5-4 Separable Interaction

The Lippmann-Schwinger equation for the T-matrix can be written as

$$T = V + VGT$$

More explicitly, we write

$$\langle \mathbb{k}' | T | \mathbb{k} \rangle = \langle \mathbb{k}' | V | \mathbb{k} \rangle + \sum_{\mathbb{k}''} \langle \mathbb{k}' | V | \mathbb{k}'' \rangle \langle \mathbb{k}'' | \frac{1}{E - H_0} | \mathbb{k}'' \rangle \times \langle \mathbb{k}'' | T | \mathbb{k} \rangle$$

Here, we define the Green's function in momentum space as

$$\boxed{G_E(\mathbb{k}) \equiv \frac{1}{E - \frac{\mathbb{k}^2}{2m} + i\varepsilon}}$$

Thus, we find

$$\boxed{\langle \mathbb{k}' | T | \mathbb{k} \rangle = \langle \mathbb{k}' | V | \mathbb{k} \rangle + \sum_{\mathbb{k}''} \langle \mathbb{k}' | V | \mathbb{k}'' \rangle G_E(\mathbb{k}'') \langle \mathbb{k}'' | T | \mathbb{k} \rangle}$$

- Separable interaction (potential)

We take

$$\boxed{\langle \mathbb{k}' | V | \mathbb{k} \rangle = \lambda g(\mathbb{k}') g(\mathbb{k})}$$

$$\begin{cases} \lambda : & \text{some constant} \\ g(\mathbb{k}) : & \text{vertex} \end{cases}$$

Therefore, we obtain

$$\langle \mathbb{k}' | T | \mathbb{k} \rangle = \lambda g(\mathbb{k}') g(\mathbb{k}) + \lambda g(\mathbb{k}') \times \int \frac{d^3k''}{(2\pi)^3} g(\mathbb{k}'') Q_E(\mathbb{k}'') \langle \mathbb{k}'' | T | \mathbb{k} \rangle$$

We further assume that

$\langle \mathbb{k}' | T | \mathbb{k} \rangle$ should be written as

$$\boxed{\langle \mathbb{k}' | T | \mathbb{k} \rangle = \beta(E) \, g(\mathbb{k}') g(\mathbb{k})}$$

In this case, the equation for the T-matrix becomes

$$\beta(E)\, g(\mathbf{k}')\, g(\mathbf{k}) = \lambda\, g(\mathbf{k}')\, g(\mathbf{k}) + \lambda\, g(\mathbf{k}')\, \beta(E)\, g(\mathbf{k})$$

$$\times \int \frac{d^3\mathbf{k}''}{(2\pi)^3} \left[g(\mathbf{k}'') \right]^2 \times G_E(\mathbf{k}'')$$

Therefore, we find

$$\boxed{\beta(E) = \lambda + \lambda\, \beta(E) \cdot \int \frac{d^3 k''}{(2\pi)^3} \left[g(\mathbf{k}'') \right]^2 G_E(\mathbf{k}'')}$$

We here define

$$\left\{ \begin{array}{l} F(E) \equiv \displaystyle\int \frac{d^3 k''}{(2\pi)^3} \left(g(\mathbf{k}'') \right)^2 G_E(\mathbf{k}'') \\ \\ E : \text{ incident energy} \quad E = \dfrac{k^2}{2m} \end{array} \right.$$

$$\therefore \quad \boxed{\beta(E) = \frac{\lambda}{1 - \lambda F(E)}}$$

Since the T-matrix is assumed to be

$$\langle \mathbf{k}' | T | \mathbf{k} \rangle = \beta(E)\, g(\mathbf{k}')\, g(\mathbf{k})$$

we find

$$\boxed{\langle \mathbf{k}' | T | \mathbf{k} \rangle = \frac{\lambda}{1 - \lambda F(E)}\, g(\mathbf{k}')\, g(\mathbf{k})}$$

<u>This is the exact solution!</u>

- Note: The separable interaction is not realistic, therefore, a toy model.