No.

Date

8-2-5 角運動量 $\mathbb{L}$ の極座標表現

<u>極座標</u>

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

よって

$$\begin{cases} \dfrac{\partial}{\partial r} = \sin\theta\cos\varphi\dfrac{\partial}{\partial x} + \sin\theta\sin\varphi\dfrac{\partial}{\partial y} + \cos\theta\dfrac{\partial}{\partial z} \\[2ex] \dfrac{\partial}{\partial\theta} = r\cos\theta\cos\varphi\dfrac{\partial}{\partial x} + r\cos\theta\sin\varphi\dfrac{\partial}{\partial y} - r\sin\theta\dfrac{\partial}{\partial z} \\[2ex] \dfrac{\partial}{\partial\varphi} = -r\sin\theta\sin\varphi\dfrac{\partial}{\partial x} + r\sin\theta\cos\varphi\dfrac{\partial}{\partial y} \end{cases}$$

[書き直す]

$$\begin{pmatrix} \dfrac{\partial}{\partial r} \\[1ex] \dfrac{1}{r}\dfrac{\partial}{\partial\theta} \\[1ex] \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\[1ex] \dfrac{\partial}{\partial y} \\[1ex] \dfrac{\partial}{\partial z} \end{pmatrix}$$

ここでこの行列を R とおく

$$R \equiv \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

この時,

$$R^t = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

よって

$$R^t \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \end{pmatrix} = R^t R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

より

$$\begin{cases} \dfrac{\partial}{\partial x} = \sin\theta\cos\varphi\dfrac{\partial}{\partial r} + \dfrac{\cos\theta\cos\varphi}{r}\dfrac{\partial}{\partial\theta} - \dfrac{\sin\varphi}{r\sin\theta}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial y} = \sin\theta\sin\varphi\dfrac{\partial}{\partial r} + \dfrac{\cos\theta\sin\varphi}{r}\dfrac{\partial}{\partial\theta} + \dfrac{\cos\varphi}{r\sin\theta}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial z} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta} \end{cases}$$

となる.

この関係式を用いて $\mathbb{L}$ が極座標表示される

[例] L_z

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\Big[r\sin\theta\cos\varphi\Big(\sin\theta\sin\varphi\frac{\partial}{\partial r}$$

$$+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\Big)$$

$$-r\sin\theta\sin\varphi\Big(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\Big)\Big]$$

$$= -i\hbar\frac{\partial}{\partial\varphi}$$

$$\therefore \boxed{L_z = -i\hbar\frac{\partial}{\partial\varphi}}$$

同様に計算して

$$\boxed{\begin{aligned} L_x &= -i\hbar\left[-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right] \\ L_y &= -i\hbar\left[\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right] \end{aligned}}$$

が求められる