5-3 Partial Wave Expansion

Partial wave : angular momentum $\underset{\sim}{l}$

- Identity equation :

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

↓

This $\boxed{l}$ - wave → partial wave !!

- Lippman - Schwinger equation

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{e^{ikr}}{r}\frac{m}{2\pi}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')d^3r'$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{ikr}}{r} \qquad (r\to\infty)$$

$$\left(f(\theta) \equiv -\frac{m}{2\pi}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')d^3r' \right)$$

$$\boxed{\psi(\mathbf{r}) \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r}}$$

$$(r\to\infty)$$

At the large distance

$$\boxed{e^{ikz} \underset{(r\to\infty)}{\Longrightarrow} \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)\left\{e^{ikr}-(-)^l e^{-ikr}\right\}P_l(\cos\theta)}$$

Therefore, the scattering wave can be written as

$$\boxed{\psi(\mathbf{r}) \underset{(r\to\infty)}{\Longrightarrow} \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)\left\{(1+2ikA_l)\,e^{ikr}-(-)^l e^{-ikr}\right\} \times P_l(\cos\theta)}$$

where we assume

$$\boxed{f(\theta) = \sum_{l=0}^{\infty}(2l+1)A_l\,P_l(\cos\theta)}$$

A_l : some constant

[Requirement]

- The scattering amplitude must have the same magnitude as the initial one except its phase

$$\Downarrow \quad \boxed{1 + 2ikA_\ell = e^{2i\delta_\ell}}$$

δ_ℓ : phase shift

$$\therefore \quad \boxed{A_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right)}$$

$$\Downarrow$$

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) P_\ell(\cos\theta)$$

The scattering amplitude is now described in terms of the phase shift δ_ℓ

[Cross section]

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

with $$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\cos\theta)$$

- total cross section :

$$\sigma = \int |f(\theta)|^2 \, d\Omega$$

$$= \frac{1}{4k^2}\sum_{l,l'}(2l+1)(2l'+1)\left(e^{2i\delta_l}-1\right)\left(e^{2i\delta_{l'}}-1\right)$$

$$\times \int P_l(\cos\theta)\, P_{l'}(\cos\theta)\, d\Omega$$

Identity : $$\int P_l(\cos\theta)\, P_{l'}(\cos\theta)\, d\Omega = \frac{4\pi}{2l+1}\delta_{ll'}$$

$$\therefore \sigma = \frac{1}{4k^2}\sum_l 16\pi(2l+1)\sin^2\delta_l$$

$$\therefore \boxed{\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l}$$

[Optical Theorem]

Total cross section :

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l}$$

On the other hand, the amplitude $f(\theta)$ can be written as

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta)$$

Here, we can easily check

$$e^{2i\delta_l} - 1 = 2i\, e^{i\delta_l} \frac{(e^{i\delta_l} - e^{-i\delta_l})}{2i}$$

$$= 2i\, e^{i\delta_l} \sin\delta_l$$

Therefore,

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) \cdot 2i\, e^{i\delta_l} \sin\delta_l \cdot P_l(\cos\theta)$$

$$\therefore f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta)$$

Here, we take $\theta = 0$.

$$f(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell$$

since $P_\ell(1) = 1$.

Thus, we find

$$\boxed{\mathrm{Im}\, f(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell}$$

Since the total cross section is

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell$$

$\Downarrow$ Therefore, we obtain

$$\boxed{\sigma = \frac{4\pi}{k}\, \mathrm{Im}\, f(0)}$$

[Optical theorem]