## 4.1 Partition Function in QCD

The "new formulation" of the path integral in QCD was introduced by Faddeev and Popov who wrote the S-matrix elements as [7]

$$\langle \text{out}|\text{in}\rangle \equiv Z = \mathcal{N} \int [\mathcal{D}A_\mu^a(x)] \exp\left[i \int \mathcal{L}_{\text{QCD}} d^4x\right] \qquad (G.4.1)$$

where the definition of the path integral volume $[\mathcal{D}A_\mu^a(x)]$ is just the same as the one explained in the previous section

$$\int [\mathcal{D}A_\mu^a(x)] \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{\mu,a} \prod_{(i,j,k,\ell)}^{n} dA_\mu^a(x_0^i, x_1^j, x_2^k, x_3^\ell). \qquad (G.4.2)$$

The Lagrangian density $\mathcal{L}_{\text{QCD}}$ for QCD is given as

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \qquad (G.4.3)$$

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where $G_{\mu\nu}$ denotes the field strength in QCD as given in chapter 6

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \qquad (G.4.4)$$

However, the expression of the path integral in QCD in eq.(G.4.1) does not correspond to the field quantization. One can also understand why the path integral formulation of QCD cannot be done, in contrast to Feynman's integrations over the parameter space in QED. In QCD, the Hamiltonian for gluons contains the fourth power of $q_{k,\lambda}^a$, and therefore one cannot carry out the Gaussian integrations over the parameters $q_{k,\lambda}^a$ in QCD. As Feynman stated repeatedly in his original papers, the path integral formulation is closely connected to the Gaussian integration where the kinetic energy term in non-relativistic quantum mechanics is always described in terms of the quadratic term $\frac{p^2}{2m}$. This naturally leads to the conclusion that the path integral formulation cannot be properly constructed in QCD, and this is consistent with the picture that QCD does not have a free Fock space due to its gauge non-invariance [8].

## 4.2   Fock Space

In quantum field theory, one must prepare Fock space since the Hamiltonian becomes an operator. The second quantized formulation is based on the creation and annihilation operators which act on the Fock space. In Feynman's path integral formulation, he prepared states which determine the number of photons in terms of $|q_{k,\lambda}\rangle$. Therefore, he started from the second quantized expression of the path integral formulation. However, Faddeev and Popov simply employed the same formula of the Lagrangian density, but integrated over the function $\mathcal{D}A_\mu^a(x)$. This cannot specify any quantum numbers of the Fock space, and therefore the integration over the function $\mathcal{D}A_\mu^a(x)$ does not correspond to the field quantization.

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