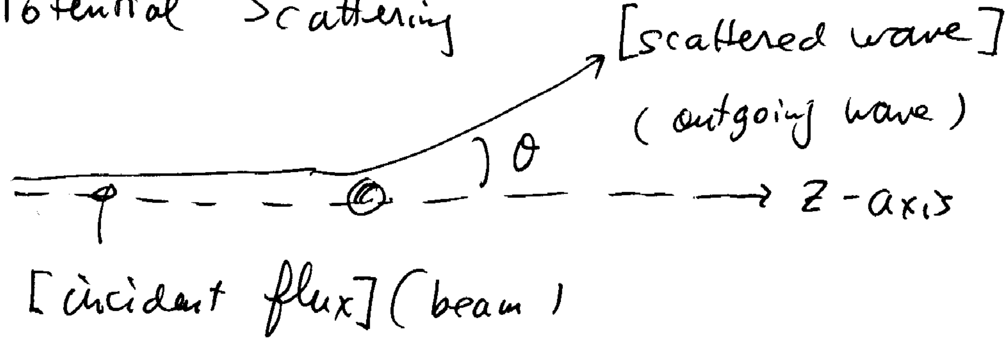
2. Scattering in Three Dimensions

2-1 Potential Scattering



Schrödinger eq.

$$\boxed{-\frac{1}{2m}\nabla^2\psi(r)+V(r)\psi(r)=E\psi(r)}$$

- We treat time-independently.
- Collision Theory : in principle it should be treated time-dependently

$$\left\{\begin{array}{c}\text{[wave packet]}\\ \Updownarrow \\ [\![\text{scattering T-matrix}]\!]\end{array}\right.$$

- Incident wave : $V(\mathbf{r}) = 0$

The incident particle cannot feel any interaction.

$\Downarrow$

free particle

$$\boxed{\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}}$$

($\frac{1}{\sqrt{V}}$ is not written.)

$\downarrow$

We always consider the probability

$\rightarrow$ $\boxed{\text{ratio} = \dfrac{\text{scattered flux}}{\text{incident flux}}}$

$\Downarrow$

This is related to the physical observable.

[Scattering is not eigenvalue problem !!]

$$(\nabla^2 + k^2)\, u(\mathbf{r}) = 0$$

$$\underline{u(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta}}$$

$$u(r) = u_\ell(r)\, Y_{\ell m}(\theta, \varphi)$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2} + k^2 \right] u_\ell\, Y_{\ell m}(\theta, \varphi) = 0$$

If we define $z \equiv kr$, then we find

$$\left[\frac{1}{z^2}\frac{d}{dz}\left(z^2 \frac{d}{dz}\right) - \frac{\ell(\ell+1)}{z^2} + 1 \right] u_\ell(z) = 0$$

$$\Downarrow$$

$$\boxed{u_\ell(z) \simeq j_\ell(z)}$$

[Mathematical formula]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^{l}(2l+1)\,\hat{j}_l(kr)\,P_l(\cos\theta)$$

At large r $(r\to\infty)$, we find

$$\hat{j}_l(kr) \approx \frac{1}{kr}\cos\left(kr - \frac{1}{2}(l+1)\pi\right)$$

$$\therefore \boxed{\hat{j}_l(kr) \approx \frac{1}{2kr}(-i)^{l+1}\left(e^{ikr} + (-)^{l+1}e^{-ikr}\right)}$$

Thus, at large r, we have

$$\boxed{\underset{(r\to\infty)}{e^{ikr\cos\theta}} \cong \sum_{l=0}^{\infty}\frac{(2l+1)}{(2ikr)}\left(e^{ikr} + (-)^{l+1}e^{-ikr}\right)P_l(\cos\theta)}$$

Legendre {Polynomial / function}

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\,Y_{l0}(\theta,\varphi)$$

Spherical harmonics

$$\begin{cases} e^{-\frac{1}{2}(l+1)\pi i} = (-i)^{l+1} \\ e^{-\frac{1}{2}\pi i} = (-i) \\ i^{l}\times(-i)^{l+1} = \frac{1}{i} \end{cases}$$

$$\left(e^{-\frac{1}{2}(l+1)\pi i}\right)^2 = \left((-i)^{l+1}\right)^2 = (-1)^{l+1}$$

- $\hat{j}_\ell(x)$: spherical Bessel function

- Bessel function :

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{du}{dz}\right) + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$$

$$u(z) = J_\nu(z)$$

Bessel function

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-)^n \left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(\nu+n+1)}$$

- spherical Bessel function

$$\hat{j}_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+\frac{1}{2}}(z)$$

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + \left(1 - \frac{n(n+1)}{z^2}\right)w = 0$$

$$w = \hat{j}_n(z)$$

$$\hat{j}_0(z) = \frac{1}{z}\sin z$$

$$\hat{j}_1(z) = \frac{1}{z^2}(\sin z - z\cos z)$$

- Outgoing wave (scattered wave)

$$\boxed{\psi_r \equiv f(\theta)\,\frac{e^{ikr}}{r}}$$

$f(\theta)$: scattering amplitude

[Note: $\frac{1}{r}e^{-ikr}$ no scattered particle]

- flux of the outgoing wave :

$$\hat{j}_r = \frac{1}{2mi}\left(\psi_r^* \frac{\partial \psi_r}{\partial r} - \left(\frac{\partial \psi_r^*}{\partial r}\right)\psi_r\right)$$

$$= \frac{1}{2mi}\left\{ \frac{e^{-ikr}}{r}\left(ik\frac{e^{ikr}}{r} - \frac{1}{r^2}e^{ikr}\right)|f|^2 - \left(-ik\frac{e^{-ikr}}{r} - \frac{1}{r^2}e^{-ikr}\right)\frac{e^{ikr}}{r}|f|^2 \right\}$$

$$\therefore \boxed{\hat{j}_r = \frac{k}{mr^2}|f(\theta)|^2}$$

- Cross section :

$$d\sigma \equiv \frac{\text{The flux in unit area at } (r^2 d\Omega)}{\text{incident flux}}$$

(outgoing flux)

(probability at $d\Omega$)

$$d\sigma = \frac{\hat{j}_r \, r^2 d\Omega}{\hat{j}_{in}} = \frac{\frac{k}{m} |f(\theta)|^2 d\Omega}{\frac{k}{m}}$$

$$\therefore \boxed{d\sigma = |f(\theta)|^2 d\Omega}$$

$\frac{d\sigma}{d\Omega}$: Differential cross section

- Dimension of $f(\theta)$:

$$\boxed{f(\theta) \sim L}$$

This is clear from the definition!!