[レポート問題] 6/2 (略解)

1. (a) $x = x(q_1 q_2 q_3)$ なので 微分の公式より

$$\frac{dx}{dt} = \frac{\partial x}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial x}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial x}{\partial q_3}\frac{dq_3}{dt}$$

$$\therefore \quad \underline{\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3}$$

($\dot{y}$, $\dot{z}$ も同様.)

(b) 上式から

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_3} = \frac{\partial x}{\partial q_3}$$

まとめると $\underline{\dfrac{\partial \dot{x}}{\partial \dot{q}_k} = \dfrac{\partial x}{\partial q_k}}$ $(k=1,2,3)$

(y, z も同様)

(c) $x = r\sin\theta\cos\varphi$ より

$$\dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$$

よって $\dfrac{\partial \dot{x}}{\partial \dot{r}} = \sin\theta\cos\varphi$, $\dfrac{\partial \dot{x}}{\partial \dot{\theta}} = r\cos\theta\cos\varphi$, $\dfrac{\partial \dot{x}}{\partial \dot{\varphi}} = -r\sin\theta\sin\varphi$

$\updownarrow \qquad \updownarrow \qquad \updownarrow$

一方. $\dfrac{\partial x}{\partial r} = \sin\theta\cos\varphi$, $\dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi$, $\dfrac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$

で確かに (b) 式がみたされている

2. $\dot{x}\dfrac{\partial x}{\partial q_k}$ を t で微分すると

$$\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_k}\right) = \ddot{x}\frac{\partial x}{\partial q_k} + \dot{x}\underbrace{\frac{d}{dt}\frac{\partial x}{\partial q_k}}_{\frac{\partial \dot{x}}{\partial q_k}}$$

$$\therefore \quad \ddot{x}\frac{\partial x}{\partial q_k} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_k}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_k}$$

$\dfrac{\partial x}{\partial q_k} = \dfrac{\partial \dot{x}}{\partial \dot{q}_k}$ を用いて

$$\ddot{x}\frac{\partial x}{\partial q_k} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_k}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_k}$$

3. (a) $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$\begin{cases} \dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi \\ \dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi \\ \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases}$$

よって $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

(b) $L = T - U$ $\quad$ $U = 0$ とすれば

$L = T$

よって

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

Lagrange の方程式は

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

よって

(i)

$$m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\varphi}^2$$

$$\therefore\ \underline{a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2}$$

(ii)

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\sin\theta\cos\theta\,\dot{\varphi}^2$$

$$\therefore\ \underline{a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2}$$

(iii)

$$\frac{d}{dt}\left(mr^2\sin^2\theta\,\dot{\varphi}\right) = 0$$

$$\therefore\ \underline{a_\varphi = 2\dot{r}\dot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\,r\cos\theta + r\sin\theta\,\ddot{\varphi}}$$