

7-2 角運動量 $\hat{L}$ の極座標表現

No.

Date

(付録)

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$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial r} = \sin \theta \cos \varphi \frac{\partial}{\partial x} + \sin \theta \sin \varphi \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} = r \cos \theta \cos \varphi \frac{\partial}{\partial x} + r \cos \theta \sin \varphi \frac{\partial}{\partial y} - r \sin \theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \varphi} = -r \sin \theta \sin \varphi \frac{\partial}{\partial x} + r \sin \theta \cos \varphi \frac{\partial}{\partial y} \end{cases}$$

書き直すと

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \varphi & \cos \varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

29 行 34 行 R とする

R は 直交行列である。すなわち

$$\boxed{R^t \cdot R = 1}$$

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$$R^t = \begin{pmatrix} \sin\theta \cos\varphi & \cos\theta \cos\varphi & -\sin\varphi \\ \sin\theta \sin\varphi & \cos\theta \sin\varphi & \cos\varphi \\ \sin\theta & -\cos\theta & 0 \end{pmatrix}$$

 $\alpha, 2$

$$R^t \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \end{pmatrix} = R^t \cdot R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

 $\alpha, 2$

$$\begin{cases} \frac{\partial}{\partial x} = \sin\theta \cos\varphi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\varphi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} = \sin\theta \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\varphi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \end{cases}$$

or $\vec{r}$ as 3 or 3

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2a式を用いて $\hat{L}_z$ を極座標表示で表す。

$$(124) \quad \hat{L}_z$$

$$\hat{L}_z = (-i\hbar) \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$$

$$= (-i\hbar) \left[r \sin\theta \cos\varphi \left(\sin\theta \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\varphi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \right.$$

$$\left. - r \sin\theta \sin\varphi \left(\sin\theta \cos\varphi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\varphi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \right]$$

$$= -i\hbar \frac{\partial}{\partial \varphi}$$

$$\therefore \boxed{\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}}$$

同様にして

$$\left\{ \begin{array}{l} \hat{L}_x = -i\hbar \left[-\sin\varphi \frac{\partial}{\partial \theta} - \cot\theta \cos\varphi \frac{\partial}{\partial \varphi} \right] \\ \hat{L}_y = -i\hbar \left[\cos\varphi \frac{\partial}{\partial \theta} - \cot\theta \sin\varphi \frac{\partial}{\partial \varphi} \right] \end{array} \right.$$

が得られる。