5-5 S-matrix Evaluation

(a) Propagator :

$$S = T\left\{ e^{-i\int \mathcal{H}' d^4x} \right\}$$

$$\mathcal{H}' = -\hat{J}_\mu A^\mu$$

$$\begin{cases} \hat{J}_\mu = \bar{\psi}\gamma_\mu\psi , \\ A^\mu : \text{vector potential} \end{cases}$$

- Second order perturbation

$$S = \frac{(-i)^2}{2}\int d^4x_1 d^4x_2 \, T\left\{ \hat{J}_\mu(x_1)\hat{J}_\nu(x_2) A^\mu(x_1) A^\nu(x_2) \right\}$$

- Propagator of Photon

$$\langle 0 | T\{ A^\mu(x_1) A^\nu(x_2) \} | 0 \rangle$$

(a) Feynman's propagator

We quantized the vector potential A_μ

$$A_\mu(x) = \sum_{\lambda,k} \frac{1}{\sqrt{2V\omega_k}} \epsilon_\mu(\lambda,k)\left[c_{k\lambda} e^{+ikx} + c^+_{k,\lambda} e^{-ikx} \right]$$

$$(kx = \omega t - k \cdot r)$$

$$\langle 0 | T\{ A_\mu(x_1) A_\nu(x_2) \} | 0 \rangle$$

$$= \sum_{\lambda,k} \sum_{\lambda',k'} \frac{1}{\sqrt{2V\omega_k}} \frac{1}{\sqrt{2V\omega_{k'}}} \epsilon_\mu(\lambda,k)\, \epsilon_\nu(\lambda',k')$$

$$\times \langle 0 | T\{ (c_{k,\lambda} e^{ikx_1} + c^+_{k\lambda} e^{-ikx_1})(c_{k'\lambda'} e^{ik'x_2} + c^+_{k'\lambda'} e^{-ik'x_2}) \} | 0 \rangle$$

$$\boxed{\delta_{kk'}\, \delta_{\lambda,\lambda'}\, e^{ik(x_1-x_2)}}$$

$$= \sum_{\lambda,k} \frac{1}{2V\omega_k} \epsilon_\mu(\lambda,k)\, \epsilon_\nu(\lambda,k) \Big\{ e^{ik(x_1-x_2)}\, \theta(t_1-t_2) + e^{ik(x_2-x_1)}\, \theta(t_2-t_1) \Big\}$$

Then, we can show that

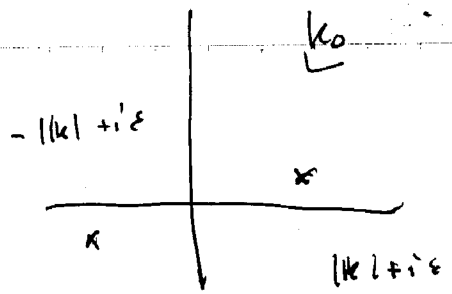
$$\sum_k \frac{1}{2\omega_k} \left\{ e^{ik(x_1-x_2)}\, \theta(t_1-t_2) + e^{-ik(x_1-x_2)}\, \theta(t_2-t_1) \right\} = (-i) \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x_1-x_2)}}{k^2+i\varepsilon}$$

$$(d^4k = d^3k \cdot dk_0)\ (k^2 = k_0^2 - k^2)$$

$t = t_1 - t_2$

k_0- integration:

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{+ik_0 t}}{k_0^2 - |k|^2 \pm i\epsilon}$$



$$k_0^2 - |k|^2 \pm i\epsilon = (k_0 - |k| \pm i\epsilon)(k_0 + |k| \mp i\epsilon)$$

(i) $t > 0$

$$k_0 = R e^{i\theta} = R\cos\theta + i R\sin\theta$$

(upper plane)

$$k_0 = +|k| + i\epsilon$$

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{ik_0 t}}{k_0^2 - |k|^2 \pm i\epsilon} = \frac{1}{2\pi} \cdot 2\pi i \frac{e^{+i|k|t}}{(+2|k|)}$$

$$= i \frac{e^{i\omega_k t}}{2\omega_k} \quad //$$

(ii) $t < 0$

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{+ik_0 t}}{k_0^2 - |k|^2 - i\epsilon} = -\frac{1}{2\pi} \cdot 2\pi i \frac{e^{-i\omega_k t}}{-2|k|}$$

$$= i \frac{e^{-i\omega_k t}}{2\omega_k} \quad //$$

• Evaluation of the polarization summation.

$$\boxed{\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda)}$$

(1) Feynman propagator :

$$\boxed{\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda) = -g_{\mu\nu}}$$

Therefore, the Feynman propagator of photon becomes

$$\boxed{D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 - i\varepsilon}}$$

• Problem of Feynman propagator :

1. This does not satisfy the Lorentz condition

$$k_\mu \sum_{\lambda=1} \varepsilon^\mu \varepsilon^\nu = 0 \qquad \text{(l.h.s.)}$$

$$-k_\mu g^{\mu\nu} = -k^\nu \qquad \text{(r.h.s.)}$$

2. The Coulomb gauge fixing is not satisfied.

(2) Correct propagator :

Coulomb gauge : $\mathbb{k}\cdot\mathbb{E} = 0$

$\Downarrow$

Therefore, we have

$$\boxed{\varepsilon_0 = 0}$$

Only the vector field $\mathbb{A}$ should be quantized.

$\Downarrow$

The polarization sum becomes

$$\boxed{\sum_{\lambda=1}^{2} \varepsilon^a(\mathbb{k},\lambda)\, \varepsilon^b(\mathbb{k},\lambda) = \left(\delta^{ab} - \frac{k^a k^b}{\mathbb{k}^2}\right)}$$

This can satisfy the Coulomb gauge condition.

$$\left\{\begin{array}{l} k^a \sum_{\lambda=1}^{2} \varepsilon^a(\mathbb{k},\lambda)\, \varepsilon^b(\mathbb{k},\lambda) = k^b - \frac{\mathbb{k}^2 k^b}{\mathbb{k}^2} = 0 \\ \text{l.h.s.} = k^a \varepsilon^a = \mathbb{k}\cdot\mathbb{\varepsilon} = 0 \end{array}\right.$$

OK.

Therefore, the correct propagator becomes

$$\begin{cases} D^{ab}(k) = \dfrac{\left(\delta^{ab} - \dfrac{k^a k^b}{\mathbb{k}^2}\right)}{k^2 - i\varepsilon} & \text{A - part} \\[2em] D^{\cdot\cdot}(k) = -\dfrac{1}{\mathbb{k}^2} \end{cases}$$

(b) Lorentz condition (Before quantization)

Free wave solution :

$$A^\mu(x) = \sum_{k,\lambda} \frac{1}{\sqrt{2V\omega_k}} \varepsilon^\mu_{(k,\lambda)} \left[c^\dagger_{k\lambda} e^{-ikx} + c_{k\lambda} e^{ikx} \right]$$

$$(kx = \omega_k t - \mathbf{k}\cdot\mathbf{r})$$

- Equation of motion :

$$\boxed{\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0} \Leftarrow \partial_\mu F^{\mu\nu} = 0$$

(Maxwell equation)

We insert the vector field solution into the above Maxwell eq.

We find

$$k^2 \varepsilon^\mu - (k_\nu \epsilon^\nu) k^\mu = 0$$

$$\therefore \sum_{\nu=0}^{3} \{ k^2 g^{\mu\nu} - k^\mu k^\nu \} \varepsilon_\nu = 0$$

$$\downarrow$$

$$\boxed{\det \{ k^2 g^{\mu\nu} - k^\mu k^\nu \} = 0}$$

$$\therefore \boxed{k^2 = 0} \Rightarrow \boxed{k_\mu \varepsilon^\mu = 0}$$ (Lorentz Condition)