3. Time-dependent Perturbation Theory

3-1 Non-static perturbation theory

We consider the system

$$\boxed{H = H_0 + H_I(t)}$$

$H_I(t)$ is assumed to be small, though it depends on time.

- Schrödinger eq.

$$\boxed{i \frac{\partial \psi}{\partial t} = (H_0 + H_I(t))\, \psi}$$

- We expand $\psi(r,t)$ in terms of the eigenfunction $u_n(r)$

$$\boxed{\begin{aligned} H_0 u_n &= E_n u_n \\ \langle u_n | u_m \rangle &= \delta_{nm} \end{aligned}}$$

$u_n(r)$ should be known functions.

Thus, we find

$$\sum_n i\left\{ \frac{da_n}{dt} u_n e^{-iE_n t} - i E_n a_n u_n e^{-iE_n t} \right\}$$

$$= \sum_n (E_n + H_I)\, a_n u_n e^{-iE_n t}$$

By operating $\langle u_k|$ from the left-hand side, we obtain

$$\boxed{\frac{da_k}{dt} = -i \sum_n \langle u_k | H_I | u_n \rangle e^{i(E_k - E_n)t} a_n(t)}$$

This can be formally solved as

$$a_k(t) = a_k(0) - i \int_0^t \sum_n \langle u_k | H_I | u_n \rangle e^{i\omega_{kn} t'} a_n(t')\, dt'$$

with $\underline{\omega_{kn} \equiv E_k - E_n}$