2-2 Lippmann - Schwinger equation

- Lippmann-Schwinger eq. is just the same as the Schrödinger eq.

But, in a different shape

- Schrodinger eq.

$$\boxed{(H_0 + V)|\psi\rangle = E|\psi\rangle}$$

potential

$$H_0 = \frac{\hat{p}^2}{2m}$$

For free Hamiltonian H_0, we have

$$\boxed{H_0|\varphi\rangle = E_0|\varphi\rangle}$$

- Lippmann - Schwinger eq.

$$\boxed{|\psi\rangle = |\varphi\rangle + \frac{1}{E - H_0 + i\epsilon} V|\psi\rangle}$$

This is just the same as the Schrödinger eq.

(proof)

$$|\psi\rangle = |\varphi\rangle + \frac{1}{E-H_0+i\varepsilon} V|\psi\rangle$$

We operate $(E-H_0)$ from the left,

$$(E-H_0)|\psi\rangle = \underbrace{(E-H_0)|\varphi\rangle}_{=0} + (E-H_0)\frac{1}{E-H_0+i\varepsilon}V|\psi\rangle$$

$$= V|\psi\rangle$$

$$\therefore \underline{(E-H_0-V)|\psi\rangle = 0}$$

$\downarrow$ Schrödinger eq.

[Solution of Lippman - Schwinger eq.]

$$|\psi\rangle = |\varphi\rangle + \frac{1}{E - H_0 + i\varepsilon} V|\psi\rangle$$

We take the coordinate representation

$$\boxed{\langle \mathbf{r}|\psi\rangle = \langle \mathbf{r}|\varphi\rangle + \int d^3r' d^3r'' \langle \mathbf{r}| \frac{1}{E - H_0 + i\varepsilon} |\mathbf{r}'\rangle \langle \mathbf{r}'|V|\mathbf{r}''\rangle \langle \mathbf{r}''|\psi\rangle}$$

Here, we assume V is diagonal

$$\underline{\langle \mathbf{r}'|V|\mathbf{r}''\rangle = V(\mathbf{r}')\,\delta(\mathbf{r}' - \mathbf{r}'')}$$

$$\therefore \psi(\mathbf{r}) = \varphi(\mathbf{r}) + \int \langle \mathbf{r}| \frac{1}{E - H_0 + i\varepsilon} |\mathbf{r}'\rangle V(\mathbf{r}')\psi(\mathbf{r}')\, d^3r'$$

This is an integral equation.

$$\underline{G(\mathbf{r}, \mathbf{r}') \equiv \langle \mathbf{r}| \frac{1}{E - H_0 + i\varepsilon} |\mathbf{r}'\rangle}$$

is called Green function.

[Evaluation of Green function]

$$G(\mathbf{r},\mathbf{r}') = \int d^3p \, \langle \mathbf{r}|\mathbf{p}\rangle \langle \mathbf{p}| \frac{1}{E-H_0+i\epsilon} |\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{r}'\rangle$$

$$\begin{cases} \langle \mathbf{r}|\mathbf{p}\rangle = \dfrac{1}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{p}\cdot\mathbf{r}} \\ \langle \mathbf{p}| \dfrac{1}{E-H_0+i\epsilon} |\mathbf{p}\rangle = \dfrac{1}{E-\frac{\mathbf{p}^2}{2m}+i\epsilon} \end{cases}$$

$$\therefore \boxed{G(\mathbf{r},\mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E-\frac{\mathbf{p}^2}{2m}+i\epsilon} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}}$$

$E = \dfrac{\mathbf{k}^2}{2m}$: incident energy

$$\therefore \boxed{G(\mathbf{r},\mathbf{r}') = \frac{2m}{(2\pi)^3} \int d^3p \, \frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2-p^2+i\epsilon}}$$

$$\underline{d^3p = p^2 dp \, d\Omega_p}$$

$$\int e^{ip|\mathbf{r}-\mathbf{r}'|\cos\theta_p}\, d\Omega_p$$

$$= 2\pi \int_{-1}^{1} dt\; e^{ip|\mathbf{r}-\mathbf{r}'|t} = \frac{2\pi}{ip|\mathbf{r}-\mathbf{r}'|}\left(e^{ip|\mathbf{r}-\mathbf{r}'|} - e^{-ip|\mathbf{r}-\mathbf{r}'|}\right)$$

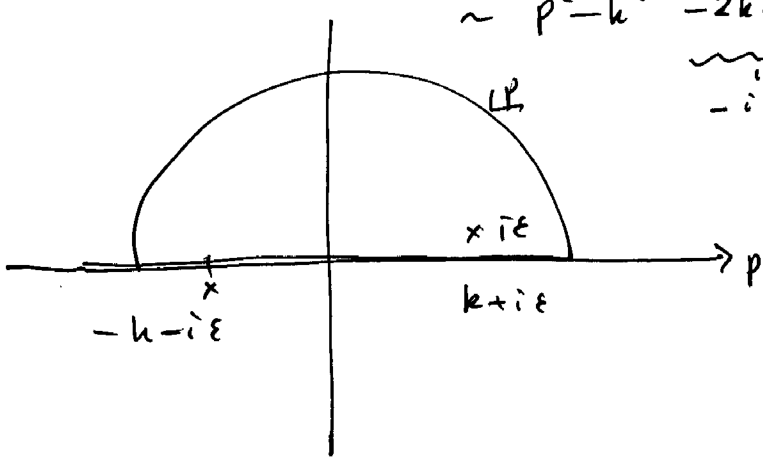
$$= \frac{4\pi}{p|\mathbf{r}-\mathbf{r}'|} \sin\left(p|\mathbf{r}-\mathbf{r}'|\right)$$

$$\therefore\ G(\mathbf{r},\mathbf{r}') = \frac{2m}{(2\pi)^3}\,\frac{4\pi}{|\mathbf{r}-\mathbf{r}'|}\,\frac{(-)}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ip|\mathbf{r}-\mathbf{r}'|}}{p^2-k^2-i\varepsilon}\, p\,dp$$

$$= -\frac{m}{2\pi^2|\mathbf{r}-\mathbf{r}'|}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ip|\mathbf{r}-\mathbf{r}'|}}{p^2-k^2-i\varepsilon}\, p\,dp$$

$$p^2-k^2-i\varepsilon = (p-k-i\varepsilon)(p+k+i\varepsilon)$$

$$\sim p^2-k^2 \underbrace{-2ki\varepsilon}_{=\,-i\varepsilon}$$



$$\therefore \quad G(\mathbf{r},\mathbf{r}') = \frac{(-m)}{2\pi^2 |\mathbf{r}-\mathbf{r}'|} \, \mathrm{Im} \, \frac{1}{2k} \, e^{ik|\mathbf{r}-\mathbf{r}'|} \cdot k \cdot 2\pi i$$

$$\therefore \quad \boxed{G(\mathbf{r},\mathbf{r}') = -\frac{m}{2\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \, e^{ik|\mathbf{r}-\mathbf{r}'|}}$$

Here, we take the large r $(r \to \infty)$.

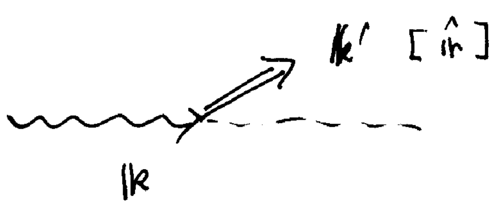
Then, we find

$$\left\{ \begin{array}{l} |\mathbf{r}-\mathbf{r}'| = r - \dfrac{(\mathbf{r}\cdot\mathbf{r}')}{r} + \cdots \\ \mathbf{r}\cdot\mathbf{r}' = r r' \cos\theta \end{array} \right.$$

$$\therefore \quad G(\mathbf{r},\mathbf{r}') = -\frac{m}{2\pi} \frac{1}{r} \, e^{ikr} \, e^{-ikr'\cos\theta}$$

$$\therefore \quad \boxed{G(\mathbf{r},\mathbf{r}') = -\frac{m}{2\pi r} \, e^{ikr} \, e^{-i\mathbf{k}'\cdot\mathbf{r}'}}$$

$$(\mathbf{k}' \equiv k\hat{\mathbf{r}})$$



$(\mathbf{k}' \equiv k\hat{\mathbf{r}})$