5-2 Born Approximation

scattering amplitude :

$$\boxed{f(\mathbf{k},\mathbf{k}') = -\frac{m}{2\pi}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')\,\psi_{\mathbf{k}}(\mathbf{r}')\,d^3r'}$$

$$(\mathbf{k}' \equiv k\hat{\mathbf{r}})$$

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')\,\psi_{\mathbf{k}}(\mathbf{r}')\,d^3r'$$

This amplitude $f(\mathbf{k},\mathbf{k}')$ is related to the T-matrix as

$$f(\mathbf{k},\mathbf{k}') = -\frac{m}{2\pi}\langle \mathbf{k}'|T|\psi_{\mathbf{k}}\rangle$$

◉ Born approximation :

$$\boxed{f(\mathbf{k},\mathbf{k}') \cong -\frac{m}{2\pi}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')\,e^{i\mathbf{k}\cdot\mathbf{r}'}\,d^3r'}$$

[Rutherford Scattering]

Coulomb potential

$$\underline{V(r) = \frac{\alpha}{r}}$$

In this case, we find

$$f(\mathbf{k},\mathbf{k}') = \frac{m\alpha}{2\pi}\int \frac{e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}}{r}\, d^3r$$

Defining the momentum transfer $\mathbf{q}$ by

$$\boxed{\mathbf{q} \equiv \mathbf{k}-\mathbf{k}'}$$

We obtain

$$f(\mathbf{q}) = \frac{m\alpha}{2\pi}\int \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{r}\, d^3r = \frac{m}{2\pi}\, 2\pi \int e^{iqr\cos\theta}\,\frac{r^2}{r}\, dr\, dt$$

$$(t = \cos\theta)$$

$$= m\int_{-\infty}^{\infty}\int_{-1}^{1} e^{iqrt}\, dt\; r\, dr$$

$$= \frac{m\alpha}{q}\, 2\int_0^{\infty} \sin qr\, dr$$

$$\therefore \boxed{f(\mathbf{q}) = \frac{2m\alpha}{q^2}}$$

Thus, the cross section for the Rutherford scatting can be written a

$$\frac{d\sigma}{d\Omega} = |f(q)|^2 = \frac{4m^2\alpha^2}{q^4} = \frac{\alpha^2}{(4E)^2}\cdot\frac{1}{\sin^4\frac{\theta}{2}}$$

when we made use of the following relations

$$\begin{cases} q^2 = |\mathbf{k}-\mathbf{k}'|^2 = 4k^2\sin^2\frac{\theta}{2} \\ E = \frac{k^2}{2m} \end{cases}$$