3-3 Transition Probability

We evaluate the transition probability from $\underbrace{\text{m-state}}_{(2p)}$ to $\underbrace{\text{k-state}}_{(1s)}$ in hydrogen atom.

$$a_k(t) = a_k(0) - i \sum_n \int_0^t H'_{kn}(t') e^{i\omega_{kn}t'} a_n(t') dt'$$

$$\begin{cases} \omega_{kn} \equiv E_k - E_n \\ H'_{kn} \equiv \langle u_k | H_I | u_n \rangle \end{cases}$$

- Initial state : At $t=0$, the state is found at $\underline{\underline{\text{m-state}}}$ (2p).

  $\therefore \underline{a_k(0) = \delta_{km}}$

- Final state : At $t \to \infty$, the state is found at $\underline{\underline{\text{k-state}}}$ (1s).

$$\therefore \boxed{a_k^{(1)}(t) = -i \int_0^t H'_{km}(t')\, e^{i\omega_{km} t'}\, dt'}$$

- The transition probability W

$$\boxed{W = \frac{1}{T} \sum_m \left| a_k^{(1)}(t) \right|^2}$$

- $H'_I = -\frac{e}{m_e} \mathbb{A} \cdot \hat{\mathbb{P}}$, $\omega_p \equiv |\mathbb{P}|$

$$\therefore \langle u_k | H'_I | u_m \rangle \equiv H'_{km}$$

$$= -\frac{e}{m_e} \langle u_k | \hat{\mathbb{P}} \cdot \epsilon | u_m \rangle \frac{1}{\sqrt{2V\omega_p}} e^{-i\omega_p t}$$

$$\equiv V_{km} \frac{1}{\sqrt{2V\omega_p}} e^{-i\omega_p t}$$

(~~there~~ Here, $\underline{V_{km} \equiv -\frac{e}{m_e} \langle u_k | \hat{\mathbb{P}} \cdot \epsilon | u_m \rangle}$)

- Further, we find

$$\sum_m = \frac{V}{(2\pi)^3} \int d^3 p$$

$$\left(\mathbb{P} = \frac{2\pi}{L} m \right)$$

Thus, W becomes

$$W = \frac{1}{T} \cdot \frac{V}{(2\pi)^3} \int d^3p \left| \int_0^T \frac{1}{\sqrt{2V\omega_p}} V_{km} e^{i(\omega_{km}-\omega_p)t'} dt' \right|^2$$

$(T \to \infty)$

First, we evaluate the integral $| \quad |^2$

$$I \equiv \left| \int_0^T e^{i(\omega_{km}-\omega_p)t'} dt' \right|^2$$

$$= 2\pi \int_0^T e^{i(\omega_{km}-\omega_p)t'} \delta(\omega_{km}-\omega_p)\, dt'$$

$$\boxed{I = 2\pi T \, \delta(\omega_{km}-\omega_p)}$$

$$\therefore \quad W = \frac{1}{T} \frac{V}{(2\pi)^3} 2\pi T \cdot \frac{1}{2V} \int d^3p \, \frac{1}{\omega_p} |V_{km}|^2 \delta(\omega_{km}-\omega_p)$$

$$\left\{ \begin{array}{l} d^3p = 4\pi p^2 dp \\ \rho(p) \equiv \frac{p}{4\pi^2} \end{array} \right.$$

$$\therefore \quad \boxed{W = 2\pi |V_{km}|^2 \rho(p)}$$ (Energy dimension)

$$V_{km} = -\frac{e}{m_e} \langle u_k | \mathbf{p} \cdot \epsilon | u_m \rangle$$