7. 角運動量とスピン

7-1 軌道角運動量

角運動量 $\hat{\mathbb{L}}$ 　　 $\hat{\mathbb{L}} \equiv \mathbb{r} \times \hat{\mathbb{p}}$

【必要性】

ポテンシャルが中心力 $V = V(r)$ なと

角運動量が保存量となる

○ 角運動量の定義： $\hat{\mathbb{L}} = \mathbb{r} \times \hat{\mathbb{p}}$, $\hat{\mathbb{p}} = -i\hbar\nabla$

すなわち

$$
\begin{cases}
\hat{L}_x = -i\hbar\left(y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}\right) \\
\hat{L}_y = -i\hbar\left(z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z}\right) \\
\hat{L}_z = -i\hbar\left(x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}\right)
\end{cases}
$$

これを極座標でかくと

$$
\begin{cases}
\hat{L}_x = -i\hbar\left(-\sin\varphi\dfrac{\partial}{\partial\theta} - \cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}\right) \\
\hat{L}_y = -i\hbar\left(\cos\varphi\dfrac{\partial}{\partial\theta} - \cot\theta\sin\varphi\dfrac{\partial}{\partial\varphi}\right) \\
\hat{L}_z = -i\hbar\dfrac{\partial}{\partial\varphi}
\end{cases}
$$

No.

Date

ここで $\hat{L}_\pm \equiv \hat{L}_x \pm i \hat{L}_y$ を導入する

極座標でかくと

$$\begin{cases} \hat{L}_+ = \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \\ \hat{L}_- = \hbar e^{-i\varphi} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \\ \hat{L}_z = -i\hbar \frac{\partial}{\partial\varphi} \end{cases}$$

また $\hat{\mathbb{L}}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ を導入する

このとき

$\hat{\mathbb{L}}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar \hat{L}_z$ とかける

$$\left(\begin{aligned} \text{これは} \quad \hat{L}_-\hat{L}_+ &= (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) \\ &= \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) \\ &= \hat{L}_x^2 + \hat{L}_y^2 - \hbar\hat{L}_z \quad \text{より あきらか} \end{aligned} \right)$$

$\hat{\mathbb{L}}^2$ を極座標でかくと

$$\begin{aligned} \hat{\mathbb{L}}^2 &= \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z \\ &= \hbar^2 e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \\ &\quad - \hbar^2\frac{\partial^2}{\partial\varphi^2} - i\hbar^2\frac{\partial}{\partial\varphi} \end{aligned}$$

よって、$\hat{\mathbb{L}}^2 = \hbar^2\Big[-\frac{\partial^2}{\partial\theta^2} + i\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2} - \cot\theta\frac{\partial}{\partial\theta}$

$$-i\cot^2\theta\frac{\partial}{\partial\varphi} - \frac{\partial^2}{\partial\varphi^2} - i\frac{\partial}{\partial\varphi}\Big]$$

$\cot^2\theta + 1 = \frac{1}{\sin^2\theta}$ より

$$\hat{\mathbb{L}}^2 = -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\therefore \boxed{\hat{\mathbb{L}}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]} \quad /\!/$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\hat{\mathbb{L}}^2$$ とかける