4. S-matrix Expansion

4-1 Interaction picture

$$i \frac{\partial \Psi_I}{\partial t} = e^{iH_0 t} H_I e^{-iH_0 t} \Psi_I$$

Now, we define

$$\boxed{H_I' \equiv e^{iH_0 t} H_I e^{-iH_0 t}}$$

Then, the Schrödinger eq. becomes

$$\boxed{i \frac{\partial \Psi_I}{\partial t} = H_I' \Psi_I}$$

4-2 Formal Solution

We define $U(t, t_0)$ by

$$\Psi_I(t) = U(t, t_0) \Psi_I(t_0)$$

$\Downarrow$ Then, we obtain

$$\boxed{i \frac{\partial U(t,t_0)}{\partial t} = H_I'(t) U(t, t_0)}$$

We can write the formal solution as

$$\boxed{U(t,t_0) = U(t_0,t_0) - i\int_{t_0}^{t} H_I'(t')\,U(t',t_0)\,dt'}$$

This can be easily confirmed by inserting it into $\underline{i\frac{\partial U}{\partial t} = H_I'U}$.

◎ Iteration Method

By iteration, we find

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I'(t')\,dt'$$
$$+ (-i)^2\int_{t_0}^{t}\int_{t_0}^{t''} H_I'(t'')\,H_I'(t')\,dt'\,dt''$$
$$+ \cdots$$

〚Problem〛 [There is a t''-dependence in the integration range.]

[Time-ordered Product]

We introduce the time-ordered product

$$T\left[H_I'(t_1)\,H_I'(t_2)\right] \equiv \begin{cases} H_I'(t_1)\,H_I'(t_2) & (t_1 > t_2) \\ H_I'(t_2)\,H_I'(t_1) & (t_2 > t_1) \end{cases}$$

$\Downarrow$

This is just the <u>symmetrization</u>.

This leads to

$$\boxed{\begin{aligned} U(t,t_0) = 1 &- i\int_{t_0}^{t} H_I'(t')\,dt' \\ &+ \frac{1}{2}(-i)^2\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\, T\left[H_I'(t')\,H_I'(t'')\right] \\ &+ \cdots \end{aligned}}$$

[Definition of S- matrix]

$$S \equiv \lim_{\substack{t \to \infty \\ t_0 \to -\infty}} U(t, t_0)$$

$$= 1 - i \int_{-\infty}^{\infty} H_I'(t')\, dt'$$

$$+ \frac{1}{2}(-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, T\left[H_I'(t_1) H_I'(t_2)\right]$$

$$+ \cdots + \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, T\left[H_I'(t_1) \cdots H_I'(t_n)\right]$$

$$+ \cdots$$

Here, $\boxed{H_I \equiv \int \mathcal{H}_I \, d^3x.}$

$\mathcal{H}_I$: interaction Hamiltonian density

$$\boxed{S \equiv T\left\{ e^{-i \int \mathcal{H}_I \, d^4x} \right\}}$$

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【S- matrix elements】

$$\boxed{S_{fi} = \langle f | S | i \rangle}$$

$$\begin{cases} |i\rangle : & \text{initial state} \\ |f\rangle : & \text{final state} \end{cases}$$

$$\underline{S_{fi} = \delta_{fi} - i\,\delta(E_f - E_i)\, T_{fi}}$$

T_{fi} : T- matrix

- Transition probability : w_{fi}

$$w_{fi} = \frac{1}{T} |S_{fi}|^2 = \delta(E_f - E_i) \frac{\delta(0)}{T} |T_{fi}|^2$$

$$\therefore \boxed{w_{fi} = \delta(E_f - E_i) |T_{fi}|^2}$$

Note ; $\quad \delta(E) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-iEt}\, dt$

$$\underline{\delta(0) = T}$$

- Transition rate :

$$\boxed{d\Gamma_{fi} = \omega_{fi}\, dN_f}$$

$$dN_f = \sum_N = \frac{V}{(2\pi)^3}\int d^3p$$

(one-body-final state)

In general

$$\boxed{dN_f = \frac{V^n}{(2\pi)^{3n}}\int d^3p_1 \cdots d^3p_n}$$